Math 114, HW 4

Due Friday, May 8

1. Show that none of the following sentences are implied by the other two; this requires giving a model where the sentence in question is false while the other two are true.
   (a) $\forall x, y, z(Pxy \to Pyz \to Pxz)$
   (b) $\forall x, y(Pxy \land Pyx \to x = y)$
   (c) $\forall x, y(Pxy \lor Pyx \lor x = y)$
2. Show that $\{\forall x(\alpha \to \beta), \forall x\alpha\} \vDash \forall x\beta$
3. Show that $\vDash \exists x[Dx \to \forall y Dy]$
4. A universal formula is a formula in the form $\forall x_1 \cdots \forall x_n \theta$ where θ is quantifier-free. (Such formulas are often called Π_1.) An existential formula is a formula of the form $\exists x_1 \cdots \exists x_n \theta$ where θ is again quantifier-free. (Such formulas are often called Σ_1.)

   Let $\mathfrak{A}$ be a substructure of $\mathfrak{B}$ and let $s : V \to |\mathfrak{A}|$.
   (a) Show that if $\vDash_{\mathfrak{B}} \phi[s]$ and ϕ is universal then $\vDash_{\mathfrak{A}} \phi[s]$.
   (b) Show that if $\vDash_{\mathfrak{A}} \phi[s]$ and ϕ is existential then $\vDash_{\mathfrak{B}} \phi[s]$.
   (c) Show that the sentence $\forall x Px$ is not logically equivalent to any existential sentence.
5. Consider a language with equality, a constant symbol 0, a unary function S, and a binary relation $<$. Consider the model $(\mathbb{N}, 0, S, <)$ where 0, S, and $<$ are given their usual meanings. Let $\mathfrak{A}$ be some other model of this language such that $\mathfrak{A} \equiv (\mathbb{N}, 0, S, <)$. Show that there is an homomorphism $h : \mathbb{N} \to |\mathfrak{A}|$ and that $\mathfrak{A}$ is an *end-extension* of $\mathbb{N}$: that is, for every $a \in \mathfrak{A}$, either there is an $n \in \mathbb{N}$ such that $a = h(n)$, or for every $n \in \mathbb{N}$, $h(n) < a$.