§2.6

2. Let $T_1 \subset T_2$ be theories. Assume T_1 is complete and T_2 is satisfiable. Let $\phi \in T_2$. Since T_1 is complete, $\phi \in T_1$ or $\neg\phi \in T_1$. If $\phi \in T_1$, we're done. If not, $\neg\phi \in T_1 \subset T_2$, so $\phi \wedge \neg\phi \in T_2$ (since T_2 is closed under deduction). But this contradicts our assumption that T_2 is satisfiable.

7. Let $T = Th(\mathfrak{N})$. Expand the language $\{<\}$ to $\{<, c_0, c_1, \ldots\}$ by adding countably many fresh constants. Let σ_i be the sentence $c_{i+1} < c_i$.

   Claim 1. *$T \cup \{\sigma_i | i \in \mathbb{N}\}$ is consistent.*

   Proof. Let $\Delta \subset T \cup \{\sigma_i | i \in \mathbb{N}\}$ be finite. Let m be the max such that $\sigma_m \in \Delta$. Then $\Delta \subset T \cup \{\sigma_0, \ldots, \sigma_m\}$, so to show that Δ is satisfiable, it is enough to show that $T \cup \{\sigma_0, \ldots, \sigma_m\}$ is satisfiable. Let $\mathfrak{N}' = (\mathbb{N}, <, c_0, c_1, \ldots)$ be a structure in the langue $\{<, c_0, c_1, \ldots\}$, with $<^{\mathfrak{N}'}$ as the usual orderin, $c_i^{\mathfrak{N}'} = m - i$ for $0 \leq i \leq m+1$, and $c_j^{\mathfrak{N}'} = j$ for $j > m+1$. Then, $\models_{\mathfrak{N}'} T$ by a previous homework exercise (since T does not contain any of the symbols in the expanded language, and $\mathfrak{N}$ and $\mathfrak{N}'$ are the same apart from those symbols). For $0 \leq i \leq m$, $\models_{\mathfrak{N}'} c_{i+1} < c_i$. Thus, $\models_{\mathfrak{N}'} T \cup \{\sigma_0, \ldots, \sigma_m\}$. So $T \cup \{\sigma_0, \ldots, \sigma_m\}$ is satisifiable. Thus, Δ is satisfiable, and hence, is consistent. So by compactness, $T \cup \{\sigma_0, \sigma_1, \ldots\}$ is consistent. $\square$

   Let $\mathfrak{A}'$ satisfy $T \cup \{\sigma_0, \sigma_1, \ldots\}$, and let $\mathfrak{A} = (|\mathfrak{A}'|, <)$ be the reduction of $\mathfrak{A}'$ to the language $\{<\}$. Since every formula in T is in this language, $\models_{\mathfrak{A}} T$. So $\models_{\mathfrak{N}} \phi \Rightarrow \models_{\mathfrak{A}} \phi$. Then, if $\models_{\mathfrak{A}} \phi$, then either $\models_{\mathfrak{N}} \phi$ or $\not\models_{\mathfrak{N}} \phi$, that is $\models_{\mathfrak{N}} \neg\phi$. If the latter case holds, then $\neg\phi \in Th(\mathfrak{N}) = T$, so $\models_{\mathfrak{A}} \neg\phi$, which is a contradiction. Hence, $\models_{\mathfrak{N}} \phi$. Thus, $\mathfrak{N}$ and $\mathfrak{A}$ are elementarily equivalent.

   Finally, since $c_0^{\mathfrak{A}} >^{\mathfrak{A}} c_1^{\mathfrak{A}} >^{\mathfrak{A}} c_2^{\mathfrak{A}} >^{\mathfrak{A}} \ldots$ (and this ordering is strict), we have an infinite descending chain.

8. Assume that σ is true in all infinite models of a theory T. Suppose for contradiction that for every finite k, there is a finite model $\mathfrak{A}$ of T with at least k elements in its universe in which σ is not true. Let ϕ_n be the sentence $\exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} \neg x_i = x_j$ for $n \geq 2$. We see that a structure $\mathfrak{A}$ satisfies ϕ_n if and only if $\mathfrak{A}$ has at least n elements. Consider the theory $T \cup \{\phi_n | n \geq 2\} \cup \{\neg\phi\}$. Let Δ be a finite subset and k be the greatest such that $\phi_k \in \Delta$. Then $\Delta \subset T \cup \{\phi_2, \ldots, \phi_k\} \cup \{\neg\sigma\}$. Let $\mathfrak{A}$ be a models of T with at least k elements in which σ is not true (this exists by assumption). Then clearly $\models_{\mathfrak{A}} \Delta$, so Δ is consistent.

   Hence, by compactness, $T \cup \{\phi_2, \phi_3, \ldots\} \cup \{\neg\sigma\}$, so it has a model $\mathfrak{A}$. Suppose $\mathfrak{A}$ has exactly k elements where k is finite. But $\models_{\mathfrak{A}} \phi_{k+2}$, and thus has at least $k+2$ elements, so this is a contradiction. So $\mathfrak{A}$ is infinite, but $\models_{\mathfrak{A}} \neg\sigma$, contradicting our original assumption.

9. By Theorem 26D, since the language is finite, the set of sentences satisfiable in a finite model is effectively enumerable. Since the satisfiable memebers of Σ are exactly the members of Σ satisfiable in a finite model, the members of Σ satisfiable in a finite model are effectively enumerable. Conversely, the set of unsatisfiable sentences is effectively

enumerable, since σ is unsatisfiable iff $\neg\sigma$ is valid, iff $\vdash \neg\sigma$, and the set of deductions from $\emptyset$ is enumerable. Since the satisfiable members of Σ are effectively enumerable, and the complement is also effectively enumerable, the satisfiable members of Σ are decidable.

10a. By Exercise 19 in Section 2.2, the sentences from $\exists_2$ in a language without function symbols have the finite model property. Since the langauge is finite, problem 9 applies, so the set of satisfiable $\exists_2$ sentences is decidable.

10b. A sentence σ is valid iff $\neg\sigma$ is not satisfiable. Given a $\forall_2$ sentence σ, $\neg\sigma$ is $\exists_2$. By the previous part, we may decide whether $\neg\sigma$ is satisfiable; if so, σ is not valid. If not, σ is valid.