MIDTERM 2

Math 33B
2/17/2010

Name: ______________________________

Section: ______________________________

Signature: ______________________________

Read all of the following information before starting the exam:

- Check your exam to make sure all pages are present.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

1	15	
2	15	
3	15	
4	20	
5	20	
6	15	
Total	100	

1. (*15 points*) Give general solutions to the following differential equations:

(**a**) $y'' + 6y' + 18y = 0$

To find the general solution, we need to solve the characteristic equation

$$\lambda^2 + 6\lambda + 18 = 0$$

using the quadratic equation,

$$\lambda_{1,2} = \frac{-6 \pm \sqrt{36 - 72}}{2} = -3 \pm 3i.$$

The general solution is thus

$$y = e^{-3x}\left(C_1 \cos 3x + C_2 \sin 3x\right).$$

(**b**) $y'' + 4y' - 12y = 0$

To find the general solution, we need to solve the characteristic equation

$$(\lambda + 6)(\lambda - 2) = \lambda^2 + 4\lambda - 12 = 0.$$

The roots are

$$\lambda_1 = -6 \qquad \text{and} \qquad \lambda_2 = 2,$$

and the general solution is

$$y = C_1 e^{-6x} + C_2 e^{2x}.$$

(**c**) $y'' + 4y' + 4y = 0$

To find the general solution, we need to solve the characteristic equation

$$(\lambda + 2)^2 = \lambda^2 + 4\lambda + 4 = 0.$$

We have a double real root $\lambda = -2$, so the general solution is

$$y = C_1 e^{-2x} + C_2 x e^{-2x}.$$

2. (*15 points*) Solve the initial value problem

$$y'' + 4y' - 12y = 0, y(0) = 1, y'(0) = 2$$

We found the general solution in Problem 1(b),

$$y = C_1e^{-6x} + C_2e^{2x}.$$

Differentiating, we get

$$y' = -6C_1e^{-6x} + 2C_2e^{2x}.$$

Plugging this into the initial conditions, we get

$$1 = y(0) = C_1 + C_2,$$
$$2 = y'(0) = -6C_1 + 2C_2.$$

This is a 2×2 system of linear equations, which we can solve using matrix methods or by substitution. Multiplying the first equation by -2 and adding it to the second, we get $0 = -8C_1$, so $C_1 = 0$ and therefore $C_2 = 1$. The solution to the initial value problem is thus

$$y = e^{2x}.$$

3. (*15 points*) Solve the initial value problem

$$y'' + 8(y')^2 = 0, y(1) = 0, y'(1) = e^{-1}.$$

This differential equation is non-linear, so we cannot apply the same technique as in Problem 2. To simplify the notation, let $z = y'$. Then the equation becomes

$$z' + 8z^2 = 0,$$

and this is separable. Thus $\frac{dz}{dx} = -8z^2$, so we have

$$\frac{dz}{z^2} = -8\ dx.$$

Integrating gives $-1/z = -8x + C_1$ and so

$$y' = z = \frac{1}{8x + C_2}.$$

Integrating again gives

$$y = \frac{1}{8}\ln|8x + C_2| + C_3.$$

We now plug y and y' into the initial conditions to get

$$0 = y(1) = \frac{1}{8}\ln|8 + C_2| + C_3,$$
$$\frac{1}{e} = y'(1) = \frac{1}{8 + C_2}.$$

The second equation gives $C_2 = e-8$, and plugging this into the first, we get $C_3 = (1/8)\ln(e) = -1/8$. Thus the solution to the initial value problem is

$$y = \frac{1}{8}\ln|8x + e - 8| - \frac{1}{8}.$$

4. (*20 points*) Find general solutions to the following differential equations:

(**a**) $y'' + 36y = 0$

To find the general solution, we need to solve the characteristic equation

$$(\lambda + 6i)(\lambda - 6i) = \lambda^2 + 36 = 0.$$

The roots are thus $\lambda_{1,2} = \pm 6i$, so the general solution is

$$y = C_1 \cos 6x + C_2 \sin 6x.$$

(**b**) $y'' + 36y = \sec 6x$

We already have the homogeneous solution from part (a), so we only need to find a particular solution and add the two. Since we do not know what form a test solution should have, the method of undetermined coefficients does not apply, and we have to use variation of parameters. From part (a), the functions $y_1 = \cos 6x$ and $y_2 = \sin 6x$ form a fundamental set of solutions, so our particular solution will be

$$y_p = v_1 \cos 6x + v_2 \sin 6x$$

for some functions v_1 and v_2. To find these functions, we solve

$$v_1' = \frac{-y_2 \sec 6x}{y_1 y_2' - y_1' y_2} = \frac{-\sin 6x}{6 \cos 6x},$$

$$v_2' = \frac{y_1 \sec 6x}{y_1 y_2' - y_1' y_2} = \frac{1}{6},$$

by integrating. Thus

$$v_1 = -\frac{1}{6} \int \frac{\sin 6x}{\cos 6x} \; dx = \frac{1}{36} \ln |\cos 6x| \qquad \text{and} \qquad v_2 = \frac{x}{6}.$$

Our general solution is thus

$$y = y_h + y_p = C_1 \cos 6x + C_2 \sin 6x + \frac{\cos 6x}{36} \ln |\cos 6x| + \frac{x \sin 6x}{6}.$$

5. (*20 points*) Find particular solutions to the following differential equations using the method of undetermined coefficients:

(**a**) $y'' + 6y' + 18y = e^x$

Since the forcing term is e^x, our test solution is $y = ae^x$. Thus

$$y = y' = y'' = ae^x,$$

so we get

$$e^x = (a + 6a + 18a)e^x = 25ae^x.$$

Therefore $a = 1/25$ and a particular solution is $y = (1/25)e^x$.

(**b**) $y'' + 6y' + 18y = x^2$

Since the forcing term is x^2, our test solution is $y = ax^2 + bx + c$. Thus

$$y' = 2ax + b \qquad \text{and} \qquad y'' = 2a.$$

We get

$$x^2 = 2a + 12ax + 6b + 18ax^2 + 18bx + 18c = 18ax^2 + (12a + 18b)x + (2a + 6b + 18c),$$

and equating coefficients gives

$$a = \frac{1}{18}, \qquad b = -\frac{1}{27}, \qquad c = \frac{1}{162}.$$

Thus a particular solution is

$$y = \frac{x^2}{18} - \frac{x}{27} + \frac{1}{162}.$$

(**c**) Find the general solution to the following differential equation:

$$y'' + 6y' + 18y = 2e^x + 3x^2$$

We found the homogeneous solution

$$y_h = e^{-3x}\left(C_1 \cos 3x + C_2 \sin 3x\right)$$

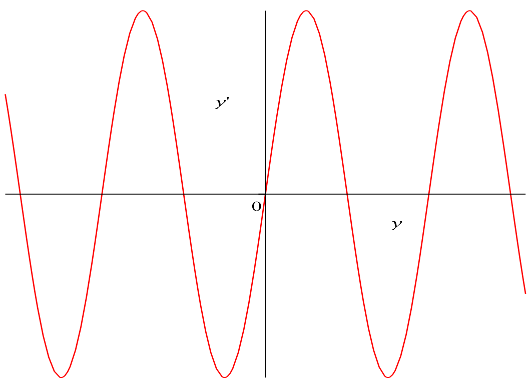
in Problem 1(a), and we found particular solutions y_{p_1} and y_{p_2} for the forcing terms e^x and x^2 in parts (b) and (c) above, respectively. Thus the general solution for the equation with forcing term $2e^x + 3x^2$ is

$$y = y_h + 2y_{p_1} + 3y_{p_2} = e^{-3x}\left(C_1 \cos 3x + C_2 \sin 3x\right) + \frac{2}{25}e^x + \frac{x^2}{6} - \frac{x}{9} + \frac{1}{54}.$$

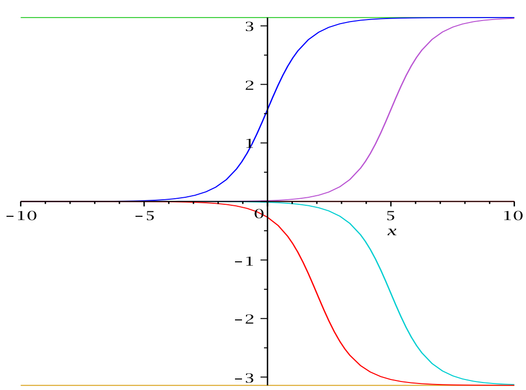
6. (*15 points*) Consider the autonomous first-order differential equation

$$y' = \sin y.$$

(**a**) Draw the phase line.



(**b**) Sketch a graph of some of the solutions which includes at least three equilibrium solutions and at least two solutions between each pair of adjacent equilibrium solutions.



(**c**) What are the equilibrium solutions, which are stable, and which are unstable?
`The equilibrium solutions are found by solving` $\sin y = 0$. `From high school trigonometry, the solutions are` $y = n\pi$ `for every integer` n. `Note that` $\frac{d}{dy}\sin y = \cos y$, `and for every integer` n,

$$\cos 2n\pi = 1 \qquad \text{and} \qquad \cos(2n+1)\pi = -1.$$

`Thus` $y = n\pi$ `is a stable equilibrium if` n `is odd and unstable if` n `is even.`