Math 114L

Homework 2 Solutions

Spring 2011

1.5.1

1.5.1a

$$(\neg A_1 \wedge \neg A_2 \wedge \neg A_3) \vee (\neg A_1 \wedge \neg A_2 \wedge A_3) \vee (\neg A_1 \wedge A_2 \wedge \neg A_3) \vee (A_1 \wedge \neg A_2 \wedge \neg A_3)$$

1.5.1a

$$(A_1 \vee A_2) \to \neg(A_3 \vee (A_1 \wedge A_2))$$

1.5.3

We will prove by induction that if α is a wff built only from $\neg$ and # and containing the sentence symbols A, B then α is tautologically equivalent to one of $A, \neg A, B, \neg B$.

Base case: Any sentence symbol is either A or B.

Inductive case for $\neg$: If α is tautologically equivalent to A then $\neg\alpha$ is tautologically equivalent to $\neg A$, and similarly if α is tautologically equivalent to one of $\neg A, B, \neg B$.

Inductive case for #: If $\alpha_1, \alpha_2, \alpha_3$ are each tautologically equivalent to one of $A, \neg A, B, \neg B$, at least one of the following must hold:

$$\begin{array}{ll} \alpha_1 \models\!\!\!\!\models \alpha_2 & \alpha_1 \models\!\!\!\!\models \neg\alpha_2 \\ \alpha_1 \models\!\!\!\!\models \alpha_3 & \alpha_1 \models\!\!\!\!\models \neg\alpha_3 \\ \alpha_2 \models\!\!\!\!\models \alpha_3 & \alpha_2 \models\!\!\!\!\models \neg\alpha_3 \end{array}$$

Let us suppose we are in one of the cases in the left column, say $\alpha_1 \models\!\!\!\!\models \alpha_3$. Then $\#\alpha_1\alpha_2\alpha_3$ is tautologically equivalent to α_1 (since α_1 and α_3 agree, and will therefore outvote α_2).

Suppose we are in one of the cases in the right column, say $\alpha_2 \models\!\!\!\!\models \neg\alpha_3$. Then $\#\alpha_1\alpha_2\alpha_3$ is tautologically equivalent to α_1 (since α_2 and α_3 will vote against each other, and α_1 will always cast the tie breaking vote).

The other cases are similar, with just the specific numbers changed. In either case, $\#\alpha_1\alpha_2\alpha_3$ is tautologically equivalent to one of $A, \neg A, B, \neg B$.

In particular, $A \wedge B$ is a formula which is not tautologically equivalent to any of $A, \neg A, B, \neg B$, so there is no way for it to be expressed with $\neg$ and #.

1.5.4

1.5.4a

- $M\alpha\alpha\alpha$ is tautologically equivalent to $\neg\alpha$
- $M(\neg\bot)(\neg\alpha)(\neg\beta)$ is tautologically equivalent to $\alpha \wedge \beta$

Since $\{\neg, \wedge\}$ is complete and $\neg$ and $\wedge$ can be represented with M and $\bot$, it follows that $\{M, \bot\}$ is complete.

1.5.4b

We will prove by induction that if α is a wff built only from M and containing the sentence symbols A, B then α is tautologically equivalent to one of $A, \neg A, B, \neg B$.

Base case: Any sentence symbol is either A or B.

Inductive case for #: If $\alpha_1, \alpha_2, \alpha_3$ are each tautologically equivalent to one of $A, \neg A, B, \neg B$, at least one of the following must hold:

$$\begin{array}{ll} \alpha_1 \vDash\Dashv \alpha_2 & \alpha_1 \vDash\Dashv \neg\alpha_2 \\ \alpha_1 \vDash\Dashv \alpha_3 & \alpha_1 \vDash\Dashv \neg\alpha_3 \\ \alpha_2 \vDash\Dashv \alpha_3 & \alpha_2 \vDash\Dashv \neg\alpha_3 \end{array}$$

Let us suppose we are in one of the cases in the left column, say $\alpha_1 \vDash\Dashv \alpha_3$. Then $M\alpha_1\alpha_2\alpha_3$ is tautologically equivalent to $\neg\alpha_1$ (since α_1 and α_3 agree, and will therefore outvote α_2).

Suppose we are in one of the cases in the right column, say $\alpha_2 \vDash\Dashv \neg\alpha_3$. Then $M\alpha_1\alpha_2\alpha_3$ is tautologically equivalent to α_1 (since α_2 and α_3 will vote against each other, and $\neg\alpha_1$ will always cast the tie breaking vote).

The other cases are similar, with just the specific numbers changed. In either case, $M\alpha_1\alpha_2\alpha_3$ is tautologically equivalent to one of $A, \neg A, B, \neg B$.

In particular, $A \wedge B$ is a formula which is not tautologically equivalent to any of $A, \neg A, B, \neg B$, so there is no way for it to be expressed with just M.

1.5.5

If α has only the sentence symbols A, B, there are four relevant truth assignments (making A T or F and B T or F). We show by induction that if α is built from $\{\top, \bot, \neg, \leftrightarrow, +\}$ and the sentence symbols A, B then $\overline{\nu}(\alpha) = T$ for an even number of the relevant truth assignments.

Base Case: $\overline{\nu}(A) = T$ for two of the four possible truth assignments, and the same is true for B.

Inductive Case: $\top$: $\overline{\nu}(\top) = T$ for all 4 possible truth assignments.

$\bot$: $\overline{\nu}(\bot) = T$ is 0 of the possible truth assignments.

$\neg$: If $\overline{\nu}(\alpha) = T$ for n of the possible truth assignments, $\overline{\nu}(\neg\alpha) = T$ for $4 - n$ of the possible truth assignments. In particular, if n is even, so is $4 - n$.

$\leftrightarrow$: Let $\overline{\nu}(\alpha) = T$ for n_a truth assignemnts and $\overline{nu}(\beta) = T$ for n_b truth assignments with n_a and n_b both even. If $n_a = 4$ then $\overline{\nu}(\alpha \leftrightarrow \beta) = \overline{\nu}(\beta)$ for every ν, so $\overline{\nu}(\alpha \leftrightarrow \beta) = T$ for n_b of the possible truth assignments, an even number. If $n_a = 0$ then $\overline{\nu}(\alpha \leftrightarrow \beta) = \overline{\nu}(\neg\beta)$ for every ν, so $\overline{\nu}(\alpha \leftrightarrow \beta) = T$ for

$4 - n_b$ of the possible truth assignments, also an even number. If $n_b = 4$ or $n_b = 0$, a symmetric argument applies.

If $n_a = n_b = 2$, we consider three subcases. If the two truth assignments ν such that $\overline{\nu}(\alpha) = T$ are the same as the two such that $\overline{\nu}(\beta) = T$ then $\overline{\nu}(\alpha \leftrightarrow \beta) = T$ for all 4 truth assignments. If there is no overlap, $\overline{\nu}(\alpha \leftrightarrow \beta) = T$ for 0 truth assignments. In the final case, there is an overlap of 1, so all four possible combinations are realized: there is a ν such that $\overline{\nu}(\alpha) = \overline{\nu}(\beta) = T$, a ν such that $\overline{\nu}(\alpha) = \overline{\nu}(\beta) = F$, a ν such that $\overline{\nu}(\alpha) = T$ while $\overline{\nu}(\beta) = F$, and a ν such that $\overline{\nu}(\alpha) = F$ while $\overline{\nu}(\beta) = T$. This gives exactly two ν satisfying $\alpha \leftrightarrow \beta$.

$+$: We can reduce this case to the previous one, since $\overline{\nu}(\alpha + \beta) = \overline{\nu}(\neg(\alpha \leftrightarrow \beta))$, so by the previous two cases, if $\overline{\nu}(\alpha) = T$ for an even number of ν and $\overline{\nu}(\beta) = T$ for an even number of ν, the same holds for $\alpha \leftrightarrow \beta$, and therefore also for $\neg(\alpha \leftrightarrow \beta) \models\dashv \alpha + \beta$.

1.5.7

1.5.7a

$+^3\top\bot\alpha$ is tautologically equivalent to $\neg\alpha$. Since $\{\neg, \wedge\}$ is complete and $\neg$ and $\wedge$ can be represented with $\{\top, \bot, \wedge, +^3\}$, it follows that $\{\top, \bot, \wedge, +^3\}$ is complete.

1.5.7b

It suffices to consider the four subsets with three of the four connectives, since every proper subset is a subset of one of them.

1.5.7b1

$\{\top, \bot, +^4\}$: For any truth assignment ν, define the *opposite* of ν, ν' by $\nu'(A) = T$ iff $\nu(A) = F$. To see that $\{\top, \bot, +^3\}$ is not complete, we show inductively that any formula α with sentence symbols A, B and connectives from $\{\top, \bot, +^3\}$ has the property that either:

- For every ν, $\overline{\nu'}(\alpha) = \overline{\nu}(\alpha)$, or
- For every ν, $\overline{\nu'}(\alpha) \neq \overline{\nu}(\alpha)$.

(In other words, either the truth value assigned to α does not depend on A at all, or flipping the truth value assigned to A always flips the truth value assigned to α, no matter which truth value was assigned to B.)

Observe that $A \wedge B$ has neither of these properties: when $\nu(A) = \nu(B) = T$, $\overline{\nu}(A \wedge B) = T \neq F = \overline{\nu'}(A \wedge B)$, while when $\nu(A) = T$ and $\nu(B) = F$, $\overline{\nu}(A \wedge B) = F = \overline{\nu'}(A \wedge B)$.

Base case: If α is the sentence symbol A then we are in the second case. If α is the sentence symbol B then we are in the first case.

Indutive case for $\bot, \top$: $\overline{\nu}(\top) = T$ for all ν, so $\overline{\nu'}(\alpha) = \overline{\nu}(\alpha)$ for all ν. Similarly for $\bot$.

Inductive case for $+^3$: Suppose $\alpha_1, \alpha_2, \alpha_3$ each have the proeprty that either

- For every ν, $\overline{\nu'}(\alpha_i) = \overline{\nu}(\alpha_i)$, or
- For every ν, $\overline{\nu'}(\alpha_i) \neq \overline{\nu}(\alpha_i)$.

Observe that in the formula $+^3ABC$, changing the truth value of an even number of A, B, C leaves the truth value of $+^3ABC$ unchanged, while changing the truth value of an odd number flips the truth value of $+^3ABC$.

If an even number of $\alpha_1, \alpha_2, \alpha_3$ are in the second case then $+^3\alpha_1\alpha_2\alpha_3$ must be in the first case. Otherwise, an odd number of $\alpha_1, \alpha_2, \alpha_3$ are in the second case, so $+^3\alpha_1\alpha_2\alpha_3$ is as well.

So $\wedge$ cannot be represented by $\top, \bot, +^3$.

1.5.7b2

$\{\top, \bot, \wedge\}$: Let $\nu_T(A_n) = T$ for all n. We prove by induction that if α is built from $\{\top, \bot, \wedge\}$ and any number of sentence symbols, either $\overline{\nu_T}(\alpha) = T$ or $\overline{\nu}(\alpha) = F$ for all ν.

Base case: $\nu_T(A_n) = T$ for any sentence symbol

Inductive case for $\top$: $\overline{\nu_T}(\top) = T$

Inductive case for $\bot$: $\overline{\nu}(\alpha) = F$ for all ν

Inductive case for $\wedge$: Suppose α_1 and α_2 both have the property that either $\overline{\nu_T}(\alpha_i) = T$ or $\overline{\nu}(\alpha_i) = F$ for all ν. If, for either i, $\overline{\nu}(\alpha_i) = F$ for all ν then $\overline{\nu}(\alpha_1 \wedge \alpha_2) = F$ for all ν. Otherwise, $\overline{\nu_T}(\alpha_1) = \overline{\nu_T}(\alpha_2)$, so $\overline{\nu_T}(\alpha_1 \wedge \alpha_2) = T$.

$\neg A$ has the property that $\overline{\nu_T}(\neg A) = F$ but there are ν such that $\overline{\nu}(\alpha) = T$, so $\neg A$ is not tautologically equivalent to any formula built from $\top, \bot, \wedge$.

1.5.7b3

$\{\bot, \wedge, +^3\}$: We prove by induction that if α is built from $\{\bot, \wedge, +^3\}$ with only the sentence symbol A then for any ν, $\overline{\nu_F}(\alpha) = F$ (where $\nu_F(A) = F$).

Base case: By definition, $\overline{\nu_F}(A) = F$

Inductive case for $\bot$: Clearly $\overline{\nu_F}(\bot) = F$

Inductive case for $\wedge$: If $\overline{\nu_F}(\alpha) = F$ then $\overline{\nu_F}(\alpha \wedge \beta) = F$

Inductive case for $+^3$: If $\overline{\nu_F}(\alpha_i) = F$ for all i then $\overline{\nu_F}(\alpha_1\alpha_2\alpha_3) = F$.

$\overline{\nu_F}(\neg A) = T$, so $\neg$ cannot be represented by $\wedge, \bot, +^3$.

1.5.7b4

$\{\top, \wedge, +^3\}$: We prove by induction that if α is built from $\{\bot, \wedge, +^3\}$ with only the sentence symbol A then for any ν, $\overline{\nu_T}(\alpha) = T$.

Base case: By definition, $\overline{\nu_T}(A) = T$

Inductive case for $\top$: Clearly $\overline{\nu_T}(\bot) = T$

Inductive case for $\wedge$: If $\overline{\nu_T}(\alpha) = \overline{\nu_T}(\beta) = T$ then $\overline{\nu_T}(\alpha \wedge \beta) = T$

Inductive case for $+^3$: If $\overline{\nu_T}(\alpha_i) = T$ for all i then $\overline{\nu_T}(\alpha_1\alpha_2\alpha_3) = T$.

$\overline{\nu_T}(\neg A) = F$, so $\neg$ cannot be represented by $\wedge, \top, +^3$.

1.5.9

1.5.9a

$$\beta = (\neg A \vee \neg B \vee C) \wedge (\neg A \vee B \vee \neg C) \wedge (A \vee \neg B \vee \neg C) \wedge (A \vee B \vee C).$$

We check the equivalence:

A	B	C	$A \leftrightarrow B \leftrightarrow C$	$\neg A \vee \neg B \vee C$	$\neg A \vee B \vee \neg C$	$A \vee \neg B \vee \neg C$	$A \vee B \vee C$	β
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	T	T	F
T	F	T	F	T	F	T	T	F
T	F	F	T	T	T	T	T	T
F	T	T	F	T	T	F	T	F
F	T	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	F	T	T	T	F	F

1.5.9b

Let α be a formula. We have already shown that $\neg\alpha$ is tautologically equivalent to a formula in disjunctive normal form; that is, a formula of the form

$$(\neg\alpha)^{DNF} = \gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_k$$

where each γ_i has the form

$$\gamma_i = \beta_{i1} \wedge \cdots \wedge \beta_{in_k}$$

and each β_{ij} is either a sentence symbol or the negation of a sentence symbol. If β_{ij} is a sentence symbol, define β'_{ij} to be $\neg\beta_{ij}$, and if β_{ij} is the negation of a sentence symbol, define β'_{ij} to be that sentence symbol. (So β'_{ij} is either a sentence symbol or the negation of a sentence symbol, and is tautologically equivalent to $\neg\beta_{ij}$).

Then α is tautologically equivalent to $\neg(\neg\alpha)^{DNF}$, which is tautologically equivalent to

$$\gamma'_1 \wedge \gamma'_2 \wedge \cdots \wedge \gamma'_k$$

where

$$\gamma'_i = \beta'_{i1} \vee \cdots \beta'_{in_k}.$$

1.5.12

No. This is part 1.5.7b2.