FINAL EXAM

Math 114
6/10/2009

Name: ______________________

Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Good luck!

1	10	
2	10	
3	20	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. (*10 points*) Let B be a binary function symbol, P a unary predicate symbol.

(**a**) Describe two different (possibly informal) models of this language. Make sure to clearly state what the universe of objects is in each model, and what the interpretations of B and P are.

Translate each of the following formulas into their English equivalent in **both** models you described above.

(**b**) $\forall x, y Px \rightarrow P(Bxy)$

(**c**) $\forall x, y Bxy = Byx$

(**d**) $\forall x(Px \vee \forall y \neg Py \rightarrow \neg P(Bxy))$

2. (*10 points*) Which of the following are well-formed formulas of a first order logic where P is a unary predicate and B is a binary function? Justify your answers.

(**a**) $\forall x \exists y Bxy$

(**b**) $\exists x \forall y P(Bxy)$

(**c**) $\forall x \forall y PBy$

(**d**) $\forall x P(B(Px)y)$

3. (*10 points*) In each of the following formulas, list the free variables:

(**a**) $\forall x \exists y Pxyz$

(**b**) $\forall x \exists y Pxyz \to \forall z Rz$

(**c**) $\exists y Pxyz \to \forall z Rz$

4. (*10 points*) Consider a language with two binary predicates, $+$ and $\cdot$, and work in the usual model of the integers. Let s be the substitution taking the variable v_i to the integer i. Compute the value of the following terms:

(a) $\overline{s}(\cdot(+v_4v_3)(v_5))$

(b) $\overline{s(v_2 \mapsto 4)}(\cdot(+v_4v_3)(v_5))$

(c) $\overline{s(v_2 \mapsto 4)}(+v_2v_3)$

(d) Does $\models_{\mathbb{N}} \forall x v_2 = x \rightarrow v_3 = x[s(v_2 \mapsto 3)]$ hold?

5. (*10 points*) Consider a nonstandard model of the theory of the reals, $^*\mathbb{R}$. Suppose that ϕ holds unboundedly—that is, above every real number there is a real number x such that $\phi(x)$ holds—where ϕ is some formula of first order logic not involving quantifiers. Prove that there is an infinite real $a \in {}^*\mathbb{R}$ such that $^*\phi(a)$ holds.

Consider the language of sentential logic with just the connectives $\neg$ and $\to$. We will define a system of deductions for this language, as we did for first order logic. The only rule will be modus ponens, and we will have the following axiom groups (where α, β, and γ are arbitrary formulas):

1. $\alpha \to (\beta \to \alpha)$
2. $(\alpha \to (\beta \to \gamma)) \to (\alpha \to \beta) \to (\alpha \to \gamma)$
3. $(\alpha \to \gamma) \to (\alpha \to \neg\gamma) \to \neg\alpha$
4. $\alpha \to \alpha$

(The fourth group can be derived from the first three, but is included to make the argument easier.)
We will write $\Gamma \vdash \phi$ if there is a sequence of formulas $\psi_1, \ldots, \psi_n$ such that $\psi_n = \phi$, and for each i, either $\psi_i \in \Gamma$, ψ_i is an axiom, or there are $j, k < i$ such that $\psi_k = \psi_j \to \psi_i$.
In the following problems, you may refer to the results of previous problems in your proofs. (In particular, if you get stuck on one, move on to the next and use the statement of the previous one.)

6. (*10 points*) Prove that if $\Sigma \vdash \phi$ then there is a finite set $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash \phi$.

7. (*10 points*) Prove that if $\{\alpha_1, \ldots, \alpha_n\} \vdash \phi$ then $\alpha_1 \to \cdots \to \alpha_n \to \phi$. (Hint: First prove that whenever $\Gamma; \alpha \vdash \phi$, also $\Gamma \vdash \alpha \to \phi$.)

8. (*10 points*) Prove that if Σ is a consistent set of formulas (that is, if $\Sigma \vdash \alpha$ then $\Sigma \nvdash \neg\alpha$) then there is a consistent set Δ extending Σ such that for every formula α, either $\alpha \in \Delta$ or $\neg\alpha \in \Delta$. (Hint: there is an enumeration of the formulas of sentential logic, $\sigma_1, \ldots, \sigma_n, \ldots$ such that for every formula α, there is an i such that $\alpha = \sigma_i$. Remember to show that the sets you construct really are consistent!)

9. (*10 points*) Let Σ be a consistent set of formulas. Construct a truth assignment ν such that whenever $\alpha \in \Sigma$, $\overline{\nu}(\alpha) = \top$.

10. (*10 points*) Show that whenever $\Sigma \vDash \alpha$, $\Sigma \vdash \alpha$. What is the name of this theorem?