MIDTERM 2

Math 3A
11/16/2009

Name: ______________________________

Signature: ______________________________

Section: ________________

Read all of the following information before starting the exam:

- Check your exam to make sure all pages are present.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Whenever you invoke a theorem to justify a result, make sure to clearly identify all premises of the theorem, show that they are true, and specify which theorem you are using.
- Circle or otherwise indicate your final answers.
- Good luck!

1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (*20 points*) Find the specified derivatives of the following fuctions:

(**a**) $\frac{d}{dx}e^{\sin\sqrt{\tan x}}$

$$\begin{aligned} e^{\sin\sqrt{\tan x}}\cdot(\sin\sqrt{\tan x})' &= e^{\sin\sqrt{\tan x}}(\cos\sqrt{\tan x})\cdot(\sqrt{\tan x})' \\ &= e^{\sin\sqrt{\tan x}}(\cos\sqrt{\tan x})\frac{1}{2\sqrt{\tan x}}(\tan x)' \\ &= e^{\sin\sqrt{\tan x}}(\cos\sqrt{\tan x})\frac{1}{2\sqrt{\tan x}}\sec^2 x \end{aligned}$$

(**b**) $\frac{d}{dt}\frac{(t^2+t+7)e^t\sin t\cos t}{\sqrt{t+3}}$

Let $f(t) = \frac{(t^2+t+7)e^t\sin t\cos t}{\sqrt{t+3}}$.
$\ln f = \ln(t^2+t+7) + t + \ln\sin t + \ln\cos t - (\ln\sqrt{t+3})$,
so $\frac{1}{f}f' = \frac{2t+1}{t^2+t+7} + 1 + \frac{\cos t}{\sin t} - \frac{\sin t}{\cos t} - \frac{1}{2(t+3)}$,
so $f' = \left(\frac{2t+1}{t^2+t+7} + 1 + \frac{\cos t}{\sin t} - \frac{\sin t}{\cos t} - \frac{1}{2(t+3)}\right)\left(\frac{(t^2+t+7)e^t\sin t\cos t}{\sqrt{t+3}}\right)$

(**c**) $\frac{d}{dx}f^{-1}(x)$ where $f(t) = \frac{\sin t}{1+t^2}$ (remember $f^{-1}(x)$ is the inverse of $f(t)$, so $f^{-1}(f(t)) = t$ and $f(f^{-1}(x)) = x$)

$f'(t) = \frac{(1+t^2)\cos t - 2t\sin t}{(1+t^2)^2}$,
and $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'[f^{-1}(x)]}$
so $\frac{d}{dx}f^{-1}(x) = \frac{(1+(f^{-1}(x))^2)^2}{(1+(f^{-1}(x))^2)\cos f^{-1}(x) - 2f^{-1}(x)\sin f^{-1}(x)}$

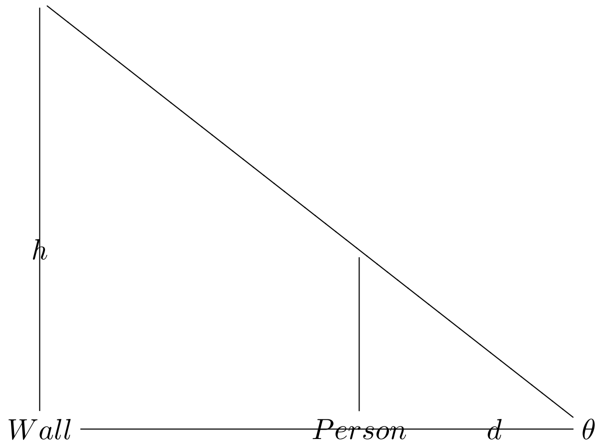
(**d**) $\frac{d^2}{dr^2}\left[r^7+7r^6+5r\right]$

$\frac{d}{dr}\left[r^7+7r^6+5r\right] = 7r^6 + 42r^5 + 5$
$\frac{d^2}{dr^2}\left[r^7+7r^6+5r\right] = 42r^5 + (5\cdot 42)r^4$

(**e**) $\frac{d^{100}}{dx^{100}}xe^x$

$\frac{d}{dx}xe^x = xe^x + e^x$
$\frac{d^2}{dx^2}xe^x = xe^x + 2e^x$
$\frac{d^3}{dx^3}xe^x = xe^x + 3e^x$
So $\frac{d^{100}}{dx^{100}}xe^x = xe^x + 100e^x$

2. (*20 points*) A spotlight is 30 feet from a wall. A 6 foot tall person is between the spotlight and the wall, walking towards the wall at 10 feet per minute. The person casts a shadow on the wall, caused by the spotlight. (Note that this question has a total of 5 parts.)



(**a**) Write an equation giving the relationship between the angle θ and the distance between the person and the spotlight.

$\tan\theta = \frac{6}{d}$

(**b**) Write an equation giving the relationship between the angle θ and the height of the person's shadow.

$\tan\theta = \frac{h}{30}$

(**c**) When the person's shadow is 18 feet tall, how far is the person from the spotlight?

$\frac{6}{d} = \frac{h}{30}$
so $\frac{6}{h} = \frac{d}{30}$
so when $h = 18$, $d = 10$

(d) Write an equation relating the rate of change of θ to the distance between the person and the spotlight and the rate the person is moving towards the wall.

$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{6}{d}$
$\sec^2\theta\frac{d\theta}{dt} = -\frac{6}{d^2}\frac{dd}{dt}$
Since $\sec^2\theta = \left(\frac{\sqrt{6^2+d^2}}{d}\right)^2$, we have
$\left(\frac{\sqrt{6^2+d^2}}{d}\right)^2\frac{d\theta}{dt} = -\frac{6}{d^2}\frac{dd}{dt}$

(e) How quickly is the angle θ changing when the person's shadow is 18 feet tall, and in which direction? (For full credit, do NOT leave unevaluated trig expressions in the answer.)

$\frac{d\theta}{dt} = -\frac{6}{d^2}\frac{d^2}{6^2+d^2}\frac{dd}{dt}$
When $h = 18$, $d = 10$, and $\frac{dd}{dt} = 10$ always, so $\frac{d\theta}{dt} = -\frac{6}{6^2+10^2}10$

3. (*20 points*) Let a be a constant. Approximate the value of $e^{0.1\cdot a}$ using a linearization.

Let $f(x) = e^{x\cdot a}$.
Then $f'(x) = ae^{x\cdot a}$
Then $f(x) \approx e^{0\cdot a} + ae^{0\cdot a}(x - 0) = 1 + ax$
So $f(0.1) \approx 1 + 0.1a = 1 + \frac{a}{10}$

4. (*20 points*) (**a**) Consider the function $f(x) = x^{2/3}$, so $f(-1) = f(1) = 1$, but there is no point where $f'(x) = 0$. Why doesn't this contradict the Mean Value Theorem?

Because f is not differentiable on the interval $(-1, 1)$, so the assumptions of the Mean Value Theorem are not satisfied.

(**b**) Sketch an example of a function which is defined on the interval $[0, 1]$ and has both a local minimum and a local maximum in this interval, but has neither a global minimum nor a global maximum on this interval. (You don't need to be able to write down a formula for the function—you can make up any function as long as you can represent it in a sketch.)

The important point is that, if the function is continuous, the extreme value theorem would guarantee global minima and maxima. So the function must be discontinuous. For one of many possible examples, the function

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1/2 \\ 0 & \text{if } 1/2 \leq x < 3/4 \\ 3/4 - x & \text{if } 1/2 \leq x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

This function has a local minimum at 0, and then a line segment reaching towards $(1/2, 1/2)$, but instead of achieving a global maximum at $x = 1/2$, the function jumps to $(1/2, 0)$, and then has a flat line segment (on which every point except the endpoint is both a local maximum and a local minimum). At $x = 3/4$, the function starts heading on a line towards $(1, -1/4)$, but instead of achieving a global minimum at $x = 1$ it again jumps back to $(1, 0)$. As a result, it has neither a global maximum nor a global minimum.

5. (*20 points*) Sketch a graph of the function

$$f(x) = \frac{x+1}{x^2 - 2x + 1}$$

Be sure to mark, and give the x-coordinates of, all inflection points, roots, minima, and maxima. Indicate all horizontal, vertical, and oblique asymptotes.

Roots when $0 = f(x) = \frac{x+1}{x^2-2x+1}$, so the only root is $x = -1$
$f(x)$ is undefined at 1
$f(x)$ is negative on $(-\infty, -1)$ and positive on $(-1, 1)$ and $(1, \infty)$
At the undefined point $x = 1$, we must check for a vertical asymptote. $\lim_{x\to 1^-} \frac{x+1}{x^2-2x+1}$ is 2 divided by a small positive number, so $\lim_{x\to 1^-} \frac{x+1}{x^2-2x+1} = \infty$. By exactly the same reasoning (since the denominator is always negative), $\lim_{x\to 1^+} \frac{x+1}{x^2-2x+1} = \infty$. So $x = 1$ is a vertical asymptote, going to positive ∞ from both sides.

$f'(x) = \frac{(x^2-2x+1)-(2x-2)(x+1)}{(x^2-2x+1)^2} = \frac{(x-1)^2-2(x-1)(x+1)}{(x-1)^4} = \frac{x-1-2(x+1)}{(x-1)^3} = -\frac{x+3}{(x-1)^3}$
Critical points are -3 (where $f'(x) = 0$) and 1 (where $f'(x)$ is undefined)
$f'(x)$ is negative on $(-\infty, -3)$, positive on $(-3, 1)$, and negative on $(1, \infty)$.
So f is decreasing on $(-\infty, -3)$, increasing on $(-3, 1)$, and decreasing on $(1, \infty)$

$f''(x) = -\frac{(x-1)^3-3(x-1)^2(x+3)}{(x-1)^6} = -\frac{(x-1)-3(x+3)}{(x-1)^4} = \frac{2x+10}{(x-1)^4}$
$f''(x) = 0$ when $x = -5$, and $f''(x)$ is undefined when $x = 1$
f'' is negative on $(-\infty, -5)$, positive on $(-5, 1)$ and $(1, \infty)$
So f is concave down on $(-\infty, -5)$ and concave up on $(-5, 1)$ and $(1, \infty)$

Finally, we check for horizontal or oblique asymptotes. $\lim_{x\to\infty} \frac{x+1}{x^2-2x+1} = 0$ since the denominator has higher degree than the top; similarly, $\lim_{x\to-\infty} \frac{x+1}{x^2-2x+1} = 0$. So $y = 0$ is the only horizontal asymptote (and therefore there can be no oblique asymptotes).