Math 3A Homework 3 Solution (Fall 2010)

Section 3.1

9. $f(x) = e^{-x^2/2}$

x	-2-0.1	-2-0.01	-2-0.001	-2+0.001	-2+0.01	-2+0.1
f(x)	0.1102505	0.1326488	0.1350648	0.1356061	0.1380623	0.1644744

Since $f(-2) = e^{-(-2)^2/2} = e^{-2} \approx 0.1353352$, from the table, we can see

$$\lim_{x\to-2} e^{-x^2/2} = e^{-2}.$$

10*. $f(x) = \frac{e^x+1}{2x+3}$

x	-0.1	-0.01	-0.001	+0.001	+0.01	+0.1
f(x)	0.680299	0.6678019	0.666778	0.6665558	0.6655795	0.6578659

Since $f(0) = \frac{e^0+1}{0+3} = \frac{2}{3} \approx 0.6667$, from the table, we can see

$$\lim_{x\to0} \frac{e^x+1}{2x+3} = \frac{2}{3}.$$

21. $f(x) = \frac{2}{x-4}$

x	4-0.1	4-0.01	4-0.001	4-0.0001	4-0.00001
f(x)	-20	-200	-2000	-20000	-200000

From the table, we can see

$$\lim_{x\to4-} \frac{2}{x-4} = -\infty.$$

22*. $f(x) = \frac{1}{x-3}$

x	3+0.00001	3+0.0001	3+0.001	3+0.01	3+0.1
f(x)	100000	10000	1000	100	10

From the table, we can see

$$\lim_{x\to3+} \frac{1}{x-3} = \infty.$$

47. $f(x) = \frac{1-x^2}{1-x}$ is a rational function, but since

$$\lim_{x\to1}(1-x) = 0,$$

we cannot use Rule 4. Hence using the similar method in Example 15 of Section 3.1 gives

$$\lim_{x\to 1}\frac{1-x^2}{1-x}=\lim_{x\to 1}\frac{(1-x)(1+x)}{1-x}=\lim_{x\to 1}(1+x)=1+1=2$$

48*. $f(u)=\frac{9-u^2}{3-u}$ is a rational function, but since

$$\lim_{u\to 3}(3-u)=0,$$

we cannot use Rule 4. Hence using the similar method in Example 15 of Section 3.1 gives

$$\lim_{u\to 3}\frac{9-u^2}{3-u}=\lim_{u\to 3}\frac{(3-u)(3+u)}{3-u}=\lim_{u\to 3}(3+u)=3+3=6$$

Section 3.2

To check whether a function is continuous at $x=c$, we need to check the following three conditions:

1. $f(x)$ is defined at $x=c$.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x)=f(c)$

11. First it is easy to see that the domain **D** of

$$f(x)=\begin{cases}\frac{x^2-3x+2}{x-2} & \text{if } x\neq 1\\ 1 & \text{if } x=1\end{cases}$$

is $\mathbf{D}=\{x\in\mathbb{R}: x\neq 2\}$.

So from Condition 1, $x=2$ is a discontinuity of $f(x)$.

Next, $\lim_{x\to 1} f(x)=\frac{1^2-3\times 1+2}{1-2}=0\neq 1=f(1)$ implies that $f(x)$ doesn't satisfy Condition 3.

In sum, the discontinuities of $f(x)$ are $x=1$ and $x=2$.

12*. It is easy to see that

$$f(x)=\begin{cases}x^2-1 & \text{if } x\leq 0\\ x & \text{if } x>0\end{cases}$$

is continuous when $x>0$ or $x<0$. But at the break point $x=0$, we have One-Sided limits:

$$\lim_{x\to 0+} f(x)=\lim_{x\to 0+} x=0$$

and

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (x^2 - 1) = -1,$$

which imply that $\lim_{x\to 0} f(x)$ doesn't exist.

Then $f(x)$ has only one discontinuity at $x = 0$.

23. First find the domain $\mathbf{D}$ of $f(x) = \tan(2\pi x)$.
Since $f(x) = \tan(2\pi x) = \frac{\sin(2\pi x)}{\cos(2\pi x)}$,
$\mathbf{D} = \{\cos(2\pi x) \neq 0\} = \{2\pi x \neq k\pi + \frac{\pi}{2}, k \in \mathbb{Z}\} = \{x \in \mathbb{R} : x \neq \frac{k}{2} + \frac{1}{4}, k \in \mathbb{Z}\}$.
Moreover, $f(x)$ is a trigonometric function, then it is continuous for all $x \in \mathbf{D}$.

24*. First find the domain $\mathbf{D}$ of $f(x) = \sin(\frac{2x}{3+x})$.
$\mathbf{D} = \{3 + x \neq 0\} = \{x \in \mathbb{R} : x \neq -3\}$.
Moreover, $f(x)$ is a composition of a trigonometric function and a rational function, then it is continuous for all $x \in \mathbf{D}$.

25(b). Obviously

$$f(x) = \begin{cases} x^2 + 2 & \text{if } x \leq 0 \\ x + c & \text{if } x > 0 \end{cases}$$

is continuous for all $x > 0$ or $x < 0$.

At $x = 0$, we have

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (x + c) = c$$

and

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x^2 + 2 = 2.$$

If $f(x)$ is continuous at $x = 0$, we should have

$$c = \lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = f(0) = 2.$$

Then if $c = 2$, $f(x)$ is continuous for all reals.

26(b)*. Obviously

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \geq 1, \\ 2x + c & \text{if } x < 1. \end{cases}$$

is continuous for all $x > 1$ or $x < 1$.

At $x = 1$, we have

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} (1/x) = 1$$

and

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (2x + c) = 2 + c.$$

If $f(x)$ is continuous at $x = 1$, we should have

$$2 + c = \lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(1) = 1,$$

which implies $c = -1$.

Then if $c = -1$, $f(x)$ is continuous for all reals.

41. Since e^x and e^{2x} are both exponential functions, and thus continuous, then

$$\lim_{x \to 0}(e^x - 1) = 0,$$

so we cannot use Rule 4. But using the similar method in Example 15 of Section 3.1 gives

$$\lim_{x \to 0} \frac{e^{2x} - 1}{e^x - 1} = \lim_{x \to 0} \frac{(e^x - 1)(e^x + 1)}{e^x - 1} = \lim_{x \to 0}(e^x + 1) = e^0 + 1 = 2$$

42*. Since e^x and e^{-x} are both exponential functions, and thus continuous, then applying Limit Laws gives

$$\lim_{x \to 0} \frac{e^{-x} - e^x}{e^{-x} + 1} = \frac{\lim_{x \to 0} e^{-x} - \lim_{x \to 0} e^x}{\lim_{x \to 0} e^{-x} + \lim_{x \to 0} 1} = \frac{e^0 - e^0}{e^0 + 1} = \frac{1 - 1}{1 + 1} = 0$$

Section 3.3

17. If we take $t = -x$, $x \to -\infty$ is equivalent to $t \to \infty$, then the original limit becomes

$$\lim_{x \to -\infty} \exp[x] = \lim_{t \to \infty} e^{-t} = 0.$$

18*. From the definition of the logarithmic functions, we know $e^{\ln x} = x$. Then

$$\lim_{x \to \infty} \exp[-\ln x] = \lim_{x \to \infty} \frac{1}{\exp[\ln x]} = \lim_{x \to \infty} \frac{1}{x} = 0$$

19.

$$\lim_{x \to \infty} \frac{3e^{2x}}{2e^{2x} - e^x} = \lim_{x \to \infty} \frac{3}{2 - e^{-x}} = \frac{3}{2 - 0} = \frac{3}{2}$$

20*.

$$\lim_{x \to \infty} \frac{3e^{2x}}{2e^{2x} - e^{3x}} = \lim_{x \to \infty} \frac{3e^{-x}}{2e^{-x} - 1} = \frac{0}{0 - 1} = 0$$

Additional Question*: Give examples of two functions, $f(x)$ and $g(x)$, such that $\lim_{x \to 1} f(x)$ does not exist, and $\lim_{x \to 1} g(x)$ does not exist, but $\lim_{x\to 1} \frac{f(x)}{g(x)} = 0$.*

One of solutions: Take $f(x) = \frac{1}{1-x}$ and $g(x) = \frac{1}{(1-x)^2}$. Then $\lim_{x\to 1} f(x)$ and $\lim_{x\to 1} g(x)$ do not exist, since they are both ∞.

But, $\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1}(1-x) = 0$.