Math 114, HW 2

Due Friday, April 17

1. Suppose that $C \subseteq U$ is the set generated inductively from $B \subseteq U$ and a family of functions $\mathcal{T}$ where for each $f \in \mathcal{T}$, there is a finite k_f such that $f : U^{k_f} \to U$. Further suppose that the functions in $\mathcal{T}$ are one-to-one, have pairwise disjoint ranges, and have ranges disjoint from B. Suppose we are given functions $h_1 : B \to V_1$, $h_2 : B \to V$, and for each $f \in \mathcal{T}$, functions $h_1^f : V_1^{k_f} \times V_2^{k_f} \to V_1$ and $h_2^f : V_1^{k_f} \times V_2^{k_f} \to V_2$.

   Prove carefully, showing every step, that there are unique functions $\overline{h_1} : C \to V_1$ and $\overline{h_2} : C \to V_2$ such that:

   - For every $b \in B$, $\overline{h_1}(b) = h_1(b)$ and $\overline{h_2}(b) = h_2(b)$
   - For any $f \in \mathcal{T}$, $a_1, \ldots, a_{k_f} \in C$,

     $$\overline{h_1}(f(a_1, \ldots, a_{k_f})) = h_1^f(\overline{h_1}(a_1), \ldots, \overline{h_1}(a_{k_f}), \overline{h_2}(a_2), \ldots, \overline{h_2}(a_{k_f}))$$

     and

     $$\overline{h_2}(f(a_1, \ldots, a_{k_f})) = h_2^f(\overline{h_1}(a_1), \ldots, \overline{h_1}(a_{k_f}), \overline{h_2}(a_2), \ldots, \overline{h_2}(a_{k_f}))$$

2. Demonstrate, using a truth table, that the following are tautologies:

   - $((A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C)))$
   - $(A \vee (\neg A))$

3. Demonstrate the the following is not a tautology:

   - $(((A \wedge B) \to C) \leftrightarrow ((A \to C) \wedge (B \to C)))$

4. Let α be a wff whose only connective symbols are $\wedge, \vee$, and $\neg$. Let α^* be the result of interchanging $\wedge$ and $\vee$ and replacing each sentence symbol by its negation. Show that α^* is tautologically equivalent to $(\neg\alpha)$.

Puzzle questions (optional):

- You are in a land inhabited by people who either always tell the truth or always tell falsehoods. You come to a fork in the road and you need to know which fork leads to the capital. There is a local resident there, but he only has time to reply to one yes-or-no question. What one question should you ask so as to learn which fork to take?

- Near the border of the same land, you find three people lined up in a row, one of whom always lies, one who always tells the truth, and one who can do either at will. You want to take one with you as a companion, but would like to avoid the one who can either lie or tell the truth, since with such a crazy companion you would never know when to believe them. What one question may you ask one of them (not knowing, of course, which one that is) so that you can then choose one of the three to be your companion, sure that they will either always tell the truth or always lie.