Math 114L

Homework 1 Solutions

Spring 2010

1.1.2

We first show that there are wffs of every length other than $2, 3, 6$ with some examples followed by induction on natural numbers. The sentence symbol A_1 is a wff of length 1. $(\neg A_1)$ is a wff of length 4. The wff $(A_1 \wedge A_1)$ is a wff of length 5. Finally, $(\neg(\neg A_1)), (\neg(A_1 \wedge A_1)), ((A_1 \wedge A_1) \wedge A_1)$ are wffs of lengths $7, 8, 9$ respectively. Now we show by induction that if $n > 9$, there is a wff of length n: let $n > 9$ be given and suppose that for all $m \in (9, n)$, there is a wff of length m. Then in particular, either $n - 3 \in (9, n)$ or $n - 3 \in \{7, 8, 9\}$, so there is a wff α of length $n - 3$. Then $(\neg\alpha)$ has length $n - 3 + 3 = n$.

Now we show by induction on α that if α is a wff then the length of α is not $2, 3,$, or 6.

Base Case: If α is a sentence symbol, the length of α is 1, which is not $2, 3$, or 6.

Inductive Case: Suppose α has length $k > 0$ where $k \neq 2, k \neq 3, k \neq 6$. Then $(\neg\alpha)$ has length $k + 3$, and therefore $k + 3 \neq 2, k + 3 \neq 3, k + 3 \neq 6$.

Suppose α has length k_α and β has length k_β where neither k_α nor k_β are $2, 3$, or 6. Then $(\alpha \triangle \beta)$, where $\triangle \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, has length $k_\alpha + k_\beta + 3$, and therefore is not $2, 3$, or 6.

1.1.5

1.1.5a

By induction, we show that for all α, the length of α is odd.

Base Case: If α is a sentence symbol, the length of α is 1, which is odd.

Inductive Case: Suppose the length of α is an odd number n_α and the length of β is an odd number n_β. Then the length of $(\alpha \triangle \beta)$, where $\triangle \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, is $n_\alpha + n_\beta + 3$, which is a sum of three odd numbers, and so also odd.

1.1.5b

We show by induction that the length of α is of the form $4k + 1$, with $k + 1$ sentence symbols. In particular, this implies that more than a quarter of the symbols are sentence symbols.

Base Case: If α is a sentence symbol then $k = 0$, so α has $0+1 = 1$ sentence symbol, and $4 \cdot 0 + 1 = 1$ total symbols.

Inductive Case: Suppose α has $4k_\alpha + 1$ symbols including $k_\alpha + 1$ sentence symbols, and β has $4k_\beta + 1$ symbols including $k_\beta + 1$ sentence symbols. Then the length of $(\alpha \triangle \beta)$ where $\triangle \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ is $4k_\alpha + 1 + 4k_\beta + 1 + 3 = 4(k_\alpha + k_\beta + 1) + 1$ and there are $k_\alpha + 1 + k_\beta + 1 = (k_\alpha + k_\beta + 1) + 1$ sentence symbols.

1.2.1

Consider the truth assignment given by $\nu(A) = \nu(B) = \nu(C) = F$. Then $\overline{\nu}((A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge \neg C)) = T$ while $\overline{\nu}(A \leftrightarrow B \leftrightarrow C) = F$, so the former does not tautologically imply the latter.

Consider the truth assignment given by $\nu(A) = T$ while $\nu(B) = \nu(C) = F$. Then $\overline{\nu}(A \leftrightarrow B \leftrightarrow C) = T$ while $\overline{\nu}((A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge \neg C)) = F$, so $A \leftrightarrow B \leftrightarrow C$ does not tautologically imply $(A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge \neg C)$, either.

1.2.3

1.2.3a

The truth table is:

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \vee (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

All rows are T, so this is a tautology.

1.2.3b

P	Q	R	$(P \wedge Q) \rightarrow R$	$(P \rightarrow R) \vee (Q \rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

Whenever the second to last column is T, so is the last, so $(P \wedge Q) \rightarrow R$ tautologically implies $(P \rightarrow R) \vee (Q \rightarrow R)$.

1.2.4

1.2.4a

For the left to right direction, assume $\Sigma; \alpha \vDash \beta$. Let ν be a truth assignment such that $\overline{\nu}(\sigma) = T$ for every $\sigma \in \Sigma$. If $\overline{\nu}(\alpha) = F$ then $\overline{\nu}(\alpha \to \beta) = T$. If $\overline{\nu}(\alpha) = T$ then $\overline{\nu}(\sigma) = T$ for every $\sigma \in \Sigma \cup \{\alpha\}$, so by assumption $\overline{\nu}(\beta) = T$ and again $\overline{\nu}(\alpha \to \beta) = T$. Therefore whenever $\overline{\nu}(\sigma) = T$ for all $\sigma \in \Sigma$, also $\overline{\nu}(\alpha \to \beta) = T$, so $\Sigma \vDash (\alpha \to \beta)$.

For the right to left direction, assume $\Sigma \vDash (\alpha \to \beta)$, and let ν be a truth assignment such that for every $\sigma \in \Sigma \cup \{\alpha\}$, $\overline{\nu}(\sigma) = T$. Then by assumption, $\overline{\nu}(\alpha \to \beta) = T$, and since $\overline{\nu}(\alpha) = T$, also $\overline{\nu}(\beta) = T$. So $\Sigma; \alpha \vDash \beta$.

1.2.4b

For the left to right direction, assume $\alpha \vDash \beta$ and $\beta \vDash \alpha$. Then for any ν, if $\overline{\nu}(\alpha) = T$ then $\overline{\nu}(\beta) = T$, and if $\overline{\nu}(\alpha) = F$ then $\overline{\nu}(\beta) = F$. In particular, $\overline{\nu}(\alpha \leftrightarrow \beta) = T$. Since this holds for any ν, $\vDash \alpha \leftrightarrow \beta$.

For the right to left direction, assumpe $\vDash \alpha \leftrightarrow \beta$. If ν is a truth assignment such that $\overline{\nu}(\alpha) = T$ then since $\overline{\nu}(\alpha \leftrightarrow \beta) = T$, $\overline{\nu}(\beta) = T$, so $\alpha \vDash \beta$. By a symmetric argument, also $\beta \vDash \alpha$.

1.2.9

By induction on α, we show that $\alpha^* \vDash (\neg\alpha)$ and $(\neg\alpha) \vDash \alpha^*$.

Base Case: If α is a sentence symbol then α^* is $(\neg\alpha)$, so the claim is trivial.

Inductive Case: Suppose $(\neg\alpha)$ is tautologically equivalent to α^* and $(\neg\beta)$ is tautologically equivalent to β^*. Then $(\alpha \vee \beta)^* = \alpha^* \wedge \beta^*$, so for any ν, $\overline{\nu}(\neg(\alpha \vee \beta)) = \overline{\nu}(\neg\alpha \wedge \neg\beta) = \overline{\nu}(\alpha^* \wedge \beta^*)$ using the de Morgan's law.

Similarly, $(\alpha \wedge \beta)^* = \alpha^* \vee \beta^*$, so for any ν, $\overline{\nu}(\neg(\alpha \wedge \beta)) = \overline{\nu}(\neg\alpha \vee \neg\beta) = \overline{\nu}(\alpha^* \vee \beta^*)$.

1.2.10

1.2.10a

By induction on the size of Σ. If $|\Sigma| = 0$, $\Sigma = \emptyset$ is automatically independent, and therefore Σ is an independent equivalent subset of itself.

Suppose that for all Δ with $|\Delta| = n$, Δ has an independent equivalent subset. Let Σ be given with $|\Sigma| = n + 1$. If Σ is independent then Σ is an independent equivalent subset of itself. If Σ is not independent, let $\alpha \in \Sigma$ be some formula such that $\Sigma \setminus \{\alpha\} \vDash \alpha$. Let $\Delta = \Sigma \setminus \{\alpha\}$. Then whenever $\Delta \vDash \beta$, also $\Sigma \vDash \beta$ since $\Delta \subseteq \Sigma$. Whenever $\Sigma \vDash \beta$, since $\Sigma = \Delta; \alpha$, also $\Delta; \alpha \vDash \beta$, so by problem 4, $\Delta \vDash \alpha \to \beta$. Since $\Delta \vDash \alpha$, also $\Delta \vDash \beta$. So Σ and Δ are equivalent. $|\Delta| = n$, so by IH, there is an independent $\Gamma \subseteq \Delta$ equivalent to Δ. Since $\Delta \subseteq \Sigma$, also $\Gamma \subseteq \Sigma$, and since Γ is equivalent to Δ and Δ is equivalent to Σ, Γ is equivalent to Σ. So Γ is an independent equivalent subset of Σ.

1.2.10b

For each n, let σ_n be the formula $A_1 \wedge A_2 \wedge \cdots \wedge A_n$. Let $\Sigma = \{\sigma_n \mid n \in \mathbb{N}\}$. Suppose $\Delta \subseteq \Sigma$ were an independent equivalent subset. If $|\Delta| > 1$ then we have $\sigma_n, \sigma_m \in \Delta$ with $n < m$, which is impossible since then $\sigma_m \vDash \sigma_n$, contradicting the fact that Δ is independent. But if $|\Delta| = 1$, $\Delta = \{\sigma_n\}$ for some n, and so, for instance, $\Delta \nvDash \sigma_{n+1}$ while $\Sigma \vDash \sigma_{n+1}$, so Δ is not equivalent to Σ.

1.2.10c

Let $\Sigma = \{\sigma_0, \sigma_1, \ldots\}$. Let $\Sigma_n = \{\sigma_i \mid i < n\}$. We recursively construct a sequence $\Delta_0 \subseteq \Delta_1 \subseteq \cdots$ of finite sets of wffs such that for all n, Δ_n is independent and equivalent to Σ_n. For convenience, whenever $\Delta_n = \{\alpha_0, \ldots, \alpha_k\}$ is one of these finite sets, let δ_n be the formula $\alpha_0 \wedge \cdots \wedge \alpha_k$.

Let $\Delta_0 = \emptyset$. Given Δ_n, if $\Delta_n \vDash \sigma_n$, let $\Delta_{n+1} = \Delta_n$. If $\Delta_n \nvDash \sigma_n$, let $\Delta_{n+1} = \Delta_n \cup \{\delta_n \to \sigma_n\}$.

We first show by induction that Δ_n is equivalent to Σ_n. For $n = 0$, both Δ_0 and Σ_0 are the empty set. Suppose that Δ_n is equivalent to Σ_n; we consider two cases. If $\Delta_n \vDash \sigma_n$ then since Δ_n is equivalent to Σ_n by the inductive hypothesis, $\Sigma_n \vDash \sigma_n$, and therefore Σ_n is equivalent to Σ_{n+1}, so $\Delta_{n+1} = \Delta_n$ is equivalent to Σ_{n+1}. Otherwise, $\Delta_n \nvDash \sigma_n$, so $\Delta_{n+1} = \Delta_n \cup \{\delta_n \to \sigma_n\}$. Since $\Delta_n \vDash \delta_n$, $\Delta_{n+1} \vDash \sigma_n$, and since Δ_n is equivalent to Σ_n, for every $\sigma \in \Sigma_{n+1}$, $\Delta_{n+1} \vDash \sigma$. Conversely, if $\Sigma_{n+1} \vDash \delta_n \to \sigma_n$, so for every $\delta \in \Delta_{n+1}$, $\Sigma_{n+1} \vDash \delta$. Therefore Δ_{n+1} is equivalent to Σ_{n+1}.

Next we show that Δ_n is independent for all n, again by induction on n. Δ_0 is independent by definition. If Δ_n is independent and $\Delta_n \vDash \sigma_n$ then $\Delta_{n+1} = \Delta_n$, so naturally Δ_{n+1} is independent. So suppose $\Delta_n \nvDash \sigma_n$, so $\Delta_{n+1} = \Delta_n \cup \{\delta_n \to \sigma_n\}$. Let $\alpha \in \Delta_{n+1}$; we must show that $\Delta_{n+1} \setminus \{\alpha\} \nvDash \alpha$. If $\alpha = \delta_n \to \sigma_n$ then since $\Delta_{n+1} \setminus \{\alpha\} = \Delta_n$ and $\Delta_n \nvDash \sigma_n$, there must be some ν such that ν satisfies Δ_n but $\overline{\nu}(\sigma_n) = F$. But then $\overline{\nu}(\delta_n) = T$, so $\overline{\nu}(\delta_n \to \sigma_n) = F$. Therefore $\Delta_n \nvDash \alpha$. Otherwise, $\alpha \in \Delta_n$. Since Δ_n is independent, there is a ν such that $\overline{\nu}(\beta) = T$ for $\beta \in \Delta_n \setminus \{\alpha\}$ and $\overline{\nu}(\alpha) = F$. Since $\overline{\nu}(\delta_n) = F$, $\overline{\nu}(\delta_n \to \sigma_n) = T$, so ν satisfies $\Delta_{n+1} \setminus \{\delta_n \to \sigma_n\}$, and therefore demonstrates that $\Delta_{n+1} \setminus \{\alpha\} \nvDash \alpha$.

Let $\Delta = \bigcup_n \Delta_n$. We claim that Δ is independent and equivalent to Σ. To see that Δ is independent, suppose there were $\alpha \in \Delta$ such that $\Delta \setminus \{\alpha\} \vDash \alpha$. Then by compactness, there would be a finite $\Gamma \subseteq \Delta \setminus \{\alpha\}$ such that $\Gamma \vDash \alpha$, and then there would be a $\Delta_n \supseteq \Gamma$ such that $\Delta_n \setminus \{\alpha\} \vDash \alpha$, contradicting the fact that Δ_n is independent.

To see that Σ is equivalent to Δ, if $\Sigma \vDash \beta$ then there is a finite $\Sigma_n \subseteq \Sigma$ such that $\Sigma_n \vDash \beta$, so $\Delta_n \vDash \beta$, and therefore $\Delta \vDash \beta$. Conversely, if $\Delta \vDash \beta$ then there is a finite $\Delta_n \subseteq \Delta$ such that $\Delta_n \vDash \beta$, so $\Sigma_n \vDash \beta$, and therefore $\Sigma \vDash \beta$.

1.2.14

Suppose ν is a truth assignment and $\overline{\nu}_1, \overline{\nu}_2$ are two functions satisfying the conditions for $\overline{\nu}$. We show by induction on α that $\overline{\nu}_1(\alpha) = \overline{\nu}_2(\alpha)$.

Base Case: If α is a sentence symbol, $\overline{\nu}_1(\alpha) = \nu(\alpha) = \overline{\nu}_2(\alpha)$.

Inductive Cases: Suppose $\overline{\nu}_1(\alpha) = \overline{\nu}_2(\alpha)$. Then $\overline{\nu}_1(\neg\alpha) = T$ iff $\overline{\nu}_1(\alpha) = T$ iff $\overline{\nu}_1(\beta) = T$ iff $\overline{\nu}_2(\neg\alpha) = T$.

Suppose $\overline{\nu}_1(\alpha) = \overline{\nu}_2(\alpha)$ and $\overline{\nu}_1(\beta) = \overline{\nu}_2(\beta)$. Then $\overline{\nu}_1(\alpha \wedge \beta) = T$ iff both $\overline{\nu}_1(\alpha) = T$ and $\overline{\nu}_1(\beta) = T$, which happens iff both $\overline{\nu}_2(\alpha) = T$ and $\overline{\nu}_2(\beta) = T$, which happens iff $\overline{\nu}_2(\alpha \wedge \beta) = T$. The cases for $\vee, \rightarrow, \leftrightarrow$ are similar.

A1

(A_1) has length 3, and we have shown that no wff has length 3.

Alternatively, (A_1) has no $\neg$, but does not have length of the form $4k + 1$.

Alternatively, we show by induction that either α is a single sentence symbol or α contains at least 1 connective.

Base Case: If α is a sentence symbol, the claim is satisfied.

Inductive Cases: Suppose α and β are either sentence symbols or contain connectives. Then $(\neg\alpha)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ all contain connectives.