Math 114L

Homework 3 Solutions

Spring 2011

2.1.1

1. $\forall x x < 0$ or $\forall x x < 0 \vee x = 0$
2. $\exists x I x \rightarrow I 0$
3. $\forall x \neg x < 0$
4. $\forall x ((\neg I x \wedge \forall y y < x \rightarrow I y) \rightarrow I x)$
5. $\forall x \neg(\forall y y < x)$ or perhaps $\forall x \neg(\forall y y < x \vee y = x)$
6. $\forall x \neg \forall y \neg y < x$

2.1.4

$(\forall x E x \rightarrow A x) \rightarrow \forall x \left((\exists y E y \wedge x = h y) \rightarrow (\exists y (A y \wedge x = h y)) \right)$

2.1.10

1. $(\neg((\neg \forall v_1 (\neg v_1 P v_1)) \rightarrow (\neg P v_1)))$, v_1 appears free
2. $((\neg(\forall v_1 A v_1 \rightarrow (\neg B v_1))) \rightarrow ((\neg(\neg \forall v_2 (\neg(\neg C v_2)))) \rightarrow D v_2))$, both v_1 and v_2 occur free.

2.2.1

a

$\Rightarrow$: Suppose $\Gamma; \alpha \vDash \phi$, and let $\mathfrak{A}, s$ be given so that for every $\gamma \in \Gamma$, $\vDash_{\mathfrak{A}} \gamma[s]$. Then either $\nvDash_{\mathfrak{A}} \alpha[s]$, in which case $\vDash_{\mathfrak{A}} \alpha \to \phi[s]$ by definition, or $\vDash_{\mathfrak{A}} \alpha[s]$, in which case, since $\Gamma; \alpha \vDash \phi$, we again have $\vDash_{\mathfrak{A}}\vDash \phi[s]$ and therefore $\vDash_{\mathfrak{A}} \alpha \to \phi[s]$.

$\Leftarrow$: Suppose $\Gamma \vDash \alpha \to \phi$, and let $\mathfrak{A}, s$ be given so that for every $\gamma \in \Gamma \cup \{\alpha\}$, $\vDash_{\mathfrak{A}} \gamma[s]$. Then this in particular holds for $\gamma \in \Gamma$, so $\vDash_{\mathfrak{A}} \alpha \to \phi[s]$. Since $\vDash_{\mathfrak{A}} \alpha[s]$, we must have $\vDash_{\mathfrak{A}} \phi[s]$.

b

$\Rightarrow$: Suppose $\phi \vDash\Dashv \psi$ and let $\mathfrak{A}, s$ be given. If $\vDash_{\mathfrak{A}} \phi[s]$ then since $\phi \vDash \psi$, $\vDash_{\mathfrak{A}} \psi[s]$, and therefore $\vDash_{\mathfrak{A}} \phi \leftrightarrow \psi[s]$. Otherwise $\nvDash_{\mathfrak{A}} \phi[s]$, and since $\psi \vDash \phi$, we must have $\nvDash_{\mathfrak{A}} \psi[s]$, so again $\vDash_{\mathfrak{A}} \phi \leftrightarrow \psi[s]$.

2.2.2

$a, b \nvDash c$: Take the universe to be $\mathbb{N}$ and interpret P by $<$.

$a, c \nvDash b$: Let the universe consist of two points (or any set with more than one element), and let P hold of every pair.

$b, c \nvDash a$: Take the universe to be $\mathbb{N}$ and let $(n, m) \in P^{\mathfrak{A}}$ exactly when $n = m + 1$. (Note that the left side of (c) fails, so (c) is true.)

2.2.3

Let $\mathfrak{A}, s$ be given so that $\vDash_{\mathfrak{A}} \forall x(\alpha \to \beta)[s]$ and $\vDash_{\mathfrak{A}} \forall x\alpha[s]$. Let $a \in |\mathfrak{A}|$. Then $\vDash_{\mathfrak{A}} \alpha \to \beta[s(a/x)]$ and $\vDash_{\mathfrak{A}} \alpha[s(a/x)]$, so $\vDash_{\mathfrak{A}} \beta[s(a/x)]$. Since this holds for all a, $\vDash_{\mathfrak{A}} \forall x\beta[s]$.

2.2.6

$\Rightarrow$: Suppose $\vDash \theta$. Let $\mathfrak{A}, s$ be given, and let $a \in |\mathfrak{A}|$. Then by assumption, $\vDash_{\mathfrak{A}} \theta[s(a/x)]$. Since this holds for any a, $\vDash_{\mathfrak{A}} \forall x\theta[s]$.

$\Leftarrow$: Suppose $\vDash \forall x\theta$. Let $\mathfrak{A}, s$ be given. Thet $\vDash_{\mathfrak{A}} \forall x\theta[s]$, and so in particular $\vdash_{\mathfrak{A}} \theta[s(s(x)/x)]$. Since $s(s(x)/x) = s$, also $\vDash_{\mathfrak{A}} \theta[s]$.

2.2.9

a

$\exists x \exists y (x \neq y \wedge \forall z (z = x \vee z = y))$

b

$\forall x \exists y Pxy \wedge \forall z (Pxz \rightarrow y = z)$

c

$(\forall x \exists y Pxy \wedge \forall z (Pxz \rightarrow y = z)) \wedge \forall y \exists x Pxy$