# Math 3B Homework5 Solutions (Winter 2011)

**Section 7.3** Evaluate following integrals

14. $\int \frac{1}{x(2x+1)}dx$

**Solution** The integrand is a proper rational function whose denominator is a product of two distinct linear functions. We claim that the integrand can be written in the form

$$\begin{aligned}\frac{1}{x(2x+1)} &= \frac{A}{x}+\frac{B}{2x+1}\\ &= \frac{A(2x+1)+Bx}{x(2x+1)}\end{aligned}$$

where $A$ and $B$ are constants that we need to determine. Then we conclude that

$$1 = A(2x+1)+Bx = (2A+B)x+A,$$

which implies

$$\begin{cases} 2A+B=0 \\ A=1 \end{cases}$$

and thus

$$\begin{cases} A=1 \\ B=-2 \end{cases}$$

Therefore

$$\begin{aligned}\int \frac{1}{x(2x+1)}dx &= \int \frac{1}{x}-\frac{2}{2x+1}dx\\ &= \int \frac{1}{x}dx - \int \frac{1}{x+\frac{1}{2}}dx\\ &= \ln|x| - \ln\left|x+\frac{1}{2}\right| + C\\ &= \ln\left|\frac{x}{x+\frac{1}{2}}\right| + C\\ &= \ln\left|\frac{2x}{2x+1}\right| + C\end{aligned}$$

□

22. $\int \frac{3x^2+4x+3}{(x^2+1)^2}dx$

**Solution** The integrand is a proper rational function, since the numerator is a polynomial of degree 2 while the denominator is of degree 4. The denominator contains the irreducible quadratic factor $x^2 + 1$ twice, so we claim that the integrand can be written in the form

$$\begin{aligned} \frac{3x^2 + 4x + 3}{(x^2 + 1)^2} &= \frac{Ax + B}{(x^2 + 1)} + \frac{Cx + D}{(x^2 + 1)^2} \\ &= \frac{(Ax + B)(x^2 + 1) + Cx + D}{(x^2 + 1)^2} \\ &= \frac{Ax^3 + Bx^2 + (A + C)x + (B + D)}{(x^2 + 1)^2} \end{aligned}$$

where $A$, $B$, $C$ and $D$ are constants that we need to determine. Then we conclude that

$$3x^2 + 4x + 3 = Ax^3 + Bx^2 + (A + C)x + (B + D),$$

which implies

$$\begin{cases} A = 0 \\ B = 3 \\ A + C = 4 \\ B + D = 3 \end{cases}$$

and thus

$$\begin{cases} A = 0 \\ B = 3 \\ C = 4 \\ D = 0 \end{cases}$$

Therefore

$$\begin{aligned}
\int \frac{3x^2+4x+3}{(x^2+1)^2}dx &= \int \frac{3}{(x^2+1)} + \frac{4x}{(x^2+1)^2}dx \\
&= \int \frac{3}{(x^2+1)}dx + \int \frac{4x}{(x^2+1)^2}dx \\
&= 3\tan^{-1}x + 2\int \frac{1}{u^2}du \\
&= 3\tan^{-1}x - \frac{2}{u} + C \\
&= 3\tan^{-1}x - \frac{2}{x^2+1} + C
\end{aligned}$$

where we use substitution $u = x^2+1$ with $du = 2xdx$. □

32. $\int \frac{1}{x^2-x+2}dx$

**Solution** The integrand is a proper rational function, and the denominator contains the irreducible quadratic factor $x^2 - x + 2$, which is irreducible, since the discriminant $\Delta = (-1)^2 - 4 \times 1 \times 2 = -7 < 0$. Then

$$\frac{1}{x^2-x+2} = \frac{1}{(x-\frac{1}{2})^2+\frac{7}{4}} = \frac{4}{7}\frac{1}{\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{7}{4}}}\right)^2+1}$$

under which setting $u = \frac{x-\frac{1}{2}}{\sqrt{\frac{7}{4}}}$ with $du = \sqrt{\frac{4}{7}}dx$ yields

$$\begin{aligned}
\int \frac{1}{x^2-x+2}dx &= \int \frac{4}{7}\frac{1}{\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{7}{4}}}\right)^2+1}dx \\
&= \sqrt{\frac{4}{7}}\int \frac{1}{1+u^2}du \\
&= \sqrt{\frac{4}{7}}\tan^{-1}u + C \\
&= \sqrt{\frac{4}{7}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{7}{4}}}\right) + C \\
&= \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right) + C
\end{aligned}$$

□

50. $\int \frac{1}{(x^2-x-2)^2}dx$

**Solution** The integrand is a proper rational function, and the denominator contains two repeated linear factors, so we claim that the integrand can be written in the form

$$\begin{aligned}
\frac{1}{(x^2-x-2)^2} &= \frac{1}{(x+1)^2(x-2)^2} \\
&= \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-2}+\frac{D}{(x-2)^2} \\
&= \frac{[A(x+1)+B](x-2)^2+[C(x-2)+D](x+1)^2}{(x+1)^2(x-2)^2} \\
&= [(A+C)x^3+(-3A+B+D)x^2+(-4B-3C+2D)x \\
&\quad +(4A+4B-2C+D)]\cdot\frac{1}{(x+1)^2(x-2)^2}
\end{aligned}$$

where $A$, $B$, $C$ and $D$ are constants that we need to determine. Then we conclude that

$$1=(A+C)x^3+(-3A+B+D)x^2+(-4B-3C+2D)x+(4A+4B-2C+D),$$

which implies

$$\begin{cases} A+C=0 \\ -3A+B+D=0 \\ -4B-3C+2D=0 \\ 4A+4B-2C+D=1 \end{cases}$$

and thus

$$\begin{cases} A=\frac{2}{27} \\ B=\frac{1}{9} \\ C=-\frac{2}{27} \\ D=\frac{1}{9} \end{cases}$$

Therefore

$$\begin{aligned}
\int \frac{1}{(x^2-x-2)^2}dx &= \int \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-2}+\frac{D}{(x-2)^2}dx \\
&= \frac{2}{27}\int \frac{1}{x+1}dx+\frac{1}{9}\int \frac{1}{(x+1)^2}dx-\frac{2}{27}\int \frac{1}{x-2}dx+\frac{1}{9}\int \frac{1}{(x-2)^2}dx \\
&= \frac{2}{27}\ln|x+1|-\frac{1}{9}\frac{1}{x+1}-\frac{2}{27}\ln|x-2|-\frac{1}{9}\frac{1}{x-2}+C \\
&= \frac{2}{27}\ln\left|\frac{x+1}{x-2}\right|-\frac{1}{9}\frac{2x-1}{(x+1)(x-2)}+C
\end{aligned}$$

□

**Section 7.4** Improper Integrals

4. $\int_e^\infty \frac{dx}{x(\ln x)^2}$

**Solution** Since the integration interval $[e,\infty)$ is unbounded,this integral is improper. Next define

$$A(z)=\int_e^z \frac{dx}{x(\ln x)^2}=-\frac{1}{\ln x}\bigg|_e^z=1-\frac{1}{\ln z}$$

Then from the definition of improper integral

$$\int_e^\infty \frac{dx}{x(\ln x)^2}=\lim_{z\to\infty}A(z)=\lim_{z\to\infty}\left(1-\frac{1}{\ln z}\right)=1$$

□

10. $\int_{-\infty}^\infty x^3e^{-x^4}dx$

**Solution** Since the integration interval $(-\infty,\infty)$ is unbounded,this integral is improper. Next splitting up the integral at $x=0$ yields

$$\int_{-\infty}^\infty x^3e^{-x^4}dx=\int_0^\infty x^3e^{-x^4}dx+\int_{-\infty}^0 x^3e^{-x^4}dx$$

and then

$$\begin{aligned}
\int_0^\infty x^3e^{-x^4}dx &= \lim_{z\to\infty}\int_0^z x^3e^{-x^4}dx \\
&= \lim_{z\to\infty}\left(-\frac{1}{4}e^{-x^4}\bigg|_0^z\right) \\
&= \lim_{z\to\infty}\frac{1}{4}\left(1-e^{-z^4}\right) \\
&= \frac{1}{4}
\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{0} x^3 e^{-x^4} dx &= \lim_{z\to-\infty}\int_z^0 x^3 e^{-x^4} dx \\ &= \lim_{z\to-\infty}\left(-\frac{1}{4}e^{-x^4}\Big|_z^0\right) \\ &= \lim_{z\to-\infty}\frac{1}{4}\left(-1+e^{-z^4}\right) \\ &= -\frac{1}{4}\end{aligned}$$

Then from the definition of improper integral

$$\int_{-\infty}^{\infty} x^3 e^{-x^4} dx = \int_0^{\infty} x^3 e^{-x^4} dx + \int_{-\infty}^{0} x^3 e^{-x^4} dx = \frac{1}{4} - \frac{1}{4} = 0$$

□

18. $\int_1^{\infty} \frac{1}{x^{\frac{1}{3}}} dx$

**Solution** Since the integration interval $[1,\infty)$ is unbounded,this integral is improper. Next define

$$A(z) = \int_1^{\infty} \frac{1}{x^{\frac{1}{3}}} dx = \frac{3}{2}x^{\frac{2}{3}}\Big|_1^z = \frac{3}{2}\left(z^{\frac{2}{3}}-1\right)$$

Then from the definition of improper integral

$$\int_1^{\infty} \frac{1}{x^{\frac{1}{3}}} dx = \lim_{z\to\infty} A(z) = \lim_{z\to\infty}\frac{3}{2}\left(z^{\frac{2}{3}}-1\right) = \infty,$$

i.e., DNE, which follows that $\int_1^{\infty} \frac{1}{x^{\frac{1}{3}}} dx$ diverges. □

32. $\int_{-\infty}^{\infty} \frac{c}{1+x^2} dx$

**Solution** Since the integration interval $(-\infty,\infty)$ is unbounded,this integral is improper. Next splitting up the integral at $x = 0$ yields

$$\int_{-\infty}^{\infty} \frac{c}{1+x^2} dx = \int_0^{\infty} \frac{c}{1+x^2} dx + \int_{-\infty}^{0} \frac{c}{1+x^2} dx$$

and then

$$\begin{aligned}\int_0^{\infty} \frac{c}{1+x^2} dx &= \lim_{z\to\infty}\int_0^z \frac{c}{1+x^2} dx \\ &= \lim_{z\to\infty}\left(c\tan^{-1} x\Big|_0^z\right) \\ &= \lim_{z\to\infty} c\tan^{-1} z \\ &= \frac{\pi}{2}c\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{0}\frac{c}{1+x^2}dx &= \lim_{z\to-\infty}\int_{z}^{0}\frac{c}{1+x^2}dx\\ &= \lim_{z\to-\infty}\left(c\tan^{-1}x\Big|_z^0\right)\\ &= \lim_{z\to-\infty}(-c\tan^{-1}z)\\ &= \frac{\pi}{2}c\end{aligned}$$

Then from the definition of improper integral

$$1=\int_{-\infty}^{\infty}\frac{c}{1+x^2}dx=\int_{0}^{\infty}\frac{c}{1+x^2}dx+\int_{-\infty}^{0}\frac{c}{1+x^2}dx=\frac{\pi}{2}c+\frac{\pi}{2}c=c\pi$$

which implies $c=\frac{1}{\pi}$ □

33. $\int_1^\infty \frac{1}{x^p}dx$ for $0<p<\infty$

**Solution** (a) For $z>1$, set

$$\begin{aligned}A(z) &= \int_1^z \frac{1}{x^p}dx\\ &= \begin{cases}\ln x\big|_1^z & p=1\\ \frac{1}{1-p}x^{1-p}\Big|_1^z & p\neq 1\end{cases}\\ &= \begin{cases}\ln z & p=1\\ \frac{1}{1-p}\left(z^{1-p}-1\right) & p\neq 1\end{cases}\end{aligned}$$

(b) When $p=1$,

$$\lim_{z\to\infty}A(z)=\lim_{z\to\infty}\ln z=\infty;$$

when $0<p<1$, i.e., $1-p>0$ implies

$$\lim_{z\to\infty}A(z)=\lim_{z\to\infty}\frac{1}{1-p}(z^{1-p}-1)=\infty.$$

(b) When $p>1$, i.e., $1-p<0$ implies

$$\lim_{z\to\infty}A(z)=\lim_{z\to\infty}\frac{1}{1-p}(z^{1-p}-1)=\frac{1}{p-1}.$$

□

**Section 8.1** Solving Differential Equations

2. $\frac{dy}{dx}=e^{-3x}$, where $y_0=10$ for $x_0=0$

**Solution**

$$\begin{aligned} y(x) &= y(0) + \int_{x_0}^{x} e^{-3u} du \\ &= 10 + \int_{0}^{x} e^{-3u} du \\ &= 10 + \left( -\frac{1}{3} e^{-3u} \right) \Big|_0^x \\ &= 10 + \frac{1}{3}(1 - e^{-3x}) \\ &= \frac{31}{3} - \frac{1}{3} e^{-3x} \end{aligned}$$

□

12. $\frac{dy}{dx} = 2(1-y)$, where $y_0 = 2$ for $x_0 = 0$

**Solution** Separating variables and then integrating follow

$$\int dx = \int \frac{dy}{2(1-y)}$$

which implies

$$-\frac{1}{2} \ln|1-y| = x + C_1$$

and thus

$$1 - y = Ce^{-2x}$$

where $C$ is a constant. Last plugging the initial condition yields

$$C = 1 - 2 = -1.$$

So

$$y = 1 + e^{-2x}$$

□

22. $\frac{dL}{dt} = k(34 - L(t))$ with $L(0) = 2$

**Solution** (a) Separating variables and then integrating follow

$$\int dt = \int \frac{dL}{k(34 - L(t))}$$

which implies

$$-\frac{1}{k} \ln|34 - L(t)| = t + C_1$$

and thus

$$34 - L(t) = Ce^{-kt}$$

where $C$ is a constant. Last plugging the initial condition yields

$$C = 34 - 2 = 32.$$

So

$$L(t) = 34 - 32e^{-kt}$$

(b) If $L(4) = 10$, $34 - 32e^{-4k} = 10$ implies

$$k = -\frac{1}{4} \ln \frac{3}{4}$$

See last page for graph.

(c) $L(10) = 34 - 32e^{-10k} = 34 - 32e^{10 \times \frac{1}{4} \ln \frac{3}{4}} = 34 - 9\sqrt{3}$

(d) Since $\frac{1}{4} \ln \frac{3}{4} < 0$,

$$\lim_{t \to \infty} L(t) = \lim_{t \to \infty} \left( 34 - 32e^{t \frac{1}{4} \ln \frac{3}{4}} \right) = 34$$

□

28. $\frac{dy}{dx} = (y-1)(y-2)$, where $y_0 = 0$ for $x_0 = 0$

**Solution** Separating variables and then integrating by partial-fraction follow

$$\int dx = \int \frac{dy}{(y-1)(y-2)} = \int \frac{1}{y-2} - \frac{1}{y-1} dx$$

which implies

$$\ln \left| \frac{y-2}{y-1} \right| = x + C_1$$

and thus

$$\frac{y-2}{y-1} = Ce^x$$

where $C$ is a constant. Last plugging the initial condition yields

$$C = \frac{0-2}{0-1} = 2.$$

So

$$y = \frac{2 - 2e^x}{1 - 2e^x}$$

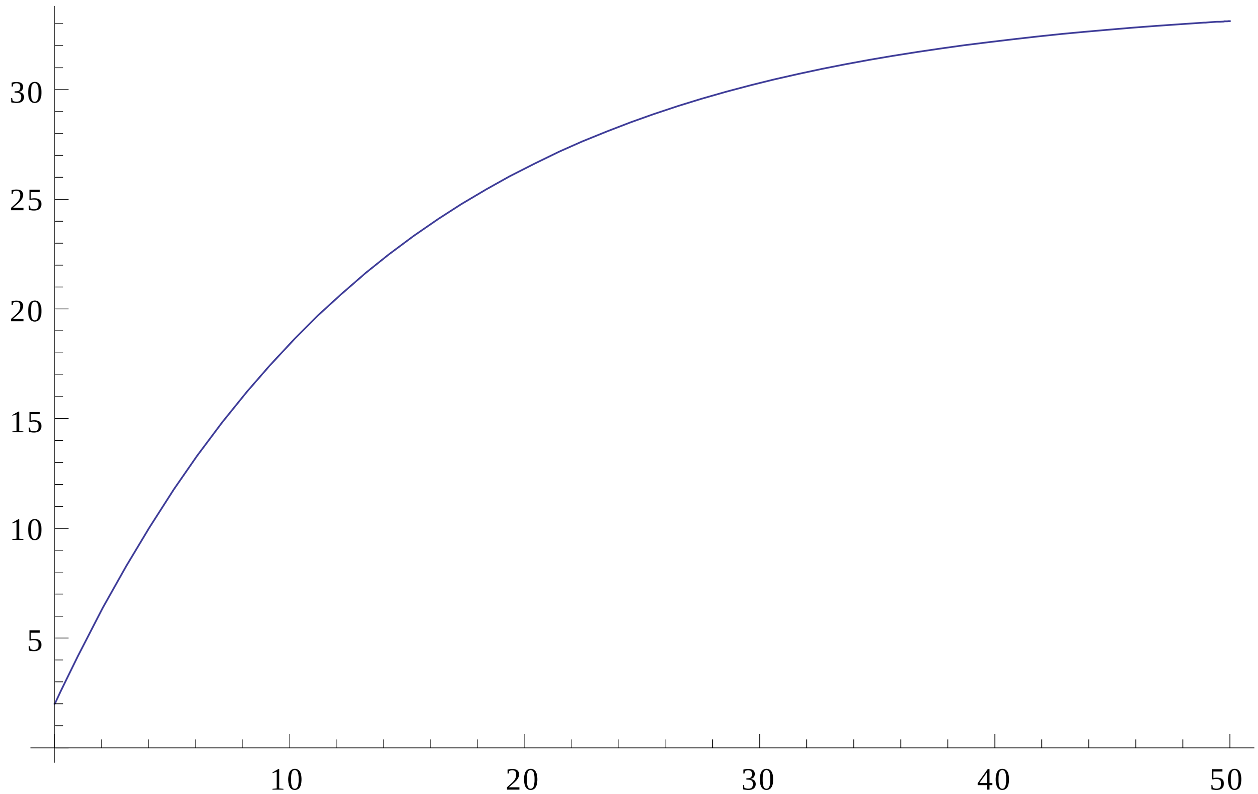
□

44. $\frac{dy}{dx} = 2\frac{y}{x}$, where $y_0 = 1$ for $x_0 = 1$

**Solution** Separating variables and then integrating follow

$$\int \frac{2}{x} dx = \int \frac{1}{y} dy$$

which implies
$$\ln|y| = 2\ln|x| + C_1$$
and thus
$$y = Cx^2$$
where $C$ is a constant. Last plugging the initial condition yields
$$C = 1.$$
So
$$y = x^2$$
□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