Final Exam, Math 114
Due 2:30, Thursday June 9th

You MAY: use your notes, use your textbook, ask *general* questions of the professor or TA
You MAY NOT: discuss problems with other people, research questions on the internet

1. Prove, using the formal inductive definition of a first-order formula, that $\forall \mathrm{P}\ \mathrm{P}x$ is not a formula.

2. Consider a language of first-order logic with a single binary function symbol R.

   (a) Given an example of a model satisfying the sentence
   $$\forall x \exists y \forall z ((\mathrm{R}xz \rightarrow \mathrm{R}yz) \wedge \mathrm{R}yx).$$

   (b) Given an example of a model satisfying the sentence
   $$\forall x \exists y \forall z ((\mathrm{R}xz \rightarrow \mathrm{R}yz) \wedge \mathrm{R}yx \wedge \neg \mathrm{R}xx).$$

3. Consider a language of first-order logic with a single binary relation R and a constant symbol $\tilde{0}$, and the model $\mathfrak{A}$ such that $|\mathfrak{A}| = \mathbb{Z}$, $\mathrm{R}^{\mathfrak{A}} = \{(n, m) \mid n - m \text{ is odd}\}$, and $\tilde{0}^{\mathfrak{A}} = 0$. (The language includes equality.)

   (a) Show that the even numbers are definable in $\mathfrak{A}$.

   (b) Show that the positive numbers are not definable in $\mathfrak{A}$.

   (c) Show that no proper subset of the odd numbers is definable in $\mathfrak{A}$. (That is, show that any definable subset of the odd numbers is either all the odd numbers, or the empty set.)

4. (a) Prove directly, using properties of derivations, that there is a deduction of $\exists x\ x = x$.

   (b) Use properties of models and the completeness theorem to show that there is a deduction of $\exists x\ x = x$.

5. Consider a language of first-order logic with a binary relation R. A model $\mathfrak{A}$ *contains a cycle* if there are finitely many elements $a_1, \ldots, a_n \in |\mathfrak{A}|$ such that for each $i < n$, $\langle a_i, a_{i+1} \rangle \in \mathrm{R}^{\mathfrak{A}}$, and also $\langle a_n, a_1 \rangle \in \mathrm{R}^{\mathfrak{A}}$. n is called the *length* of the cycle.

   (a) Show that the models which contain a cycle of length 2 form an *EC* class.

   (b) Show that the models which contain *no* cycles form an EC_Δ class.

   (c) Show that the models which contain a cycle are *not* an *EC* class. (Hint: use compactness.)

(continued on next page)

6. Suppose we expand the language of first-order logic with a new quantifier Q. The new quantifier is given the semantics:

$$\models_{\mathfrak{A}} Qx\phi[s] \Leftrightarrow \text{For infinitely many } a \in \mathfrak{A},\ \models_{\mathfrak{A}} \phi[s(a/x)].$$

(Informally, $Qx\phi$ holds if there are infinitely many values making ϕ true.)

(a) Give two examples of formulas involving Q which are valid in this semantics, using any function or predicate symbols you like. (These should actually involve the semantics of Q—if the definition of $\models$ were different, it should be possible for these sentences to stop being true; for instance, $QxPx \to QxPx$ is not a good answer. If you're genuinely unsure whether your answer "involves the semantics", I can give you a precise definition.)

(b) Give an example of a sentence involving Q and two models so that the sentence is true in one and not true in the other. (Again, the sentence should involve the semantics of Q.)

(c) Suppose we extended the set of axioms Λ to include additional axioms for the quantifier Q. Show that no choice of axioms can be both sound and complete. (Hint: show that the logic with Q cannot satisfy the compactness theorem.)