Math 3A, Fall 2010 — Homework 2 [due Oct 4th] — **Solutions**

SECTION 2.2

3. Determine the values of the sequence $\{a_n\}$, $a_n = \frac{1}{n+2}$, for $n = 0, 1, 2, 3, 4, 5$.

Solution.

n	0	1	2	3	4	5
a_n	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$

4*. Determine the values of the sequence $\{a_n\}$, $a_n = \frac{1}{1+n^2}$, for $n = 0, 1, 2, 3, 4, 5$.

Solution.

$a_0 = \frac{1}{1+0^2} = \frac{1}{1+0} = \frac{1}{1} = 1, \quad a_1 = \frac{1}{1+1^2} = \frac{1}{1+1} = \frac{1}{2}, \quad a_2 = \frac{1}{1+2^2} = \frac{1}{1+4} = \frac{1}{5},$
$a_3 = \frac{1}{1+3^2} = \frac{1}{1+9} = \frac{1}{10}, \quad a_4 = \frac{1}{1+4^2} = \frac{1}{1+16} = \frac{1}{17}, \quad a_5 = \frac{1}{1+5^2} = \frac{1}{1+25} = \frac{1}{26}$

n	0	1	2	3	4	5
a_n	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$	$\frac{1}{26}$

31. Find an expression for a_n on the basis of the values of a_0, a_1, a_2,...

$$-1, 2, -3, 4, -5, \ldots$$

Solution.

The signs are alternating, so each term will be a positive number multiplied by $(-1)^{n+1}$ [or $(-1)^{n-1}$, which is the same]. Absolute values of the terms are $1, 2, 3, 4, 5, \ldots$, and since the sequence starts with 0th term, we recognize them as $|a_n| = n + 1$. Thus, the formula is $a_n = (-1)^{n+1}(n + 1)$, and it is easy to verify it for $n = 0, 1, \ldots, 5$.

32*. Find an expression for a_n on the basis of the values of a_0, a_1, a_2,...

$$2, -4, 6, -8, 10 \ldots$$

Solution.

The signs are alternating, so each term will be a positive number multiplied by $(-1)^n$. Absolute values of the terms are $2, 4, 6, 8, 10 \ldots$, and since the sequence starts with 0th term, we recognize them as $|a_n| = 2n+2$. Thus, the formula is $a_n = (-1)^n(2n+2)$, and it is easy to verify it for $n = 0, 1, \ldots, 5$.

41. Write the first five terms of the sequence $\{a_n\}$, $a_n = \frac{1}{n^2+1}$, and find $\lim\limits_{n\to\infty} a_n$.

Solution.

n	0	1	2	3	4
a_n	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$

It seems that the sequence converges to 0, i.e. $\lim_{n\to\infty} a_n = 0$.

If we wanted to show it rigorously, we would use the limit laws and write:

$$\lim_{n\to\infty} \frac{1}{n^2+1} = \lim_{n\to\infty} \frac{1}{n^2+1} \div \frac{n^2}{n^2} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{1+\frac{1}{n^2}} = \frac{\lim\limits_{n\to\infty} \frac{1}{n^2}}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n^2}}$$

$$= \frac{\lim\limits_{n\to\infty} \left(\frac{1}{n} \cdot \frac{1}{n}\right)}{1 + \lim\limits_{n\to\infty} \left(\frac{1}{n} \cdot \frac{1}{n}\right)} = \frac{\lim\limits_{n\to\infty} \frac{1}{n} \cdot \lim\limits_{n\to\infty} \frac{1}{n}}{1 + \lim\limits_{n\to\infty} \frac{1}{n} \cdot \lim\limits_{n\to\infty} \frac{1}{n}} = \frac{0 \cdot 0}{1 + 0 \cdot 0} = 0$$

42*. Write the first five terms of the sequence $\{a_n\}$, $a_n = \frac{1}{\sqrt{n+1}}$, and find $\lim_{n\to\infty} a_n$.

Solution.

n	0	1	2	3	4
a_n	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{5}}$

It seems that the sequence converges to 0, i.e. $\lim_{n\to\infty} a_n = 0$.

If we wanted to show it rigorously, we would use the limit laws and write:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{1}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{1}{n+1} \div \frac{n}{n}}$$

$$= \sqrt{\lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n}}} = \sqrt{\frac{\lim\limits_{n\to\infty} \frac{1}{n}}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n}}} = \sqrt{\frac{0}{1+0}} = \sqrt{0} = 0$$

Above we used the property $\lim_{n\to\infty} \sqrt{b_n} = \sqrt{\lim_{n\to\infty} b_n}$, which we didn't learn yet.

47. Write the first five terms of the sequence $\{a_n\}$, $a_n = \sqrt{n}$, and determine whether $\lim_{n\to\infty} a_n$ exists. If the limit exists, find it.

Solution.

n	0	1	2	3	4
a_n	0	1	$\sqrt{2}$	$\sqrt{3}$	2

It seems that the sequence "diverges to ∞", i.e. the limit $\lim_{n\to\infty} a_n$ does not exist.

48*. Write the first five terms of the sequence $\{a_n\}$, $a_n = n^2$, and determine whether $\lim_{n\to\infty} a_n$ exists. If the limit exists, find it.

Solution.

n	0	1	2	3	4
a_n	0	1	4	9	16

It seems that the sequence "diverges to ∞", i.e. the limit $\lim_{n\to\infty} a_n$ does not exist.

75. Use the limit laws to determine the limit.

$$\lim_{n\to\infty} \frac{n^2+1}{n^2}$$

Solution.

$$\lim_{n\to\infty} \frac{n^2+1}{n^2} = \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right) = \lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n^2}$$
$$= 1 + \lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1}{n}\right) = 1 + \lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{1}{n} = 1+0\cdot 0 = 1$$

76*. Use the limit laws to determine the limit.

$$\lim_{n\to\infty} \frac{3n^2-5}{n^2}$$

Solution.

$$\lim_{n\to\infty} \frac{3n^2-5}{n^2} = \lim_{n\to\infty}\left(3-\frac{5}{n^2}\right) = \lim_{n\to\infty} 3 - \lim_{n\to\infty}\frac{5}{n^2} = 3-5\lim_{n\to\infty}\frac{1}{n^2}$$
$$= 3-5\lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1}{n}\right) = 3-5\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{1}{n} = 3-5\cdot 0\cdot 0 = 3$$

V.K.