Final Exam Review problems

Construct a set Σ of formulas of sentential logic, and for each integer n, a formula τ_n, such that $\Sigma \vDash \tau_n$ but whenever $\Sigma_0 \subseteq \Sigma$ and $\Sigma_0 \vDash \tau_n$, $|\Sigma_0| \geq n$. (For instance, τ_3 is not implied by any subset of Σ of size 2. It will be helpful to start by constructing examples Σ and τ_n for small n before addressing the general case.)

We have seen examples before for small n, for instance setting

$$\Sigma = \{Px_1 \vee Px_2 \vee Px_3 \vee Px_4, \neg Px_1, \neg Px_2, \neg Px_3\},$$

clearly $\Sigma \vDash Px_4$ but there is no $\Sigma_0 \subseteq \Sigma$ with $|\Sigma_0| \leq 3$ such that $\Sigma_0 \vDash Px_4$.

To solve the problem, we simply generalize this idea for all n:

$$\Sigma = \{Px_{1,1} \vee Px_{1,2}, \neg Px_{1,1}, Px_{2,1} \vee Px_{2,2} \vee Px_{2,3}, \neg Px_{2,1}, \neg Px_{2,2}, \ldots\}$$

or, more compactly,

$$\Sigma = \{\bigvee_{j \leq n+1} Px_{n,j} \mid n \in \mathbb{N}\} \cup \{\neg Px_{n,j} \mid j \leq n \in \mathbb{N}\}.$$

Clearly $\Sigma \vDash Px_{n,n+1}$ for every n, but if $\Sigma_0 \subseteq \Sigma$ and $\Sigma_0 \vDash Px_{n,n+1}$ then $\{Px_{n,1} \vee \cdots \vee Px_{n,n+1}, \neg Px_{n,1}, \ldots, \neg Px_{n,n+1}\} \subseteq \Sigma_0$, so $|\Sigma_0| \geq n$.

Consider a first order language with a binary predicate symbol $<$ and a unary function symbol f. Express the statement

$$\lim_{x \to x_0} f(x) = y$$

as a formula of first order logic using the standard $\epsilon - \delta$ definition of a limit, in such a way that the formula will be true if and only if the limit does go to y when the underlying model is the usual real numbers.

$$\forall \epsilon \epsilon > 0 \to \exists \delta \delta > 0 \wedge \forall x \left[|x - x_0| < \delta \to |f(x) - y| < \epsilon\right].$$

Consider a language with a constant symbol 0, a function symbol S, and a binary predicate $<$, and the standard model given by $\mathbb{N}$. Describe an elementarily equivalent model properly extending $\mathbb{N}$ and a homomorphism from $\mathbb{N}$ into this model.

We obtain such a model by taking $Th\mathbb{N}$ and finding, by completeness, a model of $Th\mathbb{N} \cup \{S^n 0 < x_1 \mid n \in \mathbb{N}\}$. Such a model exists by compactness since $Th\mathbb{N} \cup \{S^n 0 < x_1 \mid n \leq m\}$ is satisfiable for each m.

It is easy to see that the map $n \mapsto S^n 0$ is a homomorphism.

Give an example showing that if we drop axiom group 4, the resulting calculus is no longer complete.

If c is a constant and P a unary predicate, $Pc \vDash \forall x Pc$, but this is not provable without axiom group 4.

Consider an expansion of first order logic by a new quantifier $\exists_\infty$, and extend $\models$ to formulas in this language by adding the clause

$$\models_{\mathfrak{A}} \exists_\infty x\phi[s] \Leftrightarrow \{a \in |\mathfrak{A}| \mid \models_{\mathfrak{A}} \phi[s(x \mapsto a)]\} \text{ is infinite.}$$

Show that there are no additional axiom groups which could be added to make the Completeness Theorem go through for the expanded language. (Hint: recall that the Completeness Theorem implies the Compactness Theorem.)

The compactness theorem fails in this expanded semantics. Let σ_n be the formula $\exists x_1 \cdots \exists x_n x_1 \neq x_2 \wedge \cdots \wedge x_1 \neg x_n \wedge x_2 \neq x_3 \wedge \cdots x_{n-1} \neq x_n$ (that is, σ_n asserts that there are at least n elements). Then $\{\sigma_n \mid n \in \mathbb{N}\} \models \exists_\infty x(x = x)$, but no finite subset of $\{\sigma_n \mid n \in \mathbb{N}\}$ implies $\exists_\infty x(x = x)$.

But if some extension of our proof system satisfied completeness, there would be a deduction of $\exists_\infty x(x = x)$ from $\{\sigma_n \mid n \in \mathbb{N}\}$, and this deduction would contain only finitely many σ_n. Therefore the proof system would not be sound.

Consider a language with a single binary predicate P. Consider the model with universe $\mathbb{N}$ such that P is interpreted by the empty set (that is, $\langle n, m\rangle \notin P^{\mathbb{N}}$ for any n, m). Show that the theory of this model is complete.

Since $\mathbb{N} \models \sigma_n$ for each n, every model of this theory is infinite. Also, this theory is $\aleph_0$-categorical, since if $\mathfrak{A}$ is a countable model, there is a bijection $\pi : \mathbb{N} \to |\mathfrak{A}|$, and π is clearly an isomorphism. By the Łoś-Vaught test, the theory is complete.

Show that for every $r \in \mathbb{R}$, there is a $q \in {}^*\mathbb{Q}$ such that $st(q) = r$. (Note that the elements of ${}^*\mathbb{Q}$ are exactly ratios n/m where $n, m \in {}^*\mathbb{N}$.)

For fixed r,

$$\models_{\mathbb{R}} \forall \delta > 0 \exists n, m \in \mathbb{N} |r - n/m| < \delta.$$

Therefore in ${}^*\mathbb{R}$, we may choose δ infinitesimal and find $n, m \in {}^*\mathbb{N}$ such that ${}^*|r^* {-} n^*/m|^* {<} \delta$, and therefore $r \sim n/m$.