Midterm Exam

Math 135, Lecture 2
2/06/2009

Name: ____________________

Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Unless otherwise specified, a solution to a differential equation means giving all possible solutions subject to the constraints of the problem.
- Good luck!

	Points	Grade
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	10	
Total	100	

1. (*15 points*) The Laplace transform of a certain function $Y_0(x)$ is

$$-\frac{2\sinh^{-1}(p/2)}{\pi\sqrt{p^2+4}}$$

Find the Laplace transform of:

(**a**) $3Y_0(x) + \sin x$

red

$$\begin{aligned} L[3Y_0(x)+\sin x] &= 3L[Y_0(x)] + \tfrac{1}{p^2+1} \\ &= -\tfrac{6\sinh^{-1}(p/2)}{\pi\sqrt{p^2+4}} + \tfrac{1}{p^2+1} \end{aligned}$$

(**b**) $\int_0^x Y_0(x)dx$

$$L[\int_0^x Y_0(x)dx] = \frac{L[Y_0(x)]}{p} = -\frac{2\sinh^{-1}(p/2)}{\pi p\sqrt{p^2+4}}$$

(**c**) $2e^{3x}Y_0(x) + \frac{1}{\sqrt{x}}$

$$\begin{aligned} L[2e^{3x}Y_0(x) + \frac{1}{\sqrt{x}}] &= 2L[Y_0(x)](p-3) + \sqrt{\tfrac{\pi}{p}} \\ &= -\tfrac{4\sinh^{-1}((p-3)/2)}{\pi\sqrt{(p-3)^2+4}} + \sqrt{\tfrac{\pi}{p}} \end{aligned}$$

2. (*15 points*) Solve the following differential equation using a Laplace transform:

$$y'' + 3y' + 2y = e^{-2x}, y(0) = 1, y'(0) = 0.$$

$$L[y''] + 3L[y'] + 2L[y] = L[e^{-2x}]$$

$$p^2Y - py(0) - y'(0) + 3pY - y(0) + 2Y = \frac{1}{p+2}$$

$$p^2Y - p + 3pY - 3 + 2Y = \frac{1}{p+2}$$

$$Y\left(p^2 + 3p + 2\right) = \frac{1}{p+2} + p + 3$$

$$Y = \frac{1}{(p+2)^2(p+1)} + \frac{p+3}{(p+2)(p+1)}$$

Partial fractions:

$$\frac{1}{(p+2)^2(p+1)} + \frac{p+3}{(p+2)(p+1)} = \frac{p^2+5p+7}{(p+2)^2(p+1)} = \frac{A}{(p+2)^2} + \frac{B}{p+2} + \frac{C}{p+1}$$

$$A = -1, B = -2, C = 3$$

$$Y = \frac{-1}{(p+2)^2} + \frac{-2}{p+2} + \frac{3}{p+1}$$

$$y = -xe^{-2x} - 2e^{-2x} + 3e^{-x}$$

3. (*15 points*) Solve the following differential equation using a Laplace transform:

$$xy'' - (2x+2)y' + (x+2)y = 0, y(0) = 0.$$

$$L[xy''] - 2L[xy'] - 2L[y'] + L[xy] + 2L[y] = 0$$
$$-p^2Y' - 2pY + 2pY' + 2Y - 2pY - Y' + 2Y = 0$$
$$Y'(-p^2 + 2p - 1) = Y(4p - 4)$$
$$\frac{1}{Y}dY = \frac{-4(p-1)}{(p-1)^2}$$
$$\ln Y = -4\ln(p-1) + C$$
$$Y = De^{-4\ln(p-1)} = De^{\ln(p-1)^{-4}} = \frac{D}{(p-1)^4}$$
$$y = Ex^3e^x$$

4. (*15 points*) Solve the following integral equation:

$$y(x) = x^2 + \int_0^x \sinh[x-t]y(t)dt.$$

$$L[y] = L[x^2] + L[\sinh x]L[y]$$
$$L[y] = \frac{2}{p^3} + \frac{1}{p^2-1}L[y]$$
$$L[y] = \frac{2}{p^3}\left(\frac{1}{1-\frac{1}{p^2-1}}\right) = \frac{2}{p^3}\left(\frac{1}{\frac{p^2-2}{p^2-1}}\right) = \frac{2p^2-2}{p^3(p^2-2)}$$

Partial Fractions:

$$\frac{2p^2-2}{p^3(p^2-2)} = \frac{A}{p^3} + \frac{B}{p^2} + \frac{C}{p} + \frac{D}{p-\sqrt{2}} + \frac{E}{p+\sqrt{2}}$$
$$2p^2 - 2 = p^2A - 2A + p^3B - 2pB + p^4C - 2p^2C + p^4D - p^3\sqrt{2}D + p^4E + p^3\sqrt{2}E$$
$$0 = D + E + C, 0 = B - D + E, 2 = A - 2C, 0 = -2B, -2 = -2A$$
$$A = 1, B = 0, C = -1/2, D = 1/4, E = 1/4$$
$$L[y] = \frac{1}{p^3} - \frac{1}{2p} + \frac{1}{4(p-\sqrt{2})} + \frac{1}{4(p+\sqrt{2})}$$
$$y = \frac{1}{2}x^2 - \frac{1}{2} + \frac{1}{4}(e^{\sqrt{2}} + e^{-\sqrt{2}})$$

5. (*15 points*) What are y_0, y_1, y_2, and y_3 in the Picard iteration associated with the following differential equation?

$$y' = y \sin x, y(0) = \pi$$

$$y_0(x) = \pi$$

$$y_1(x) = \pi + \int_0^x \pi \sin t dt = \pi - \pi \cos t|_0^x = 2\pi - \pi \cos x$$

$$y_2(x) = \pi + \int_0^x 2\pi \sin t - \pi \cos t \sin t dt = \pi - 2\pi \cos t|_0^2 + \pi \cos^2 t dt\Big|_0^x = \frac{5}{2}\pi - 2\pi \cos x + \frac{\pi}{2} \cos^2 x$$

$$\begin{aligned} y_3(x) \qquad &= \pi + \textstyle\int_0^x (\frac{5}{2}\pi - 2\pi \cos t + \frac{\pi}{2} \cos^2 t) \sin t dt \\ &= \pi + \textstyle\int_0^x \frac{5\pi}{2} \sin t - 2\pi \cos t \sin t + \frac{\pi}{2} \cos^2 t \sin t dt \\ &= \pi - \tfrac{5\pi}{2} \cos t\Big|_0^x + \pi \cos^2 t|_0^x - \tfrac{\pi}{6} \cos^3 t|_0^x \\ &= \tfrac{10\pi}{3} - \tfrac{5\pi}{2} \cos t + \pi \cos^2 x - \tfrac{\pi}{6} \cos^3 x \end{aligned}$$

6. (*15 points*) For each of the following, state whether the claim is true or false. If false, explain why Picard's Theorem does not apply. (That is, which assumption of Picard's Theorem is violated.)

(**a**) The differential equation $y' = x^2 y$ has a unique solution on the interval $0 \leq x \leq 1$ such that $y(0) = 1$.

True

(**b**) The differential equation $y' = xy^3$ has a unique solution on the interval $0 \leq x \leq 1$ such that $y(0) = 1$.

False. $f(x, y) = xy^3$ is not Lipschitz on the rectangle $0 \leq x \leq 1, -\infty < y < \infty$.

(**c**) There is some $h > 0$ such that the differential equation $y' = x\sqrt{y}$ has a unique solution on the interval $[-h, h]$ such that $y(0) = 1$.

True. For example, $f(x, y) = x\sqrt{y}$ is Lipschitz on the rectangle $-1 \leq x \leq 1, 1/2 \leq y \leq 3/2$.

(**d**) The differential equation $y' = x^3 y$ has a unique solution on the interval $0 \leq x \leq 1$ such that $y(-1) = 1$.

False. -1 is not in the interval $[0, 1]$.

7. (*10 points*) (**a**) Recall that the Dirac δ "function" is characterized by the property that $\int_0^\infty f(x)\delta(x)dx = f(0)$ for any function f. The "function" δ_2 is similar, except that δ_2 has the property that $\int_0^\infty f(x)\delta_2(x)dx = f(2)$ for any function f.
Find the Laplace Transform of δ_2.

$$L[\delta_2] = \int_0^\infty e^{-px}\delta_2(x)dx = e^{-p2}$$

(**b**) An unknown "function" μ has the property that $\int_0^\infty f(x)\mu(x)dx = f'(2)$ for any function f. Find the Laplace Transform of μ.

$$L[\mu] = \int_0^\infty e^{-px}\mu(x)dx = -pe^{-2p}$$

(**c**) Assuming that the properties of Laplace Transforms apply to "functions," what is the relationship between μ and δ_2?
In general, $L[f'] = pL[f]$, so in particular, $L[\delta_2'] = pL[\delta_2] = pe^{-2p} = -L[\mu]$, so μ is $-\delta_2$.