Math 114L

Homework 2 Solutions

Spring 2011

1.7.2

Let Δ be finitely satisfiable and complete, and let ν be as given in the problem. We show by induction on α that $\overline{\nu}(\alpha) = T$ iff $\alpha \in \Delta$.

Base Case: If α is a sentence symbol, $\overline{\nu}(\alpha) = \nu(\alpha) = T$ iff $\alpha \in \Delta$ by the definition of ν.

Inductive Case for $\neg$: If $\overline{\nu}(\neg\alpha) = T$ then $\overline{\nu}(\alpha) = F$, so by IH, $\alpha \notin \Delta$, and since Δ is complete, we must have $\neg\alpha \in \Delta$. If $\overline{\nu}(\neg\alpha) = F$ then $\overline{\nu}(\alpha) = T$, so by IH, $\alpha \in \Delta$; if $\neg\alpha \in \Delta$ then $\{\alpha, \neg\alpha\}$ is a finite unsatisfiable subset of Δ, and since this is impossible, $\neg\alpha \notin \Delta$.

Inductive Case for $\wedge$: If $\overline{\nu}(\alpha \wedge \beta) = T$ then $\overline{\nu}(\alpha) = \overline{\nu}(\beta) = T$, so by IH $\{\alpha, \beta\} \subseteq \Delta$; if $\alpha \wedge \beta \notin \Delta$ then $\neg(\alpha \wedge \beta) \in \Delta$, and so $\{\alpha, \beta, \neg(\alpha \wedge \beta)\}$ is a finite unsatisfiable subset of Δ, and since this is impossible, $\alpha \wedge \beta \in \Delta$. If $\overline{\nu}(\alpha \wedge \beta) = F$ and $\overline{\nu}(\alpha) = F$ then by IH, $\alpha \notin \Delta$, so $\neg\alpha \in \Delta$. If we had $\alpha \wedge \beta \in \Delta$ then $\{\neg\alpha, \alpha \wedge \beta\}$ would be a finite unsatisfiable subset of Δ, and since there are none, $\alpha \wedge \beta \notin \Delta$. If $\overline{\nu}(\alpha \wedge \beta) = F$ and $\overline{\nu}(\alpha) = T$ then $\overline{\nu}(\beta) = F$, and a similar argument applies.

The other inductive cases are simialr.

1.7.3

Suppose Corollary 17A holds and that Σ is not satisfiable. If Σ were unsatisfiable, we would have $\Sigma \models A_1 \wedge \neg A_1$. By the corollary, there must be a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models A_1 \wedge \neg A_1$, and therefore Σ_0 is a finite unsatisfiable subset of Σ, so Σ is not finitely satisfiable. This is the contrapositive of the compactness theorem, and therefore equivalent to the compactness theorem.

1.7.10

a

Let Δ be the set of tautological consequences of Σ. Consider the following procedure: For an expression τ, first check if τ is a wff. By Theorem 17B, there is an effective procedure for this, so in a finite number of steps, this will produce "yes" if τ is a wff, "no" if τ is not. If "no", we will output "no". If "yes", we will continue as follows.

Σ is effectively enumerable, so we have an effective way to enumerate Σ, $\{\sigma_0, \sigma_1, \sigma_2, \ldots\}$. Let $\Sigma_n = \{\sigma_i \mid i \leq n\}$. Since each Σ_n is finite, there is an effective procedure for determining whether $\Sigma_n \vDash \tau$. We successively check if $\Sigma_0 \vDash \tau$, then if $\Sigma_0 \vDash \neg\tau$, then if $\Sigma_1 \vDash \tau$, then if $\Sigma_1 \vDash \neg\tau$, and so on. If we find some n such that $\Sigma_n \vDash \tau$, we output "yes", if we find some n such that $\Sigma_n \vDash \neg\tau$, we output "no".

By assumption, either $\Sigma \models \tau$ or $\Sigma \models \neg\tau$ but not both, so at some n, we will have either $\Sigma_n \models \epsilon$ or $\Sigma_n \models \neg\epsilon$, so this procedure always eventually stops with the correct answer.

b

Suppose that there is τ such that $\Sigma \models \tau$ and $\Sigma \models \neg\tau$. Then Σ is unsatisfiable, so $\Sigma \vDash \sigma'$ for every wff σ'. Therefore we take the decision procedure which outputs "yes" on every wff.

Otherwise, there is no such τ, so we are in the case of part a.

(Note that the procedure here is *non-uniform*, in the sense that we can't decide, given a description of Σ, which of the two prcedures to use. But that doesn't change the fact that the set is decidable, we just don't know how to decide it!)

1.7.11

a

Let A and B be effectively enumerable. By Theorem 17E, they are both semidecidable. Let $C = A \cup B$ and let τ be an expression. We dovetail the two semidecision procedures: we first spend 1 minute checking if $\tau \in A$, then 1 minute checking if $\tau \in B$, then 2 minutes checking if $\tau \in A$, then 2 minutes checking if $\tau \in B$, and so on. If either $\tau \in A$ or $\tau \in B$, this process will eventually stop, and we output "yes". If $\tau \notin A \cup B$, this process runs forever. This is a semidecision procedure, so C is effectively enumerable.

b

Again, let A and B be effectively enumerable, and note that by Theorem 17E they are each semidecidable. Given an expression τ, first run the semidecision procedure checking if $\tau \in A$. If the procedure runs forever, $\tau \notin A$, so $\tau \notin A \cap B$, and we run forever.

Otherwise, the semidecision procedure eventually tells us $\tau \in A$. Then we run the semidecision procedure for B. If this procedure runs forever, $\tau \notin B$, so $\tau \notin A \cap B$, and we run forever. If this procedure stops, we output "yes", since $\tau \in A$ and $\tau \in B$.

1.7.12

a

$\Gamma = \{A_1, \neg A_1\}$

b

$\Gamma = \{A_1, A_2, \neg(A_1 \wedge A_2)\}$

c

$\Gamma = \{A_1, A_2, A_3, \neg(A_1 \wedge A_2 \wedge A_3)\}$