Math 114L

Homework 3 Solutions

Spring 2011

Solution to 5b

Let ϕ_n be the formula

$$\mathbf{R}x_1x_2 \wedge \cdots \wedge \mathbf{R}x_{n-1}x_n \wedge \mathbf{R}x_nx_1$$

and let σ_n be

$$\neg\exists x_1 \exists x_2 \cdots \exists x_n \phi_n.$$

Let $\Sigma = \{\sigma_n \mid n \geq 1\}$. If $\mathfrak{A}$ contains a cycle $a_1, \ldots, a_n$ then,

$$\mathfrak{A} \models \phi_n[[a_1, \ldots, a_n]]$$

and therefore

$$\mathfrak{A} \models \neg\sigma_n.$$

Conversely, if $\mathfrak{A} \not\models \Sigma$ then $\mathfrak{A} \not\models \sigma_n$ for some n, so there must be elements $a_1, \ldots, a_n$ such that

$$\mathfrak{A} \models \phi_n[[a_1, \ldots, a_n]]$$

and therefore these elements form a cycle.

Solution to 5c

Suppose the class of structures without cycles were an *EC* class; let τ be the formula defining this class. Since $\Sigma \models \tau$, by compactness there is a finite subset Σ_0 of Σ such that $\Sigma_0 \models \tau$, and therefore for some n, $\{\sigma_1, \ldots, \sigma_n\} \models \tau$.

Consider the structure $\mathfrak{A}$ consisting of a cycle $n+1$ points $a_1, \ldots, a_{n+1}$. Then $\mathfrak{A} \models \{\sigma_1, \ldots, \sigma_n\}$, so $\mathfrak{A} \models \tau$. But this structure has a cycle, so τ does not define the structures without cycles.