Math 135, Extra Problems

for Final Exam

1. What is $\int_0^\infty e^{-px}(\sin x + x^2)dx$?
2. Find the Laplace transform of:
   (a) $e^{-2x}\cos x$
   (b) $x^{-3/2}$
3. Find the inverse Laplace transform of $\frac{p+1}{p^2+4}$
4. Find $L[x^2]$ in three ways: by using the definition of the Laplace transform, by using the Laplace transform of x and the integral rule, and by using the Laplace transform of x^3 and the derivative rule.
5. Solve the differential equation $xy'' - (x+1)y' - y = 0$ with $y(0) = 0$
6. Solve the integral equation $e^x = y(x) - \int_0^x \sin(x-t)y(t)dt$
7. What are the first three members of the Picard iteration for the equation $y(x) = 1 + \int_0^x \sqrt{y(t)}dt$?
8. Given an integral equation equivalent to $y' = x + y, y(0) = 0$, and find the first three steps of the corresponding Picard iteration.
9. Which of the following functions are Lipschitz on the interval $[0, \infty)$: $1/x, \frac{1}{x+1}, x^7, e^x$? Which are Lipschitz on the interval $[0, 2]$?
10. Find any two Fourier coefficients of e^{-x}
11. Find any two Fourier coefficients of x
12. Which of the following functions does Dirichlet's Theorem that the Fourier series converges pointwise at every point in the interval $[-\pi, \pi)$:
    - $f(x) = x$
    - $f(x) = \sin\frac{1}{x}$
    - $f(x) = x^3 \sin\frac{1}{x}$
    - $f(x) = \begin{cases} x^2 & if - \pi \leq x < 0 \\ \cos x & if 0 \leq x < \pi \end{cases}$

For which of these functions does Dirichlet's Theorem imply that the Fourier series converges pointwise at all but a finite number of points in the same interval? For which of these functions does the Fourier series converge in the mean?

13. Give the first two terms of the Fourier sine series for x

14. Give the first two terms of the Fourier cosine series for x^2

15. If $\theta_1, \ldots, \theta_n, \ldots$ is an orthonormal series on $[a, b]$ and $n \neq m$, what is

$$\int_a^b [\theta_n + \theta_m]^2 dx.$$

16. Which of the following sequences might be the first elements of an orthonormal series:

    - $\cos x, \sin x + \sin 2x, \sin 3x$ on $[-\pi, \pi]$
    - x^2, x^4 on $[-1, 1]$
    - $1/\sqrt{2}, \frac{|x|}{x}\sqrt{|x|}$ on $[-1, 1]$

17. Find a, b so that $\int_{-1}^{1} (x - a/\sqrt{2} - b\frac{|x|}{x}\sqrt{|x|})^2 dx$ is minimal.

18. Recall that $w(x,t)$, the heat of a thin cylindrical rod, is given by $w(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2a^2t} \sin nx$ with $x \in [0, \pi]$. If the initial temperature $w(x, 0) = -x$, what is $w(x,t)$?

19. Given an example of two elements of an orthonormal series with respect to to $\cos x$ on $[-\pi, \pi]$

20. If $\sin xy'' + \cos xy' + \lambda\sqrt{x}y = 0$ with $y(0) = y(\pi) = 0$ has two known solutions, y_1 when $\lambda = 1$ and y_2 when $\lambda = 4$, write down an integral involving y_1 and y_2 which is 0 in this case, but need not be 0 for arbitrary functions.

21. Find the stationary functions of $I(y) = \int_0^1 y(x) \cdot y'(x) + y(x) dx$ with $y(0) = y(1) = 1$.

22. Find the stationary functions of $I(y) = \int_0^1 y(x) \cdot (y'(x))^2 dx$ with $y(0) = 1$, $y(1) = 0$

23. Find the stationary functions of $I(y, z) = \int_1^e xy'(x)z'(x) dx$ with $y(1) = 2$, $y(e) = \pi$, $z(1) = 3$, $z(e) = -1$.