1. (*15 points*) Give general solutions to the following differential equations:

(**a**) $z' = \frac{x^2}{\cos z}$

The equation $\frac{dz}{dx} = \frac{x^2}{\cos z}$ is separable. We first separate variables and then integrate, $\int \cos z \ dz = \int x^2 \ dx$. This gives $\sin z = \frac{x^3}{3} + C$.

(**b**) $tu' = e^u$

The equation $t \ \frac{du}{dt} = e^u$ is separable. We first separate variables and then integrate, $\int e^{-u} \ du = \int \frac{dt}{t}$. This gives $-e^{-u} = \ln|t| + C_1$. To solve for u, we multiply through by -1, take the natural log, and multiply through by -1 again, $u(t) = -\ln(-\ln|t| + C)$.

2. (*15 points*) (**a**) Solve the following initial value problem

$$t^2u' = e^u, u(1) = 0$$

The equation $t^2 \ \frac{du}{dt} = e^u$ is separable. We first separate variables and then integrate, $\int e^{-u} \ du = \int \frac{dt}{t^2}$. This gives $-e^{-u} = -\frac{1}{t} + C_1$. To solve for u, we multiply through by -1, take the natural log, and multiply through by -1 again, $u = -\ln\left(\frac{1}{t} + C\right)$. We now use the initial condition $u(1) = 0$ to get $0 = u(1) = -\ln(1 + C)$, and this implies $1 + C = 1$. Thus $C = 0$, so the solution to the initial value problem is $u(t) = -\ln\left(\frac{1}{t}\right) = \ln(t)$.

(**b**) What is the interval of existence of this solution?

The natural log function is defined for all $x > 0$, so the interval of existence is $(0, \infty)$.

3. (*20 points*) Indiana Jones has fallen into a deep pit containing $100m^3$ of pure water. Corrosive acid is pouring into the pit at a rate of $3m^3/min$, while $2m^3/min$ of liquid drains from the pit.

(**a**) Let $A(t)$ be the amount of acid in the pit after t minutes. Write a differential equation expressing $A'(t)$ in terms of $A(t)$ and t (assuming instantaneous mixing).

Using that

$$\texttt{rate of change} = \texttt{rate in} - \texttt{rate out},$$

the differential equation is $A'(t) = 3 - \frac{2A(t)}{100+t}$.

(**b**) Find a general solution of the equation in part a.

The equation $A' + \frac{2A}{100+t} = 3$ is linear, and the general integrating factor is given by

$$\mu(t) = e^{\int \frac{2}{100+t}\ dt} = e^{2\ln|100+t|+C_1} = C_2(100+t)^2.$$

The constant C is arbitrary, so we can set $C_2 = 1$ and use $\mu = (100+t)^2$. We multiply through by μ, $\left((100+t)^2 A\right)' = 3(100+t)^2$, then integrate, $(100+t)^2 A = \int 3(100+t)^2\ dt = (100+t)^3 + C$. Solving for A gives $A(t) = 100 + t + \frac{C}{(100+t)^2}$.

(**c**) Find the particular solution to the equation in part a, taking into account information about the initial situation.

Since the pit contains pure water initially, the initial condition is $A(0) = 0$. This implies $0 = A(0) = 100 + \frac{C}{100^2}$, so $C = -100^3$ and the particular solution to the equation in part (a) is $A(t) = 100 + t - \frac{100^3}{(100+t)^2}$.

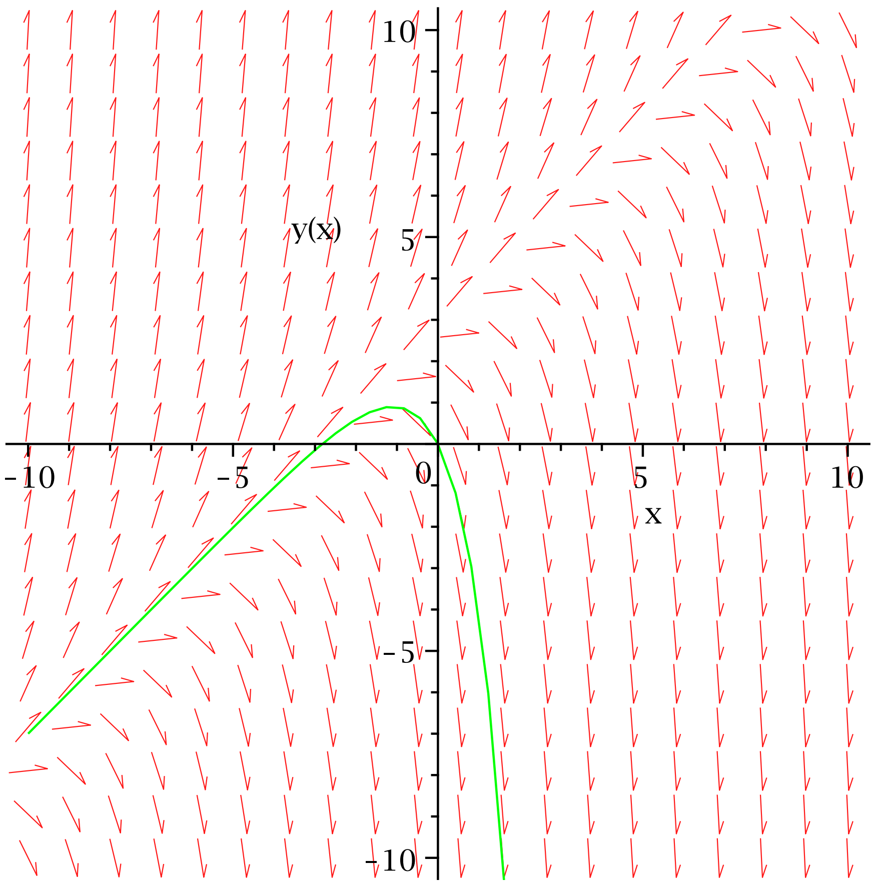
(**d**) When the concentration of the acid (that is, $A(t)$ divided by $V(t)$) reaches $1/4$, Indiana Jones will die. How long does he have to escape? (That is, at what t does $A(t)/V(t) = 1/4$. You need not evaluate any numerical calculations; it suffices to solve the appropriate equation for t.)

We solve

$$\frac{1}{4} = \frac{A(t)}{V(t)} = \frac{100 + t - 100^3\ (100+t)^{-2}}{100+t} = 1 - \frac{100^3}{(100+t)^3}$$

for t, getting $\frac{100^3}{(100+t)^3} = \frac{3}{4}$, and therefore $(100+t)^3 = \frac{4 \cdot 100^3}{3}$. The solution is $t = 100\left(\frac{4}{3}\right)^{1/3} - 100$.

4. (*15 points*) Consider the following direction field of an unknown differential equation:



(**a**) Give the equation for a particular solution of this differential equation.

`We can see from the direction field that a particular solution is given by the straight line through the points` $(-2, 0)$ `and` $(0, 2)$`, and its equation is therefore`

$$y(x) = x + 2.$$

(**b**) Draw (on the direction field above) the solution through $(0, 0)$.

(c) Is this differential equation autonomous? How can you tell? (NO credit without an explanation.)

`The equation is not autonomous. The slope for a fixed value of` $y(x)$ `depends on` x. `For example, the slope for` $y(x) = 5$ `decreases from a positive number to a negative number as` x `increases from` -10 `to` 10.

5. (*20 points*) Consider the differential equation

$$5y + (3 + y)xy' = 0$$

or, written with differentials, $5ydx + (3 + y)xdy = 0$.

(**a**) For which values of a, b, c is $x^a y^b e^{cy}$ an integrating factor? (Give two equations relating a, b, c; you need not get them in a particular form.)

Multiply the differential form $5y\ dx + (3 + y)x\ dy = 0$ by $x^a y^b e^{cy}$ to get

$$5x^a y^{b+1} e^{cy}\ dx + \left(3x^{a+1} y^b e^{cy} + x^{a+1} y^{b+1} e^{cy}\right)\ dy = 0.$$

Now let $P(x, y) = 5x^a y^{b+1} e^{cy}$ and $Q(x, y) = 3x^{a+1} y^b e^{cy} + x^{a+1} y^{b+1} e^{cy}$. The requirement for $x^a y^b e^{cy}$ to be an integrating factor is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

that is, $5(b + 1)x^a y^b e^{cy} + 5cx^a y^{b+1} e^{cy} = 3(a + 1)x^a y^b e^{cy} + (a + 1)x^a y^{b+1} e^{cy}$. Simplifying a bit, $5x^a y^b e^{cy} ((b + 1) + cy) = (a + 1)x^a y^b e^{cy} (3 + y)$, and dividing by the integrating factor $x^a y^b e^{cy}$, we get $5(b + 1) + 5cy = (a + 1)(3 + y)$. Two equations relating a, b, c are

$$3(a + 1) = 5(b + 1) \quad \text{and} \quad a + 1 = 5c.$$

(**b**) Using an integrating factor from the first part, give the general solution. (Hint: try setting $b = 0$.)

Setting $b = 0$, the equations from part (a) imply that $a = 2/3$ and $c = 1/3$, so

$$\mu(x, y) = x^{2/3} e^{y/3}$$

is an integrating factor. We multiply the given differential form by this integrating factor, $5x^{2/3} y e^{y/3}\ dx + x(3 + y)x^{2/3} e^{y/3}\ dy = 0$, and we get an exact equation. Let $\tilde{P}(x, y) = 5x^{2/3} y e^{y/3}$ and $\tilde{Q}(x, y) = x(3+y)x^{2/3} e^{y/3}$. We integrate to solve $\partial F/\partial x = \tilde{P}$,

$$F(x, y) = \int 5x^{2/3} y e^{y/3}\ dx + \phi(y) = 3x^{5/3} y e^{y/3} + \phi(y),$$

and differentiate to solve $\partial F/\partial y = \tilde{Q}$,

$$x(3 + y)x^{2/3} e^{y/3} = \frac{\partial}{\partial y}\left(3x^{5/3} y e^{y/3} + \phi(y)\right) = 3x^{5/3} e^{y/3} + x^{5/3} y e^{y/3} + \phi'(y).$$

It follows that $3 + y = 3 + y + \phi'(y)$, so $\phi'(y) = 0$ and therefore $\phi(y) = C$. Thus the general solution is $F(x, y) = 3x^{5/3} y e^{y/3} = D$.

6. (*15 points*) Suppose y is a solution to the initial value problem

$$y' = (y-5)e^{t^2 y} \text{ and } y(1) = 2$$

Show that $y(t) < 5$ for all t for which y is defined. First, note that

$$f(y,t) = (y-5)e^{t^2 y} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = e^{t^2 y} - 5t^2 e^{t^2 y}$$

are continuous everywhere in t, y-plane, so by the uniqueness theorem, the solution for any initial condition exists and is unique wherever it is defined.
By substitution, we see that $y_1(t) = 5$ is a particular solution to the differential equation

$$y' = (y-5)e^{t^2 y}.$$

Suppose $y_2(t)$ is a solution to the initial value problem

$$y' = (y-5)e^{t^2 y}, \qquad y(1) = 2,$$

and $y_2(t) \geq 5$ for some t. Since y_2 is continuous, the intermediate value theorem implies that there is a value t_0 such that $y_2(t_0) = 5$ (that is, the solution curve y_2 ''crosses'' the solution curve y_1 at $t = t_0$). But then $y_1(t)$ and $y_2(t)$ are distinct solutions of the inital value problem

$$y' = (y-5)e^{t^2 y}, \qquad y(t_0) = 5,$$

contradicting the uniquness theorem. Hence any solution $y(t)$ of the initial value problem

$$y' = (y-5)e^{t^2 y}, \qquad y(1) = 2$$

must satisfy $y(t) < 5$ for all t for which y is defined.