Math 114L

Extra Credit Assignment

Spring 2010

1. Consider the model $\mathfrak{A}$ with $|\mathfrak{A}| = \mathbb{N}$, $<^{\mathfrak{A}} = <$, and $+^{\mathfrak{A}} = +$. Determine:

   (a) For which variable assignments s and which d_1, d_2, d_3 does

   $$\models_{\mathfrak{A}} v_1 + v_3 = v_2[s(v_1 \mapsto d_1, v_2 \mapsto d_2, v_3 \mapsto d_3)]?$$

   (b) For which variable assignments s and which d_1, d_2, d_3 does

   $$\models_{\mathfrak{A}} \neg v_1 + v_3 = v_2[s(v_1 \mapsto d_1, v_2 \mapsto d_2, v_3 \mapsto d_3)]?$$

   (c) For which variable assignments s and which d_1, d_2 does

   $$\models_{\mathfrak{A}} \forall v_3 \neg v_1 + v_3 = v_2[s(v_1 \mapsto d_1, v_2 \mapsto d_2)]?$$

   (d) For which variable assignments s and which d_1, d_2 does

   $$\models_{\mathfrak{A}} \neg \forall v_3 \neg v_1 + v_3 = v_2[s(v_1 \mapsto d_1, v_2 \mapsto d_2)]?$$

   (e) For which variable assignments s and which d_1, d_2 does

   $$\models_{\mathfrak{A}} \exists v_3 v_1 + v_3 = v_2[s(v_1 \mapsto d_1, v_2 \mapsto d_2)]?$$

   (f) For which variable assignments s and which d_1, d_2 does

   $$\models_{\mathfrak{A}} v_1 < v_2[s(v_1 \mapsto d_1, v_2 \mapsto d_2)]?$$

   (g) For which variable assignments s and which d_1, d_2 does

   $$\models_{\mathfrak{A}} v_1 < v_2 \rightarrow \exists v_3 v_1 + v_3 = v_2[s(v_1 \mapsto d_1, v_2 \mapsto d_2)]?$$

   (h) For which variable assignments s and which d_1 does

   $$\models_{\mathfrak{A}} \forall v_2 (v_1 < v_2 \rightarrow \exists v_3 v_1 + v_3 = v_2)[s(v_1 \mapsto d_1)]?$$

   (i) For which variable assignments s does

   $$\models_{\mathfrak{A}} \forall v_1 \forall v_2 (v_1 < v_2 \rightarrow \exists v_3 v_1 + v_3 = v_2)[s]?$$

(j) Does
$$\models_{\mathfrak{A}} \forall v_1 \forall v_2 (v_1 < v_2 \rightarrow \exists v_3 v_1 + v_3 = v_2)?$$

2. Does $\models_{\mathfrak{A}} \exists v_1 \forall v_2 v_2 < v_2 + v_1$? Break the question into steps, as in the previous part.

3. Let α be the formula $\forall x \forall y x < y \rightarrow \exists z x < z \wedge z < y$, let β be the formula $\forall x \exists y x < y$, and let γ be the formula $\forall x \forall y x < y \rightarrow \neg x = y$. Show that:

   (a) $\alpha; \gamma \nvdash \beta$
   (b) $\alpha; \gamma \nvdash \neg\beta$
   (c) $\beta; \gamma \nvdash \alpha$
   (d) $\beta; \gamma \nvdash \neg\alpha$

4. Write down a deduction showing that $\forall x \forall y \phi \vdash \forall x \phi$. (Don't just prove that the deduction exists; actually write it down as a list of formulas.)

5. Write down a deduction showing that $\forall x \forall y \phi \vdash \forall y \forall x \phi$. (Hint: the fact that there is such a deduction follows from the previous part and the generalization theorem. Use the proof of the generalization theorem to transform your deduction from the previous part, step by step, into the deduction for this part.)

6. Write down a deduction showing that $\vdash \forall x \forall y \phi \rightarrow \forall y \forall x \phi$. (Hint: similar to the last part, but with the deduction theorem.)

7. Prove, carefully, that if Γ is a consistent set of first-order formulas in a language $\mathcal{L}$ (and $\mathcal{L}$ has only countably many predicate and function symbols) then there is a consistent set $\Delta \supseteq \Gamma$ such that for every formula ϕ, either $\phi \in \Delta$ or $\neg\phi \in \Delta$.