MIDTERM

Math 32B
8/18/2010

Name: ______________________

Section: ______________________

Signature: ______________________

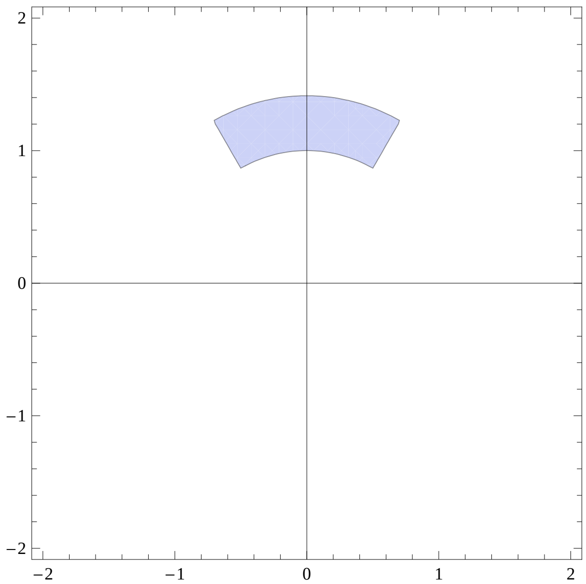
Read all of the following information before starting the exam:

- Check your exam to make sure all pages are present.
- NO CALCULATORS!
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

1	5	
2	5	
3	9	
4	9	
5	9	
6	9	
7	9	
8	6	
9	9	
10	8	
11	8	
12	8	
13	6	
Total	100	

1. (*5 points*) On the graph provided, indicate the region satisfying the conditions

$$1 \le r \le 2, \pi/3 \le \theta \le 2\pi/3.$$

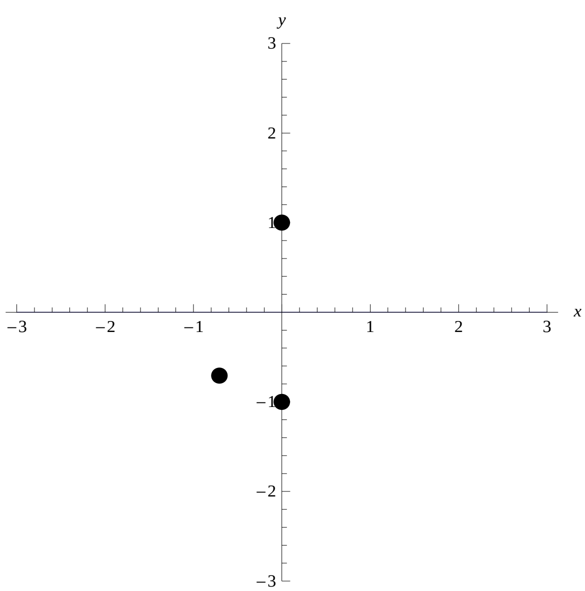


2. (*5 points*) On the graph provided, indicate (and label) the following points (which are given in the form (r, θ)):

(a) $(1, \pi/2)$

(b) $(-1, 5\pi/2)$

(c) $(-1/2, 17\pi/4)$



3. (*9 points*) Solve the following integral. You may use any coordinate system you choose.

$$\iint_B (xe^y)dA$$

where $B = [1,2] \times [4,6]$.

$$\int_1^2 \int_4^6 \mathtt{xe^y\ dy\ dx} = \int_1^2 \mathtt{xe^y}|_4^6 \mathtt{dx} = \int_1^2 \mathtt{x(e^6 - e^4)dx} = \frac{1}{2}\mathtt{x^2(e^6 - e^4)}|_1^2 = \frac{3}{2}\mathtt{(e^6 - e^4)}$$

4. (*9 points*) Solve the following integral. You may use any coordinate system you choose.

$$\iiint_E (x^2 + y^2)(x^2 + y^2 + z)dV$$

where $E = \{(x, y, z) \mid x^2 + y^2 \leq z, 0 \leq z \leq 1\}$.

$$\begin{aligned}
&\int_0^{2\pi} \int_0^1 \int_{r^2}^1 r^2(r^2 + z)r\ dz\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^1 \int_{r^2}^1 r^5 + r^3 z\ dz\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^1 r^5 z + \frac{1}{2} r^3 z^2 \Big|_{r^2}^1\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^1 r^5 + \frac{1}{2} r^3 - \frac{3}{2} r^7\ dr\ d\theta \\
&= \int_0^{2\pi} \frac{1}{6} r^6 + \frac{1}{8} r^4 - \frac{3}{16} r^8 \Big|_0^1\ d\theta \\
&= 2\pi \left(\frac{1}{6} + \frac{1}{8} - \frac{3}{16}\right) \\
&= \frac{5\pi}{24}
\end{aligned}$$

5. (*9 points*) Solve the following integral. You may use any coordinate system you choose.

$$\iiint_D \sqrt{x^2+y^2+z^2}dV$$

where $D = \{(x,y,z) \mid 0 \le x^2+y^2+z^2 \le 4 \text{ and } z^2 \le x^2+y^2\}$.

`This is naturally expressed in spherical coordinates. The first condition says that` $0 \le \rho \le 2$ `while the second says` $\rho^2\cos^2\phi \le \rho^2\sin^2\phi$`; that is,` $\cos^2\phi \le \sin^2\phi$`. This occurs whenever` $\pi/4 \le \phi \le 3\pi/4$`. So the integral is:`

$$\int_0^{2\pi}\int_0^2\int_{\pi/4}^{3\pi/4} \rho^3 \sin\phi \, d\phi \, d\rho \, d\theta = 2\pi \int_0^2 \rho^3 \, d\rho \int_{\pi/4}^{3\pi/4} \sin\phi \, d\phi = 2\pi \cdot 4 \cdot \frac{\sqrt{2}}{2}$$

6. (*9 points*) Solve the following integral. You may use any coordinate system you choose.

$$\iiint_C y \, dV$$

where $C = \{(x,y,z) \mid 0 \le x^2+y^2+z^2 \le 4 \text{ and } x \le z\}$.

`The region is symmetric around the` xz`-plane, and the function is simply` y`, so the integral must be` 0`: the` $-y$ `part will exactly cancel the` $+y$ `part.`

7. (*9 points*) Solve the following integral. You may use any coordinate system you choose.

$$\iiint_F x + 2z\ dV$$

where $F = \{(x,y,z) \mid 0 \le x \le \cos y \le z \le 1, -\frac{\pi}{2} \le y \le \frac{\pi}{2}\}$.

$$\int_{-\pi/2}^{\pi/2}\int_0^{\cos y}\int_{\cos y}^{1} x + 2z\ dz\ dx\ dy = \int_{-\pi/2}^{\pi/2}\int_0^{\cos y} xz + z^2\Big|_{\cos y}^{1}\ dx\ dy =$$

$$\int_{-\pi/2}^{\pi/2}\int_0^{\cos y} x + 1 - x\cos y - \cos^2 y\ dx\ dy = \int_{-\pi/2}^{\pi/2}\int_0^{\cos y} x - x\cos y + \sin^2 y\ dx\ dy =$$

$$\int_{-\pi/2}^{\pi/2} x^2/2 - x^2/2\cos y + x\sin^2 y\Big|_0^{\cos y}\ dy = \int_{-\pi/2}^{\pi/2} \cos^2 y/2 - \cos^3 y/2 + \cos y\sin^2 y\ dy =$$

$$\int_{-\pi/2}^{\pi/2} \cos^2 y/2 - \cos y(1-\sin^2 y)/2 + \cos y \sin^2 y\ dy =$$

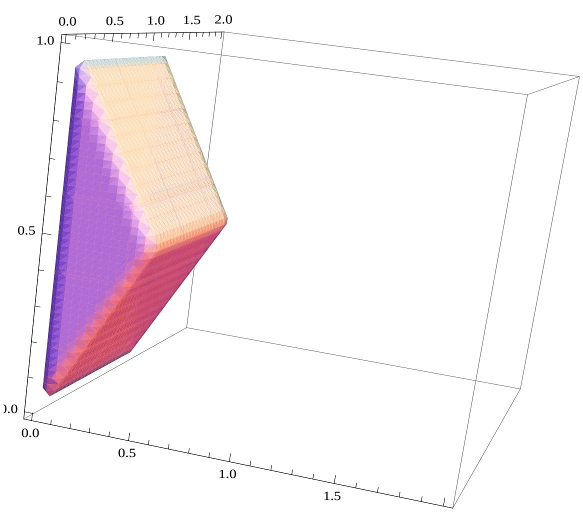
$$\frac{1}{4}\left(y + \frac{1}{2}\sin 2y\right) - \frac{1}{8}\left(3\sin y + \frac{1}{3}\sin 3y\right) + \frac{1}{3}\sin^3 y\Big|_{-\pi/2}^{\pi/2} =$$

$$\frac{\pi}{8} - \frac{1}{8}\left(3 - \frac{1}{3}\right) - \frac{1}{3} - \left(-\frac{\pi}{8} - \frac{1}{8}\left(-3 + \frac{1}{3}\right) - \frac{1}{3}\right) = \frac{\pi}{4} - \frac{2}{9} + \frac{2}{9} = \frac{\pi}{4}$$

8. (*6 points*) Consider the integral

$$\int_0^1\int_0^2\int_x^{1-x} f(x,y)dzdydx.$$

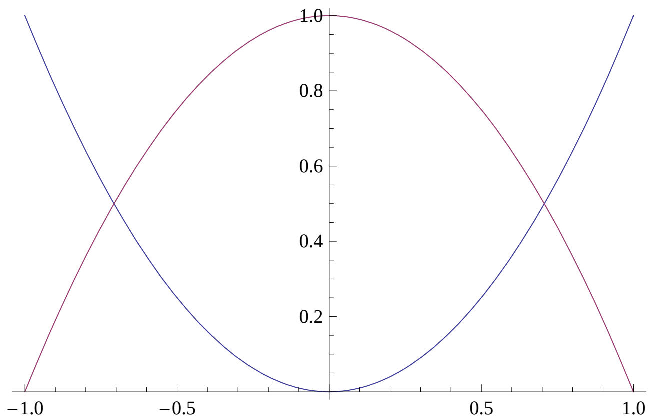
Describe the region being integrated over.



Alternatively, a verbal discription that covered the major features sufficed. A perfect description should have indicated: 1) that the shape was a triangular wedge (an appropriate comparison---say, to a roof or another object with a similar shape, worked as well), 2) that one side was flush with the yz-plane in the $+y, +z$ quadrant, 3) that the other two sides converged to an apex pointing in the $+x$ direction, above the xz-plane. (We weren't looking for a description in precisely those terms, of course; evidence that the writer knew that was what was happening was sufficient.) The most common problem was not realizing (or not indicating) that the middle corner was lifted off the xz-plane.

9. (*9 points*) Consider the region D between the functions $y = x^2$ and $y = 1 - x^2$ (as shown on the diagram). Consider the double integral $\iint_D xy \, dA$.

(**a**) Write down an iterated integral which is equal to this double integral, and where x is the variable in the outermost equation. (That is, an equation of the form $\iint xy \, dy \, dx$.)



$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \int_{x^2}^{1-x^2} xy \ dy \ dx$$

(**b**) Write down an iterated integral which is equal to this double integral, and where y is the variable in the outermost equation. (That is, an equation of the form $\iint xy \, dx \, dy$.)

$$\int_{0}^{\sqrt{2}/2} \int_{-\sqrt{y}}^{\sqrt{y}} xy \ dx \ dy + \int_{\sqrt{2}/2}^{1} \int_{-\sqrt{1-y}}^{\sqrt{1-y}} xy \ dx \ dy$$

(**c**) Solve either of the equations in the previous parts to give a value for this integral.

$$\int_{-1}^{1} \int_{x^2}^{1-x^2} xy \ dy \ dx = \frac{1}{2} \int_{-1}^{1} xy^2 \Big|_{x^2}^{1-x^2} dydx = \frac{1}{2} \int_{-1}^{1} x((1-x^2)^2 - x^4)dx = \frac{1}{2} \int_{-1}^{1} x - 2x^3 \ dx = \frac{1}{4}(x^2 - x^4)_{-1}^{1} = 0$$

10. (*8 points*) Consider the integral

$$\iint_G (x^2 + y^2) dA$$

where $G = \{(x, y) \mid 0 \le x^2 + y^2 \le 1 \text{ and } x \ge 0\}$.

(**a**) Write this as an integral in polar coordinates. You do not need to solve this integral.

$\int_{-\pi/2}^{\pi/2} \int_0^1 \mathtt{r}^2 \cdot \mathtt{r}\ \mathtt{dr}\ \mathtt{d}\theta$

(**b**) Approximate your integral from the previous part using a Riemann sum with $m = n = 3$. You should evaluate this integral to the point that you are adding up a long string of numbers, but you do not need to actually calculate their sum.

$\sum_{i=1}^{3} \sum_{j=1}^{3} (r_i^*)^2 \cdot r_i^* \Delta r \Delta\theta = \left((\frac{1}{6})^3 + (\frac{1}{2})^3 + (\frac{5}{6})^3\right) 3 \cdot \frac{1}{3} \cdot \frac{\pi}{3}$
`(Here we have used the midpoints,` $1/6, 1/2, 5/6$`, but other choices for the sample points were also correct.)`

11. (*8 points*) Suppose we have a thin circular disc of radius 1 centered on the origin. Suppose the density of this disc at each point (x, y) is $x^2 + y^4$.

(**a**) Write (but you need not solve or simplify) an integral giving the total mass of this disc.

$$\int_0^{2\pi} \int_0^1 \mathtt{r}^2 \cos^2\theta + \mathtt{r}^4 \sin^4\theta\ \mathtt{dr}\ \mathtt{d}\theta$$

(**b**) Write (but you need not solve or simplify) an integral giving the moment about the x-axis.

$$\int_0^{2\pi} \int_0^1 (\mathtt{r}\sin\theta)(\mathtt{r}^2 \cos^2\theta + \mathtt{r}^4 \sin^4\theta)\ \mathtt{dr}\ \mathtt{d}\theta$$

(**c**) Write (but you need not solve or simplify) an integral giving the moment of inertia about the origin.

$$\int_0^{2\pi} \int_0^1 \mathtt{r}^2(\mathtt{r}^2 \cos^2\theta + \mathtt{r}^4 \sin^4\theta)\ \mathtt{dr}\ \mathtt{d}\theta$$

12. (*8 points*) The random variable X is distributed with probability density function $e^{(x-1)^2}$. The random variable Y is always ≥ 1, and is distributed with probability density function $\frac{1}{y^2}$. Write down (but do not solve) an integral expressing the probability that $|X - Y| \leq 10$.

$\int_1^\infty \int_{y-10}^{y+10} \frac{e^{(x-1)^2}}{y^2} dx\ dy$

13. (*6 points*) Consider the coordinate transformation $u = -\frac{1}{2}\ln\frac{y}{x}$, $v = \sqrt{xy}$. Let R be the triangle bounded by the lines $y = 1$, $y = x$, and $y = 4x$. What is the image of R under the transformation?

The line $y = 1$ corresponds to $u = -\frac{1}{2}\ln\frac{1}{x} = \ln\sqrt{x}$ while $v = \sqrt{x}$, so $v = e^u$; since x ranges from $1/4$ to 1 on this line, u ranges from $-\frac{1}{2}\ln 4 = -\ln 2$ to $-\frac{1}{2}\ln 1 = 0$.

The line $y = x$ corresponds to $u = -\frac{1}{2}\ln\frac{x}{x} = 0$ while $v = \sqrt{x^2} = x$. Since x ranges from 0 to 1 on this line, this is a vertical line at $u = 0$ with v ranging from 0 to 1.

The line $y = 4x$ corresponds to $u = -\frac{1}{2}\ln 4 = -\ln 2$ while $v = \sqrt{4x^2} = 2x$. This corresponds to the vertical line at $u = -\ln 2$ with v ranging from 0 to $1/2$.

Combined, these three lines form three of the four lines of a trapezoid to the left of the u-axis. Combined with the fact that $v \geq 0$ (since $v = \sqrt{xy}$ is never negative), the region R maps to the resulting trapezoid.