Math 3A, Fall 2010 — **Homework 5** [due Nov 1st] — **Solutions**

Section 4.4

3. Differentiate the function with respect to the independent variable.
   $f(x) = (1 - 3x^2)^4$

   Solution. One can decompose f as the composition $f(x) = f_2(f_1(x))$, where $f_1(x) = 1 - 3x^2$ and $f_2(y) = y^4$. Using the chain rule we get:

   $$f'(x) = f_2'(f_1(x))f_1'(x) = 4(1 - 3x^2)^3(-6x)$$

   or after simplification

   $$f'(x) = -24x(1 - 3x^2)^3$$

4*. Differentiate the function with respect to the independent variable.
   $f(x) = (5x^2 - 3x)^3$

   Solution. One can decompose f as the composition $f(x) = f_2(f_1(x))$, where $f_1(x) = 5x^2 - 3x$ and $f_2(y) = y^3$. Using the chain rule we get:

   $$f'(x) = f_2'(f_1(x))f_1'(x) = 3(5x^2 - 3x)^2(10x - 3)$$

15. Differentiate the function with respect to the independent variable.
   $f(s) = \sqrt{s + \sqrt{s}}$

   Solution. One can decompose f as the composition $f(s) = f_2(f_1(s))$, where $f_1(s) = s + \sqrt{s} = s + s^{1/2}$ and $f_2(t) = \sqrt{t} = t^{1/2}$. Using the chain rule we get:

   $$f'(s) = f_2'(f_1(s))f_1'(s) = \frac{1}{2}(s + s^{1/2})^{-1/2}(1 + \frac{1}{2}s^{-1/2})$$

   or after simplification

   $$f'(s) = \frac{2\sqrt{s} + 1}{4\sqrt{s}\sqrt{s + \sqrt{s}}}$$

16*. Differentiate the function with respect to the independent variable.
   $g(t) = \sqrt{t^2 + \sqrt{t + 1}}$

   Solution. One can decompose g as the composition $g(t) = g_2(g_1(t))$, where $g_1(t) = t^2 + \sqrt{t + 1} = t^2 + (t + 1)^{1/2}$ and $g_2(u) = \sqrt{u} = u^{1/2}$. Using the chain rule we get:

   $$g'(t) = g_2'(g_1(t))g_1'(t) = \frac{1}{2}(t^2 + (t + 1)^{1/2})^{-1/2}(2t + \frac{1}{2}(t + 1)^{-1/2})$$

   or after simplification

   $$g'(t) = \frac{4t\sqrt{t + 1} + 1}{4\sqrt{t + 1}\sqrt{t^2 + \sqrt{t + 1}}}$$

61. Assume that x and y are differentiable functions of t. Find $\frac{dy}{dt}$ when $x^2+y^2=1$, $\frac{dx}{dt}=2$ for $x=\frac{1}{2}$, and $y>0$.

Solution. Observe that $x=\frac{1}{2}$ and $y>0$ together with the equation give $(\frac{1}{2})^2+y^2=1$, i.e. $y=\frac{\sqrt{3}}{2}$. Differentiating the equation $x^2+y^2=1$ with respect to t and using the chain rule we obtain

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

Plugging in $x=\frac{1}{2}$, $y=\frac{\sqrt{3}}{2}$, $\frac{dx}{dt}=2$ we get

$$2\cdot\frac{1}{2}\cdot 2+2\cdot\frac{\sqrt{3}}{2}\cdot\frac{dy}{dt}=0$$

and thus $\frac{dy}{dt}=-\frac{2}{\sqrt{3}}=-\frac{2\sqrt{3}}{3}$.

62*. Assume that x and y are differentiable functions of t. Find $\frac{dy}{dt}$ when $y^2=x^2-x^4$, $\frac{dx}{dt}=1$ for $x=\frac{1}{2}$, and $y>0$.

Solution. Observe that $x=\frac{1}{2}$ and $y>0$ together with the equation give $y^2=(\frac{1}{2})^2-(\frac{1}{2})^4=\frac{3}{16}$, i.e. $y=\frac{\sqrt{3}}{4}$. Differentiating the equation $y^2=x^2-x^4$ with respect to t and using the chain rule we obtain

$$2y\frac{dy}{dt}=2x\frac{dx}{dt}-4x^3\frac{dx}{dt}$$

Plugging in $x=\frac{1}{2}$, $y=\frac{\sqrt{3}}{4}$, $\frac{dx}{dt}=1$ we get

$$2\cdot\frac{\sqrt{3}}{4}\cdot\frac{dy}{dt}=2\cdot\frac{1}{2}\cdot 1-4\cdot\left(\frac{1}{2}\right)^3\cdot 1$$

i.e.

$$\frac{\sqrt{3}}{2}\cdot\frac{dy}{dt}=\frac{1}{2}$$

and thus $\frac{dy}{dt}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$.

69. Suppose that water is stored in a cylindrical tank of radius 5m. If the height of the water in the tank is h, then the volume of the water is $V=\pi r^2h=25\pi h\mathrm{m}^2$. If we drain the water at a rate of 250 liters per minute, what is the rate at which the water level inside the tank drops?

Solution. Differentiating $V=25\pi h$ with respect to t we get

$$\frac{dV}{dt}=25\pi\frac{dh}{dt}$$

Since $\frac{dV}{dt}=250\frac{\mathrm{l}}{\mathrm{min}}=0.25\frac{\mathrm{m}^3}{\mathrm{min}}$ we get from above

$$\frac{dh}{dt}=\frac{0.25}{25\pi}=\frac{1}{100\pi}$$

and so the rate is $\frac{1}{100\pi}\frac{\mathrm{m}}{\mathrm{min}}$ [meters per minute].

70*. Suppose that we pump water into an inverted right circular conical tank at the rate of 5 cubic feet per minute. The tank has height of 6 ft and the radius on top is 3 ft. What is the rate at which the water level is rising when the water is 2 ft deep? (Note that the volume of a right circular cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

Solution. When the water level is h and the radius on top of the water level is r, we can equal the proportions

$$\frac{h}{6} = \frac{r}{3}$$

to get $r = \frac{1}{2}h$. Therefore the formula becomes

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{\pi}{12}h^3$$

Differentiating with respect to t gives

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

Since $\frac{dV}{dt} = 5$ and we are interested in the moment when $h = 2$, we finally obtain

$$\frac{dh}{dt} = \frac{5}{(\pi/4)2^2} = \frac{5}{\pi}$$

and so the rate is $\frac{5}{\pi}\frac{\text{ft}}{\text{min}}$.

75. Find the first and the second derivative of the function.
$g(x) = \frac{x-1}{x+1}$

Solution. Using the quotient rule we get:

$$g'(x) = \frac{1 \cdot (x+1) - (x-1) \cdot 1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

and once again:

$$g''(x) = \frac{0 \cdot (x+1)^2 - 2 \cdot 2(x+1)}{(x+1)^4} = \frac{-4}{(x+1)^3}$$

76*. Find the first and the second derivative of the function.
$h(s) = \frac{1}{s^2+2}$

Solution. Using the quotient rule we get:

$$h'(s) = \frac{0 \cdot (s^2+2) - 1 \cdot 2s}{(s^2+2)^2} = \frac{-2s}{(s^2+2)^2}$$

and once again, together with the chain rule for $((s^2+2)^2)' = 2(s^2+2) \cdot 2s$:

$$h''(s) = \frac{(-2) \cdot (s^2+2)^2 - (-2s) \cdot 2(s^2+2) \cdot 2s}{(s^2+2)^4} = \frac{6s^2-4}{(s^2+2)^3}$$

83. Find the first 10 derivatives of $y = x^5$.

Solution.

$$
\begin{aligned}
y' &= 5x^4 \\
y'' &= 5 \cdot 4 \cdot x^3 = 20x^3 \\
y^{(3)} &= 5 \cdot 4 \cdot 3 \cdot x^2 = 60x^2 \\
y^{(4)} &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot x = 120x \\
y^{(5)} &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\
y^{(6)} &= 0 \\
y^{(7)} &= 0 \\
y^{(8)} &= 0 \\
y^{(9)} &= 0 \\
y^{(10)} &= 0
\end{aligned}
$$

84*. Find $f^{(n)}(x)$ and $f^{(n+1)}(x)$ of $f(x) = x^n$.

Solution.

$$
\begin{aligned}
f'(x) &= nx^{n-1} \\
f''(x) &= n(n-1)x^{n-2} \\
f^{(3)}(x) &= n(n-1)(n-2)x^{n-3} \\
\cdots & \quad \cdots \\
f^{(n)}(x) &= n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1 \cdot x^0 = n! \\
f^{(n+1)}(x) &= 0
\end{aligned}
$$

Section 4.5

5. Find the derivative with respect to the independent variable.
$f(x) = \tan x - \cot x$

Solution.

$$f'(x) = \sec^2 x - (-\csc^2 x) = \sec^2 x + \csc^2 x$$

6*. Find the derivative with respect to the independent variable.
$f(x) = \sec x - \csc x$

Solution.

$$f'(x) = \sec x \tan x - (-\csc x \cot x) = \sec x \tan x + \csc x \cot x$$

47. Find the derivative with respect to the independent variable.
$g(x) = \frac{1}{\csc^3(1-5x^2)}$

Solution. It is convenient to use $\csc\theta = \frac{1}{\sin\theta}$ and rewrite the function as

$$g(x) = \sin^3(1-5x^2) = \Big(\sin(1-5x^2)\Big)^3$$

Now we apply the chain rule:

$$g'(x) = 3\Big(\sin(1-5x^2)\Big)^2(-10x) = -30x\sin^2(1-5x^2)$$

48*. Find the derivative with respect to the independent variable.
$h(x) = \cot(3x)\csc(3x)$

Solution. We apply the product rule:

$$h'(x) = -\csc^2(3x)\cdot 3\cdot\csc(3x) + \cot(3x)\cdot(-\csc(3x)\cot(3x))\cdot 3$$

and this result can be simplified as

$$h'(x) = -3\csc^3(3x) - 3\cot^2(3x)\csc(3x)$$

63. Use the quotient rule to show that

$$\frac{d}{dx}\sec x = \sec x\tan x$$

Solution.

$$\begin{aligned}\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{0\cdot\cos x - 1\cdot(-\sin x)}{\cos^2 x}\\ &= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x\tan x\end{aligned}$$

64*. Use the quotient rule to show that

$$\frac{d}{dx}\csc x = -\csc x\cot x$$

Solution.

$$\begin{aligned}\frac{d}{dx}\csc x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{0\cdot\sin x - 1\cdot\cos x}{\sin^2 x}\\ &= \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} = -\csc x\cot x\end{aligned}$$

Section 4.6

1. Differentiate the function with respect to the independent variable.
$f(x) = e^{3x}$

Solution. Using the chain rule we get:

$$f'(x) = e^{3x} \cdot 3 = 3e^{3x}$$

2*. Differentiate the function with respect to the independent variable.
$f(x) = e^{-2x}$

Solution. Using the chain rule we get:

$$f'(x) = e^{-2x} \cdot (-2) = -2e^{-2x}$$

45. Differentiate the function with respect to the independent variable.
$f(x) = 2^{\sqrt{x^2-1}}$

Solution. One can decompose f as the composition $f(x) = f_3(f_2(f_1(x)))$, where $f_1(x) = x^2 - 1$, $f_2(y) = \sqrt{y}$, $f_3(z) = 2^z$. Using the chain rule we get:

$$f'(x) = f_3'(f_2(f_1(x)))f_2'(f_1(x))f_1'(x) = 2^{\sqrt{x^2-1}} \ln 2 \cdot \frac{1}{2}(x^2-1)^{-1/2} \cdot 2x$$

or after simplification

$$f'(x) = \frac{x\, 2^{\sqrt{x^2-1}} \ln 2}{\sqrt{x^2-1}}$$

46*. Differentiate the function with respect to the independent variable.
$f(x) = 4^{\sqrt{1-2x^3}}$

Solution. One can decompose f as the composition $f(x) = f_3(f_2(f_1(x)))$, where $f_1(x) = 1 - 2x^3$, $f_2(y) = \sqrt{y}$, $f_3(z) = 4^z$. Using the chain rule we get:

$$f'(x) = f_3'(f_2(f_1(x)))f_2'(f_1(x))f_1'(x) = 4^{\sqrt{1-2x^3}} \ln 4 \cdot \frac{1}{2}(1-2x^3)^{-1/2} \cdot (-6x^2)$$

or after simplification

$$f'(x) = \frac{-3x^2\, 4^{\sqrt{1-2x^3}} \ln 4}{\sqrt{1-2x^3}}$$

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