Final Exam Guidelines

The final exam will cover:

- Sections 1.0-1.5, 1.7
- Section 2.0-2.2, 2.4-2.6, 2.8
- Section 3.1 up to Corollary 31E

In addition to all textbook problems from the chapters above, here are a selection of additional practice problems. Sample answers to these problems will be posted prior to the exam.

1. Construct a set Σ of formulas of sentential logic, and for each integer n, a formula τ_n, such that $\Sigma \models \tau_n$ but whenever $\Sigma_0 \subseteq \Sigma$ and $\Sigma_0 \models \tau_n$, $|\Sigma_0| \geq n$. (For instance, τ_3 is not implied by any subset of Σ of size 2. It will be helpful to start by constructing examples Σ and τ_n for small n before addressing the general case.)

2. Consider a first order language with a binary predicate symbol $<$ and a unary function symbol f. Express the statement
$$\lim_{x \to x_0} f(x) = y$$
as a formula of first order logic using the standard $\epsilon - \delta$ definition of a limit, in such a way that the formula will be true if and only if the limit does go to y when the underlying model is the usual real numbers.

3. Consider a language with a constant symbol 0, a function symbol S, and a binary predicate $<$, and the standard model given by $\mathbb{N}$. Describe an elementarily equivalent model properly extending $\mathbb{N}$ and a homomorphism from $\mathbb{N}$ into this model.

4. Give an example showing that if we drop axiom group 4, the resulting calculus is no longer complete.

5. Consider an expansion of first order logic by a new quantifier $\exists_\infty$, and extend $\models$ to formulas in this language by adding the clause
$$\models_{\mathfrak{A}} \exists_\infty x \phi[s] \Leftrightarrow \{a \in |\mathfrak{A}| \mid \models_{\mathfrak{A}} \phi[s(x \mapsto a)]\} \text{ is infinite.}$$
Show that there are no additional axiom groups which could be added to make the Completeness Theorem go through for the expanded language. (Hint: recall that the Completeness Theorem implies the Compactness Theorem.)

6. Consider a language with a single binary predicate P. Consider the model with universe $\mathbb{N}$ such that P is interpreted by the empty set (that is, $\langle n, m\rangle \notin P^{\mathbb{N}}$ for any n, m). Show that the theory of this model is complete.

7. Show that for every $r \in \mathbb{R}$, there is a $q \in {}^*\mathbb{Q}$ such that $st(q) = r$. (Note that the elements of ${}^*\mathbb{Q}$ are exactly ratios n/m where $n, m \in {}^*\mathbb{N}$.