Extra Credit

1. (a) For any $s : V \to |\mathfrak{A}|$, $\models_{\mathfrak{A}} v_1 + v_3 = [s(v_1|d_1)(v_2|d_2)(v_3|d_3)]$
   iff $\overline{s(v_1|d_1)(v_2|d_2)(v_3|d_3)}(v_1) +^{\mathfrak{A}} \overline{s(v_1|d_1)(v_2|d_2)(v_3|d_3)}(v_3) = \overline{s(v_1|d_1)(v_2|d_2)(v_3|d_3)}(v_2)$
   iff $d_1 + d_3 = d_2$.

   (b) For any $s : V \to |\mathfrak{A}|$, $\models_{\mathfrak{A}} \neg v_1 + v_3 = v_2[s(v_1|d_1)(v_2|d_2)(v_3|d_3)]$
   iff $\not\models_{\mathfrak{A}} \neg v_1 + v_3 = v_2[s(v_1|d_1)(v_2|d_2)(v_3|d_3)]$
   iff $d_1 + d_3 \neq d_2$ by part (a).

   (c) For any $s : V \to |\mathfrak{A}|$, $\models_{\mathfrak{A}} \forall v_3 \neg v_1 + v_3 = v_2[s(v_1|d_1)(v_2|d_2)]$
   iff $\models_{\mathfrak{A}} \neg v_1 + v_3 = v_2[s(v_1|d_1)(v_2|d_2)(v_3|c)]$ for all $c \in |\mathfrak{A}|$
   iff $\overline{s(v_1|d_1)(v_2|d_2)(v_3|c)}(v_1) +^{\mathfrak{A}} \overline{s(v_1|d_1)(v_2|d_2)(v_3|c)}(v_3) \neq \overline{s(v_1|d_1)(v_2|d_2)(v_3|c)}(v_2)$ for all $c \in |\mathfrak{A}|$
   iff $d_1 + c \neq d_2$ for all $c \in |\mathfrak{A}|$
   iff $d_1 > d_2$.

   (d) For any $s : V \to |\mathfrak{A}|$, $\models_{\mathfrak{A}} \neg\forall v_3 \neg v_1 + v_3 = v_2[s(v_1|d_1)(v_2|d_2)]$
   iff $\models_{\mathfrak{A}} \exists v_3 v_1 + v_3 = v_2[s(v_1|d_1)(v_2|d_2)]$
   iff $\models_{\mathfrak{A}} v_1 + v_3 = v_2[s(v_1|d_1)(v_2|d_2)(v_3|c)]$ for some $c \in |\mathfrak{A}|$
   iff $\overline{s(v_1|d_1)(v_2|d_2)(v_3|c)}(v_1) +^{\mathfrak{A}} \overline{s(v_1|d_1)(v_2|d_2)(v_3|c)}(v_3) = \overline{s(v_1|d_1)(v_2|d_2)(v_3|c)}(v_2)$ for some $c \in |\mathfrak{A}|$
   iff $d_1 + c = d_2$ for some $c \in |\mathfrak{A}|$
   iff $d_1 \leq d_2$.

   (e) For any $s : V \to |\mathfrak{A}|$, $\models_{\mathfrak{A}} \exists v_3 v_1 + v_3 = v_2[s(v_1|d_1)(v_2|d_2)]$
   iff $d_1 \leq d_2$ by part (d).

   (f) For any $s : V \to |\mathfrak{A}|$, $\models_{\mathfrak{A}} v_1 < v_2[s(v_1|d_1)(v_2|d_2)]$
   iff $\langle \overline{s(v_1|d_1)(v_2|d_2)}(v_1), \overline{s(v_1|d_1)(v_2|d_2)}(v_2) \rangle \in <^{\mathfrak{A}}$
   iff $d_1 < d_2$.

   (g) For any $s : V \to |\mathfrak{A}|$, $\models_{\mathfrak{A}} v_1 < v_2 \to \exists v_3 v_1 + v_3 = v_2[s(v_1|d_1)(v_2|d_2)]$
   iff $\models_{\mathfrak{A}} \neg v_1 < v_2[s(v_1|d_1)(v_2|d_2)]$ or $\models_{\mathfrak{A}} \exists v_3 v_1 + v_3 = v_2[s(v_1|d_1)(v_2|d_2)]$
   iff $\langle \overline{s(v_1|d_1)(v_2|d_2)}(v_1), \overline{s(v_1|d_1)(v_2|d_2)}(v_2) \rangle \notin <^{\mathfrak{A}}$ or $d_1 \leq d_2$ by (e)
   iff $d_1 \not< d_2$ or $d_1 \leq d_2$
   iff $d_2 \leq d_1$ or $d_1 \leq d_2$,
   which is true for all d_1 and d_2.

   (h) For any $s : V \to |\mathfrak{A}|$, $\models_{\mathfrak{A}} \forall v_2 (v_1 < v_2 \to \exists v_e v_1 + v_e = v_2)[s(v_1|d_1)]$
   iff $\models_{\mathfrak{A}} v_1 < v_2 \to \exists v_3 v_1 | v_3 = v_2[s(v_1|d_1)(v_2|d_2)]$ for all $d_2 \in |\mathfrak{A}|$,
   which is true for all d_1, by (g).

   (i) For any $s : V \to |\mathfrak{A}|$, $\models_{\mathfrak{A}} \forall v_1 \forall v_2 (v_1 < v_2 \to \exists v_3 v_1 + v_3 = v_2)[s]$
   iff $\models_{\mathfrak{A}} v_1 < v_2 \to \exists v_3 v_1 + v_3 = v_2[s(v_1|d_1)(v_2|d_2)]$ for all $d_1, d_2 \in |\mathfrak{A}|$,
   which is true by (g).

   (j) Yes, since by (i), this holds for every $s : V \to |\mathfrak{A}|$.

2. $\models_{\mathfrak{A}} \exists v_1 \forall v_2 v_2 < v_2 + v_1$

   iff for every $s : V \to |\mathfrak{A}|$, $\models_{\mathfrak{A}} \exists v_1 \forall v_2 v_2 < v_2 + v_1[s]$

   iff there is $d_1 \in |\mathfrak{A}|$ such that for all $d_2 \in |\mathfrak{A}|$, $\models_{\mathfrak{A}} v_2 < v_2 + v_1[s(v_1|d_1)(v_2|d_2)]$

   iff there is $d_1 \in |\mathfrak{A}|$ such that for all $d_2 \in |\mathfrak{A}|$, $\langle \overline{s(v_1|d_1)(v_2|d_2)}(v_2), \overline{s(v_1|d_1)(v_2|d_2)}(v_2) +^{\mathfrak{A}} \overline{s(v_1|d_1)(v_2|d_2)}(v_2) \rangle \in <^{\mathfrak{A}}$

   iff there is $d_1 \in \mathbb{N}$ such that for all $d_2 \in \mathbb{N}$, $d_2 < d_2 + d_1$.

   This is true, take $d_1 = 1$. $d_2 < d_2 + 1$ for all $d_2 \in \mathbb{N}$.

3. First note that α says that the ordering is dense, β says the the ordering has no maximum, and γ says that the ordering is strict. Further note that $\neg\alpha$ is $\exists x \exists y (x < y \wedge \neg \exists z (x < z \wedge z < y))$(there are two elements with no elements "between" them) and $\neg\beta$ is $\exists x \forall y \neg x < y$ (there is a max).

   Soundness and completeness tell us that $\Gamma \vdash \phi \Leftrightarrow \Gamma \models \phi$, so $\Gamma \nvdash \phi \Leftrightarrow \Gamma \not\models \phi$. So, to show $\Gamma \nvdash \phi$, it is enough to show $\Gamma \not\models \phi$. Thus, we just need a structure $\mathfrak{A}$ which models Γ and not ϕ.

   (a) $\alpha, \gamma \nvdash \beta$

   $\mathfrak{A} = ([0,1], <)$ with the usual ordering. This is dense and strict, but does not have a max.

   (b) $\alpha, \gamma \nvdash \neg\beta$

   $\mathfrak{A} = (\mathbb{Q}, <)$ with the usual ordering. This is dense and strict, but has no max (so $\neg\beta$ does not hold).

   (c) $\beta, \gamma \nvdash \alpha$

   $\mathfrak{A} = (\mathbb{N}, <)$ with the usual ordering. This has no max, is strict, but is not dense.

   (d) $\beta, \gamma \nvdash \neg\alpha$

   $\mathfrak{A} = (\mathbb{Q}, <)$ with the usual ordering. It has no max, is strict, and is dense (so $\neg\alpha$ does not hold).

4. (1) $\forall x(\forall y \phi \to \phi)$ (Ax 2)

   (2) $\forall x(\forall y \phi \to \phi) \to (\forall x \forall y \phi) \to \forall x \phi$ (Ax 3)

   (3) $\forall x \forall y \phi \to \forall x \phi$ (mp, lines 1 and 2)

   (4) $\forall x \forall y \phi$ (assumption)

   (5) $\forall x \phi$ (mp, lines 3 and 4)

   Therefore $\forall x \forall y \phi \vdash \forall x \phi$.

5. (1) $\forall y \forall x(\forall y \phi \to \phi)$ (Ax 2)

   (2) $\forall y[\forall x(\forall y \phi \to \phi) \to (\forall x \forall y \phi) \to \forall x \phi]$ (Ax 3)

   (3) $\forall y[\forall x(\forall y \phi \to \phi) \to (\forall x \forall y \phi) \to \forall x \phi] \to \forall y \forall x(\forall y \phi \to \phi) \to \forall y((\forall x \forall y \phi) \to \forall x \phi)$ (Ax 3)

   (4) $\forall y \forall x(\forall y \phi \to \phi) \to \forall y((\forall x \forall y \phi) \to \forall x \phi)$ (mp, lines 2 and 3)

   (5) $\forall y((\forall x \forall y \phi) \to \forall x \phi)$ (mp, lines 1 and 4)

(6) $\forall x \forall y \phi$ (assumption)

(7) $\forall x \forall y \phi \to \forall y \forall x \forall y \phi$ (Ax 4)

(8) $\forall y \forall x \forall y \phi$ (mp, lines 6 and 7)

(9) $\forall y((\forall x \forall y \phi) \to \forall x \phi) \to \forall y \forall x \forall y \phi \to \forall y \forall x \phi$ (Ax 3)

(10) $\forall y \forall x \forall y \phi \to \forall y \forall x \phi$ (mp, lines 5 and 9)

(11) $\forall y \forall x \phi$ (lines 8 and 10)

Therefore $\forall x \forall y \phi \vdash \forall y \forall x \phi$.

6. (1) $\forall y \forall x(\forall y \phi \to \phi)$ (Ax 2)

(2) $\forall y[\forall x(\forall y \phi \to \phi) \to (\forall x \forall y \phi) \to \forall x \phi]$ (Ax 3)

(3) $\forall y[\forall x(\forall y \phi \to \phi) \to (\forall x \forall y \phi) \to \forall x \phi] \to \forall y \forall x(\forall y \phi \to \phi) \to \forall y((\forall x \forall y \phi) \to \forall x \phi)$ (Ax 3)

(4) $\forall y \forall x(\forall y \phi \to \phi) \to \forall y((\forall x \forall y \phi) \to \forall x \phi)$ (mp, lines 2 and 3)

(5) $\forall y((\forall x \forall y \phi) \to \forall x \phi)$ (mp, lines 1 and 4)

(6) $\forall y((\forall x \forall y \phi) \to \forall x \phi) \to \forall y \forall x \forall y \phi \to \forall y \forall x \phi$ (Ax 3)

(7) $\forall y \forall x \forall y \phi \to \forall y \forall x \phi$ (mp, lines 5 and 6)

(8) $\forall x \forall y \phi \to \forall y \forall x \forall y \phi$ (Ax 4)

(9) $(\forall x \forall y \phi \to \forall y \forall x \forall y \phi) \to (\forall y \forall x \forall y \phi \to \forall y \forall x \phi) \to (\forall x \forall y \phi \to \forall x \forall y \phi)$ (Ax 1)

(10) $(\forall y \forall x \forall y \phi \to \forall y \forall x \phi) \to (\forall x \forall y \phi \to \forall x \forall y \phi)$ (mp, lines 8 and 9)

(11) $\forall x \forall y \phi \to \forall x \forall y \phi$ (mp, lines 7 and 10)

Therefore $\vdash \forall x \forall y \phi \to \forall x \forall y \phi$

7. Let Γ be a consistent set of first order formulas in a countable language. Then there are countably many first order formulas in this language: Consider a formula of length n. The language is countable, so in each of the n positions, there are countably many symbols which can be used. Thus, there are at most $\omega^n = \omega$ possible formulas of length n. Furthermore, formulas are finite in length, so there are at most $\omega \cdot \omega = \omega$ many formulas.

Hence, we can enumerate the formulas $\{\phi_1, \phi_2, \ldots\}$.

Define $\Gamma_0 := \Gamma$. Γ_0 is consistent (by assumption) and for every $i \leq 0$, $\phi_i \in \Gamma_0$ or $\neg\phi_i \in \Gamma_0$ (vacuously).

Suppose Γ_n has been defined and assume for all $i \leq 1$ that $\phi_i \in \Gamma_n$ or $\neg\phi_i \in \Gamma_n$, and that Γ_n is consistent.

Let $\mathfrak{A}_n$ be a structure in the language such that $\models_{\mathfrak{A}_n} \Gamma_n$ (this exists since Γ_n is consistent). Then $\models_{\mathfrak{A}_n} \phi_{n+1}$ or $\not\models_{\mathfrak{A}_n} \neg\phi_{n+1}$, so either $\models_{\mathfrak{A}_n} \phi_{n+1}$ or $\models_{\mathfrak{A}_n} \neg\phi_{n+1}$. In the first case, define $\Gamma_{n+1} := \Gamma_n \cup \{\phi_{n+1}\}$, in the second case, let $\Gamma_{n+1} := \Gamma_n \cup \{\neg\phi_{n+1}\}$. So, $\models_{\mathfrak{A}_n} \Gamma_{n+1}$, which means that Γ_{n+1} is consistent, and for all $i \leq n+1$, $\phi_i \in \Gamma_{n+1}$ or $\neg\phi_i \in \Gamma_{n+1}$. Now, define $\bigcup_{n \in \mathbb{N}} \Gamma_n$.

Let σ be given. Then, for some $i \in \mathbb{N}$, $\sigma = \phi_i$ (since the ϕ_i's enumerate all formulas). So either $\sigma \in \Gamma_i \subset \Delta$, or $\neg\sigma \in \Gamma_i \subset \Delta$. Hence, Δ is complete.

Let $\Delta_0 \subset \Delta$ be finte. Let i be the max such that $\phi_i \in \Delta_0$. Then $\Delta_0 \subset \Gamma_i$. By construction, Γ_i is consistent. Thus, Δ_0 is consistent. Hence, since $\Delta_0 \subset \Delta$ finite was arbitrary, by compactness, Δ is consistent.

Finally, $\Gamma \subset \Delta$. Thus, Δ is as required.