Suggested practice problems from recent sections:

- 10.4: 3, 4, 7, 8, 13, 14, 19, 20
- 10.5: 17, 18, 27, 28, 35, 36
- 10.6: 1, 2, 3, 4, 40, 41, 42, 43

a

Approximate $\int_1^7 e^x dx$ as a Riemann sum with 3 equal intervals, choosing the left endpoint of each rectangle to be its height.

Solution: $2(e + e^3 + e^5)$

b

Give the general solution of the differential equation

$$\frac{dy}{dt} = \frac{e^y - 1}{e^y} t^2.$$

Solution: If $y = 0$, the right side is 0, so $y = 0$ is a solution. If $y \neq 0$, we have

$$\frac{e^y}{e^y - 1} dy = t^2 dt.$$

Integrating both sides (and using $u = e^y$ to integrate the left), we have

$$\ln |e^y - 1| = \frac{t^3}{3} + C.$$

Therefore

$$|e^y - 1| = e^{\frac{t^3}{3} + C}$$

and so

$$e^y - 1 = e^{\frac{t^3}{3} + C} \text{ or } e^y = -e^{\frac{t^3}{3} + C}.$$

Solving for y, we get

$$y = \ln\left(e^{\frac{t^3}{3} + C} + 1\right) \text{ or } y = \ln\left(-e^{\frac{t^3}{3} + C} + 1\right) \text{ or } y = 0.$$

c

Recall that $\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt$. Show that when $y > 0$,

$$\arcsin \sqrt{1 - \frac{1}{y^2}} \leq y\sqrt{1 - \frac{1}{y^2}}.$$

Solution: $\arcsin \sqrt{1 - \frac{1}{y^2}} = \int_0^{\sqrt{1-\frac{1}{y^2}}} \frac{1}{\sqrt{1-t^2}} dt$. Since $\frac{1}{\sqrt{1-t^2}}$ is increasing, the area under the curve $\frac{1}{\sqrt{1-t^2}}$ as t goes from 0 to $\sqrt{1 - \frac{1}{y^2}}$ is contained inside the rectangle with corners $(0, 0)$ and $(\sqrt{1 - \frac{1}{y^2}}, y)$. This box has area $y\sqrt{1 - \frac{1}{y^2}}$.

d

You know that $2 \leq f(x) \leq 3$ for all x. Is it possible that $\int_2^5 f(x)dx = 4$?

Solution: No. $\int_0^3 f(x)dx$ contains the rectangle with corners $(2,0)$ and $(5,2)$, which has area 6, so $6 \leq \int_0^3 f(x)dx$.

e

What is $\int_{-1}^{-1} \frac{\cos x}{x} dx$?

Solution: 0, since the bounds are equal.

f

Find a value $a > 0$ such that $\int_1^a \frac{\sin(x-2)}{(x-2)^2} dx = 0$.

Solution: We can't find the indefinite integral, so we must use geometry. This function is anti-symmetric around 2: $\frac{\sin(2-c-2)}{(2-c-2)} = \frac{-\sin c}{c^2} = -\frac{\sin(2+c-2)}{(2+c-2)^2}$. Therefore we need 1 and a to be symmetric around 2, so $a = 3$.

g

Define $F(x) = \int_0^x \frac{\sin t}{t} dt$. What is $\frac{d}{dx} F(\ln x)$?

Solution: By FTC, $F'(x) = \frac{\sin x}{x}$, so by the chain rule, $\frac{d}{dx} F(\ln x) = \frac{\sin \ln x}{x \ln x}$.

h

Water is flowing into a container at a rate of $W(t) gal/sec$ (where t is the time). Express the amount of water that enters the container between $t = 0$ and $t = 4$.

Solution: $\int_0^4 W(t)dt$.

i

What is the partial fraction decomposition of

$$\frac{1}{(x^2+4)^3(x^2+1)^2(x-1)^3(x+2)}$$

Solution:

$$\frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{Ex+F}{(x^2+4)^3} + \frac{Gx+H}{x^2+1} + \frac{Ix+J}{(x^2+1)^2} + \frac{K}{x-1} + \frac{L}{(x-1)^2} + \frac{M}{(x-1)^3} + \frac{N}{x+2}.$$

j

Find and solve the partial fraction decomposition for

$$\frac{1}{(x^2+1)(x^2-1)}$$

Solution

$$\begin{aligned}\frac{1}{x^4-1} &= \frac{1}{(x^2+1)(x+1)(x-1)} \\ &= \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}\end{aligned}$$

so we get

$$\begin{aligned}1 &= (Ax+B)(x^2-1) + C(x-1)(x^2+1) + D(x+1)(x^2+1) \\ &= Ax^3 - Ax + Bx^2 - B + Cx^3 + Cx - Cx^2 - C + Dx^3 + Dx + Dx^2 + D\end{aligned}$$

This give four equations:

$$\begin{aligned}0x^3 &= (A+C+D)x^3 \ (1) \\ 0x^2 &= (B-C+D)x^2 \ (2) \\ 0x &= (-A+C+D)x \ (3) \\ 1 &= -B-C+D \ (4)\end{aligned}$$

We combine these to get:

$$\begin{aligned}(1)+(2) :& 0 = A+B+2D && (5) \\ (2)+(3) :& 0 = -A+B+2D && (6) \\ (3)+(4) :& 1 = -A-B+2D && (7) \\ (5)-(6) :& 0 = 2A && \\ & A = 0 && (8) \\ (8) \text{ and } (6) :& 0 = B+2D && (9) \\ (8) \text{ and } (7) :& 1 = -B+2D && (10) \\ (9)-(10) :& -1 = 2B && \\ & B = -1/2 && (11) \\ (11) \text{ and } (10) :& 1 = 1/2+2D && (12) \\ & D = 1/4 && (13) \\ (13), (8), \text{ and } (1) :& 0 = 0+C+1/4 && \\ & C = -1/4 &&\end{aligned}$$

So

$$\frac{1}{x^4-1} = -\frac{1}{2(x^2+1)} - \frac{1}{4(x+1)} + \frac{1}{4(x-1)}$$

k

Integrate:

1. $\int x^2 \ln x^3 dx$

2. $\int \frac{x}{\sqrt{1-x^2}} dx$

3. $\int \arcsin x \ dx$

4. $\int \frac{1}{x^4-1} dx$

5. $\int \frac{1}{4x^2+8x+29} dx$

6. $\int_1^\infty \frac{\ln x}{x} dx$

7. $\int_1^{-\infty} e^x dx$

Solution:

1. $u = \ln x^3, dv = x^2 dx, du = 3x^2/x^3 = 3/x, v = x^3/3, \int x^2 \ln x^3 dx = (x^3/3)\ln x^3 - \int x^2 dx = (x^3/3)\ln x^3 - x^3/3$

2. $u = 1-x^2, du = -2x, \int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2}\int u^{-1/2} du = -\sqrt{u} = -\sqrt{1-x^2}$

3. $u = \arcsin x, dv = dx, du = \frac{1}{\sqrt{1-x^2}}, v = x, \int \arcsin x \ dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = x \arcsin x + \sqrt{1-x^2}$

4. $\int \frac{1}{x^4-1} dx = \int -\frac{1}{2(x^2+1)} - \frac{1}{4(x+1)} + \frac{1}{4(x-1)} dx = -\frac{1}{2}\arctan x - \frac{1}{4}\ln(x+1) + \frac{1}{4}\ln(x-1)$

5. Since $(2x+2)^2 = 4x^2+8x+4$, $\int \frac{1}{4x^2+8x+29} dx = \int \frac{1}{(2x+2)^2+25} dx = \frac{1}{25}\int \frac{1}{\frac{(2x+2)^2}{25}+1} dx = \frac{1}{25}\int \frac{1}{(\frac{2x+2}{25})^2+1} dx$. Substituting $u = \frac{2x+2}{5}, du = \frac{2}{5} dx$, we get $\int \frac{1}{4x^2+8x+29} dx = \frac{1}{10}\int \frac{1}{u^2+1} du = \frac{1}{10}\arctan u = \frac{1}{10}\arctan \frac{2x+2}{5}$.

6. $\int_1^\infty \frac{\ln x}{x} dx = \lim_{a\to\infty} \int_1^a \frac{\ln x}{x} dx$. $u = \ln x, du = dx/x$, so

$$\begin{aligned}\lim_{a\to\infty}\int_1^a \frac{\ln x}{x} dx &= \lim_{a\to\infty}\int_0^{\ln a} u du \\ &= \lim_{a\to\infty} u^2/2\Big|_0^{\ln a} \\ &= \lim_{a\to\infty} (\ln^2 a/2 - 0)\end{aligned}$$

   Since $\lim_{a\to\infty} \ln^2 a/2 = \infty$, this integral does not exist.

7. $\int_1^{-\infty} e^x dx = -\int_{-\infty}^1 e^x dx = \lim_{a\to\infty} -\int_{-a}^1 e^x dx = \lim_{a\to\infty} - e^x\Big|_{-a}^1 = \lim_{a\to\infty} e^{-a} - e = -e$.

1

Describe the domain, range, and level curves of $\ln(x^2+y^2-1)$.

Solution: $\ln u$ is undefined for $u \le 0$, so this is only defined when $x^2+y^2-1 > 0$, which is when $x^2+y^2 > 1$. The level curves are solutions $c = \ln(x^2+y^2-1)$, so $x^2+y^2 = e^c+1$, which are circles of radius $\sqrt{e^c+1}$.

m

Find the following partial derivatives:

1. $\frac{\partial}{\partial x}(x^3 + xy + \ln x)$
2. $\frac{\partial}{\partial y} e^{xe^{xy}}$
3. $\frac{\partial^2}{\partial x \partial y} e^{xe^{xy}}$
4. $\frac{\partial}{\partial y} \ln xy$
5. $\frac{\partial^3}{\partial y \partial x \partial y} e^{x^2y^2}$
6. $\frac{\partial}{\partial z} \ln(xy + xz + yz)$

Solution:

1. $3x^2 + y + 1/x$
2. $x^2 e^{xy} e^{xe^{xy}}$
3. $2xe^{xy}e^{xe^{xy}} + x^2ye^{xy}e^{xe^{xy}} + x^2e^{xy}e^{xe^{xy}}(e^{xy} + xye^{xy})$
4. $1/y$
5. $\frac{\partial}{\partial y} e^{x^2y^2} = 2x^2ye^{x^2y^2}$, $\frac{\partial^2}{\partial x \partial y} e^{x^2y^2} = 4xye^{x^2y^2} + 4x^3y^3e^{x^2y^2}$, $\frac{\partial^3}{\partial y \partial x \partial y} e^{x^2y^2} = 4xe^{x^2y^2} + 8x^3y^2e^{x^2y^2} + 12x^3y^2e^{x^2y^2} + 8x^5y^4e^{x^2y^2}$
6. $\frac{x+y}{xy+xz+yz}$

n

Indicate whether the following statements are *(A)lways True*, *(S)ometimes True*, or *(N)ever True*.

1. A function that is continuous at (x, y) is also differentiable at (x, y)
2. If f is differentiable at (x, y) then the partial derivative $\frac{\partial f}{\partial x}$ is exists at (x, y)
3. If f is differentiable and $\nabla f \neq 0$, ∇f is the direction in which f decreases most rapidly
4. If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at (x, y) then $\nabla f(x, y)$ is defined
5. If $f, f_x, f_y, f_{xy}, f_{yx}$ are both defined and continuous at (x, y) then the mixed partials are equal at (x, y)

Solutions:

1. (S): Some functions are continuous but not differentiable.

2. (A): if f is differentiable (x, y) then all directional derivatives, including the partial derivatives, exist at (x, y).

3. (N): If f is differentiable and $\nabla f \neq 0$ then ∇f is the direction in which f *increases* most rapidly.

4. (A): $\nabla f(x, y)$ is just the vector $\begin{bmatrix} \frac{\partial f}{\partial x}(x, y) \\ \frac{\partial f}{\partial y}(x, y) \end{bmatrix}$, which is defined if both its components are defined.

5. (S): We need these functions to be continuous in a ball containing (x, y), not just at (x, y), to be sure the mixed partials are equal

o

Find and classify all critical points of $x^3y - 4xy^3 + y$

Solution: $f_x = 3x^2y - 4y^3$ and $f_y = x^3 - 12xy^2 + 1$, so if $\nabla f = 0$ then

$$0 = 3x^2y - 4y^3, 0 = x^3 - 12xy^2 + 1.$$

The first equation is $0 = y(3x^2 - 4y^2)$, so either $y = 0$ or $3x^2 = 4y^2$. If $y = 0$, the second equation is $0 = x^3 + 1$, so one critical points is $(-1, 0)$. In the second case, we substitue $y^2 = \frac{3}{4}x^2$ into the second equation to get $0 = x^3 - 9x^3 + 1$ so $-1 = -8x^3$, so $x = 1/2$ is a solution. Since $y^2 = \frac{3}{4}x^2$ in this case, $(1/2, \pm\sqrt{3}/4)$ are two more critical points.

$f_{xx} = 6xy$, $f_{yy} = -24xy$, $f_{xy} = 3x^2 - 12y^2$, so

$$D = -144x^2y^2 - (3x^2 - 12y^2)^2.$$

This is always negative, so all three critical points are saddle points.

p

Find and classify all critical points of $e^{xy} - e^{2xy}$.

Solution: $f_x = ye^{xy} - 2ye^{2xy} = y(e^{xy} - 2e^{2xy})$ and $f_y = xe^{xy} - 2xe^{2xy} = x(e^{xy} - 2e^{2xy})$. If $\nabla f = 0$ then either $y = 0$ or $e^{xy} - 2e^{2xy} = 0$, and either $x = 0$ or $e^{xy} - 2e^{2xy} = 0$. So one critical point is at $(0, 0)$. When $y = 0$, $e^{xy} - 2e^{2xy} = e^0 - 2e^0 = -1 \neq 0$, so the other solutions are when $e^{xy} - 2e^{2xy} = 0$. This is equivalent to $e^{xy} = 2e^{2xy}$, and dividing both sides by e^{xy} gives $1 = 2e^{xy}$. Solving, we get $xy = -\ln 2$. So the critical points are $(0, 0)$ and the hyperbola $(x, -\ln 2/x)$.

To classify, we find $f_{xx} = y^2e^{xy} - 4y^2e^{2xy}$, $f_{yy} = x^2e^{xy} - 4x^2e^{2xy}$, and $f_{xy} = f_{yx} = e^{xy} - 2e^{2xy} + xye^{xy} - 4xye^{2xy}$, and so

$$D = (y^2e^{xy} - 4y^2e^{2xy})(x^2e^{xy} - 4x^2e^{2xy}) - (e^{xy} - 2e^{2xy} + xye^{xy} - 4xye^{2xy})^2.$$

At the origin, almost all these terms disappear, and we have

$$D(0,0) = -(e^0 - 2e^0)^2 < 0$$

so the origin is a saddle point.

On the hyperbola, we can simplify a bit. We have

$$D(x,y) = x^2y^2e^{2xy} - 8x^2y^2e^{3xy} + 16x^2y^2e^{4xy} - (e^{xy} - 2e^{2xy} + xye^{xy} - 4xye^{2xy})^2.$$

In particular, we never use x or y alone, just xy or $x^2y^2 = (xy)^2$. Note that $e^{-\ln 2} = e^{\ln 2^{-1}} = 1/2$ and that $e^{2xy} = (e^{xy})^2 = 1/4$, $e^{3xy} = 1/8$ and $e^{4xy} = 1/16$. So on the hyperbola we have

$$D(x,y) = (\ln^2 2)/4 - (\ln^2 2) + (\ln^2 2) - (1/2 - 1/2 - \ln 2/2 + \ln 2)^2 = (\ln^2 2)/4 + 2 - (\ln 2/2)^2 = 0$$

so the points on the hyperbola cannot be classified.

q

Find the candidates for where e^{xy} achieves its minimum on the circle $x^2+y^2 = 1$.

Solution: $[ye^{xy}, xe^{xy}]' = c[2x, 2y]'$, which gives the equations $ye^{xy} = 2cx$ and $xe^{xy} = 2cy$. Multiplying by y and x respectively, $y^2e^{xy} = 2cxy = x^2e^{xy}$. Since e^{xy} is never 0, $x^2 = y^2$, and therefore the candidates are $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$.

r

Find the candidates for where e^{xy} achieves its minimum on the hyperbola $x = 1/y$.

Solution: We solve the second equation to give $xy = 1$. But e^{xy} is constantly equal to e when $xy = 1$, so all points are maxima and minima! (If we used Lagrange multipliers, we'd find that $[ye^{xy}, xe^{xy}]' = c[y, x]'$, so $ye^{xy} = cy, xe^{xy} = cx$, and therefore $e^{xy} = c$ and $xy = 1$. Then, wtih $c = e$, this is true everywhere $xy = 1$.)

s

Find the candidates for where $x^2 + y^2$ achieves its minimum on the hyperbola $x = 1/y$.

Solution: We solve the second equation to give $xy = 1$. $[2x, 2y] = c[y, x]$ so $2x = cy, 2y = cx$, and so $2x^2 = cxy = 2y^2$, and therefore $x^2 = y^2$. Since also $x = 1/y$, $1/y^2 = y^2$, so $1 = y^4$, so $y = \pm 1$. Since $xy = 1$, the candidates are $(1, 1)$ and $(-1, -1)$.

t

1. Find and classify as stable or unstable the equilibria of

$$\frac{dy}{dt} = (y-3)(e^y - e).$$

2. y_0 is a solution with $y_0(0) = 0$. What is $\lim_{t\to\infty} y_0$?

3. y_1 is a solution with $y_1(0) = 1$. What is $\lim_{t\to\infty} y_1$?

4. y_2 is a solution with $y_2(0) = 2$. What is $\lim_{t\to\infty} y_2$?

5. y_3 is a solution with $y_3(0) = 3$. What is $\lim_{t\to\infty} y_3$?

6. y_4 is a solution with $y_4(0) = 4$. What is $\lim_{t\to\infty} y_4$?

Solution:

1. $g(y) = (y-3)(e^y - e)$. $g(y) = 0$ means $y = 3$ or $e^y = e$, which means $y = 1$. $g'(y) = (y-3)e^y + e^y - e$. $g'(3) = e^3 - e > 0$, so 3 is an unstable equilibrium. $g'(1) = -3e < 0$, so 1 is a stable equilibrium.

2. When $y_0(0) = 0$, $y' = g(0) > 0$, so y_0 is increasing towards the equilibrium, so $\lim_{t\to\infty} y_0 = 1$.

3. When $y_1(0) = 1$, $y' = g(1) = 0$, so y_1 is constantly equal to 1, so $\lim_{t\to\infty} y_1 = 1$.

4. When $y_2(0) = 2$, $y' = g(2) < 0$, so y_2 is decreasing towards the equilibrium, so $\lim_{t\to\infty} y_2 = 1$.

5. When $y_3(0) = 3$, $y' = g(3) = 0$, so y_3 is constantly equal to 3, so $\lim_{t\to\infty} y_3 = 3$.

6. When $y_4(0) = 4$, $y' = g(4) > 0$, so y_4 is increasing away from the equilibrium, so $\lim_{t\to\infty} y_0 = \infty$.

u

Give an example of an autonomous differential equation which has x^3 as a solution.

Solution: When $y = x^3$, $y' = 3x^2$. We must express y' as a function of y, which is easily done by setting $y' = 3y^{2/3}$.