MATH 3B HOMEWORK2 SOLUTIONS (WINTER 2011)

Section 6.1

33. Define $f(x) = 1 - x^2$, and we partition $[-1, 1]$ into five equal subintervals, each of length $\Delta x_k = \frac{1-(-1)}{5} = 0.4$:
$[-1, -0.6]$, $[-0.6, -0.2]$, $[-0.2, 0.2]$, $[0.2, 0.6]$, and $[0.6, 1]$.
Then the corresponding Riemann sum is given by

$$S_P = \Sigma_{k=1}^5 f(c_k)\Delta x_k = 0.4\Sigma_{k=1}^5 f(c_k).$$

The midpoints are $c_1 = -0.8, c_2 = -0.4, c_3 = 0, c_4 = 0.4$ and $c_5 = 0.8$, and then
$S_P = 0.4[f(-0.8) + f(-0.4) + f(0) + f(0.4) + f(0.8)] = 1.36.$

34. Define $f(x) = 2 + x^2$, and we partition $[-1, 1]$ into five equal subintervals, each of length $\Delta x_k = \frac{1-(-1)}{5} = 0.4$:
$[-1, -0.6]$, $[-0.6, -0.2]$, $[-0.2, 0.2]$, $[0.2, 0.6]$, and $[0.6, 1]$.
Then the corresponding Riemann sum is given by

$$S_P = \Sigma_{k=1}^5 f(c_k)\Delta x_k = 0.4\Sigma_{k=1}^5 f(c_k).$$

The right endpoints are $c_1 = -0.6, c_2 = -0.2, c_3 = 0.2, c_4 = 0.6$ and $c_5 = 1$, and then
$S_P = 0.4[f(-0.6) + f(-0.2) + f(0.2) + f(0.6) + f(1)] = 4.72.$

39. The area S_T of a trapezoid is given $S_T = \frac{1}{2}(a + b)h$, where h is the height, and a and b are the lengths of the parallel sides. See Figure 1(Last page). Then from Geometric interpretation of definite integrals,

$$\int_a^b x dx = S_{\text{shade}} = S_{\text{trapezoid}} = \frac{1}{2}(a + b)(b - a) = \frac{b^2 - a^2}{2}.$$

41. $\lim_{\|P\|\to 0} \Sigma_{k=1}^n 2c_k^3 \Delta x_k = \int_1^2 2x^3 dx$

42. $\lim_{\|P\|\to 0} \Sigma_{k=1}^n \sqrt{c_k}\Delta x_k = \int_1^4 \sqrt{x} dx$

49.

$$\int_2^6 (x + 1)^{1/3} dx = \lim_{\|P\|\to 0} \Sigma_{k=1}^n (c_k + 1)^{1/3} \Delta x_k$$

where $x_0 = 2 < x_1 < x_2 < \cdots < x_n = 6$, $n = 1, 2, ...$, is a sequence of partitions of $[2, 6]$, $c_k \in [x_{k-1}, x_k]$, $\Delta x_k = x_k - x_{k-1}$.

50.

$$\int_1^3 e^{-2x}dx = \lim_{\|P\|\to 0} \Sigma_{k=1}^n e^{-2c_k}\Delta x_k$$

where $x_0 = 1 < x_1 < x_2 < \cdots < x_n = 3,\ n = 1, 2, ...,$ is a sequence of partitions of $[1, 3]$, $c_k \in [x_{k-1}, x_k]$, $\Delta x_k = x_k - x_{k-1}$.

53.

$$\int_0^5 g(x)dx = \lim_{\|P\|\to 0} \Sigma_{k=1}^n g(c_k)\Delta x_k$$

where $x_0 = 0 < x_1 < x_2 < \cdots < x_n = 5,\ n = 1, 2, ...,$ is a sequence of partitions of $[0, 5]$, $c_k \in [x_{k-1}, x_k]$, $\Delta x_k = x_k - x_{k-1}$.

57. See Figure 2(Last page). Let A_S denote the area of the shade region. Then $\int_0^5 e^{-x}dx = A_S$.

58. See Figure 3(Last page). Let A_U denote the area of the shade region above the x-axis and A_L be the total area of the shade regions below the x-axis. Then $\int_{-\pi}^{\pi} \cos x dx = A_U - A_L = 0$.

64. Calculate $\int_{1/2}^1 \sqrt{1-x^2}dx$. See Figure 4.

Setting $y = \sqrt{1-x^2}$ gives $x^2 + y^2 = 1$, which denotes a unit circle. Let $O = (0, 0)$, $A = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $B = (\frac{1}{2}, 0)$ and $C = (1, 0)$. Then $\int_{1/2}^1 \sqrt{1-x^2}dx$ is the area of the shade region(See Figure 4), which is enclosed by line segments AB, BC and the arc $\widehat{AC}$. Thus if let $S_{\widehat{OAC}}$ and $S_{\Delta OAB}$ denote the areas of SectorOAC and Triangle ΔOAB respectively, from $\cos\angle AOB = \frac{|OB|}{|OA|} = \frac{1}{2}$, we know $\angle AOB = \frac{\pi}{3}$. So

$$\begin{aligned}
\int_{1/2}^1 \sqrt{1-x^2}dx &= S_{\widehat{OAC}} - S_{\Delta OAB} \\
&= \frac{1}{2}|OA|^2\angle AOB - \frac{1}{2}|OA||OB|\sin\angle AOB \\
&= \frac{\pi}{6} - \frac{\sqrt{3}}{8}.
\end{aligned}$$

65. Calculate $\int_{-2}^2 (\sqrt{4-x^2} - 2)dx$. See Figure 5.

Setting $y = \sqrt{4-x^2} - 2$ gives $x^2 + (y+2)^2 = 4$, which denotes a circle. Let $A = (-2, 0)$, $B = (-2, -2)$, $C = (2, -2)$ and $D = (2, 0)$. Since the shade region is below x-axis, then

$$\begin{aligned}\int_{-2}^{2}(\sqrt{4-x^2}-2)dx &= -S_{\text{shade}} \\ &= -S_{\square \text{A}BCD} + S_{\text{ half circle}} \\ &= -|AD||AB| + \frac{1}{2}\pi 2^2 \\ &= 2\pi - 8.\end{aligned}$$

68. Given that $\int_0^a x^2dx = \frac{1}{3}a^3$, from properties of integrals, we get

(a)

$$\int_0^2 \frac{1}{2}x^2dx = \frac{1}{2}\int_0^2 x^2dx = \frac{1}{2}\times\frac{1}{3}\times 2^3 = \frac{4}{3}$$

(b)

$$\begin{aligned}\int_{-3}^{-2} 3x^2dx &= 3(\int_{-3}^{0} x^2dx + \int_0^{-2} x^2dx) \\ &= 3(-\int_0^{-3} x^2dx + \int_0^{-2} x^2dx) \\ &= 3[-\frac{1}{3}(-3)^3 + \frac{1}{3}(-2)^3] \\ &= 19\end{aligned}$$

(c)

$$\begin{aligned}\int_{-1}^{3} \frac{1}{3}x^2dx &= \frac{1}{3}(\int_{-1}^{0} x^2dx + \int_0^{3} x^2dx) \\ &= \frac{1}{3}(-\int_0^{-1} x^2dx + \int_0^{3} x^2dx) \\ &= \frac{1}{3}[-\frac{1}{3}(-1)^3 + \frac{1}{3}3^3] \\ &= \frac{28}{9}\end{aligned}$$

(d)

$$\int_1^1 3x^2dx = 0$$

(e) From the similar discussion in Question 39, we obtain

$$\begin{aligned}
\int_{-2}^{3}(x+1)^2dx &= \int_{-2}^{3}(x^2+2x+1)dx \\
&= \int_{-2}^{3} x^2dx + \int_{-2}^{3} 2xdx + \int_{-2}^{3} 1dx \\
&= [\int_{0}^{3} x^2dx - \int_{0}^{-2} x^2dx] + 2\int_{-2}^{3} xdx + \int_{-2}^{3} 1dx \\
&= \frac{3^3-(-2)^3}{3} + 2\frac{3^2-(-2)^2}{2} + (3-(-2)) \times 1 \\
&= \frac{65}{3}
\end{aligned}$$

(f) Similar as above,

$$\begin{aligned}
\int_{2}^{4}(x-2)^2dx &= \int_{2}^{4}(x^2-4x+4)dx \\
&= \int_{2}^{4} x^2dx - \int_{2}^{4} 4xdx + \int_{2}^{4} 4dx \\
&= [\int_{0}^{4} x^2dx - \int_{0}^{2} x^2dx] - 4\int_{2}^{4} xdx + 4\int_{2}^{4} 1dx \\
&= \frac{4^3-2^3}{3} - 4\frac{4^2-2^2}{2} + (4-2) \times 4 \\
&= \frac{8}{3}
\end{aligned}$$

70. From the first property of the integral

$$\int_{a}^{a} f(x)dx = 0,$$

we have

$$\int_{-3}^{-3} e^{-x^2/2}dx = 0.$$

73. See Figure 6. We have a fact that $\tan(-x) = -\tan x$, which implies $\tan x$ is odd and thus symmetric about $x = 0$. Then the area of the region below the graph of $f(x) = \tan x$ and above the x-axis between 0 and 1(denoted by A_+) is same as the area of the region above the graph of f and below the x-axis between -1 and 0(denoted by A_-). Therefore $A_+ = A_-$ and

$$\int_{-1}^{1} \tan xdx = A_+ - A_- = 0.$$

Remark: Actually for any odd function $f(x)$, and $a \geq 0$ such that $(-a, a)$ is in the domain of $f(x)$, we always have

$$\int_{-a}^{a} f(x)dx = 0.$$

81. See Figure 7 for $a \in [0, 2\pi]$. Using the interpretation of the definite integral as the signed area, we see from the graph of $f(x) = \cos x$ that the graph of $f(x)$ is positive for $0 \leq x < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x \leq 2\pi$, while negative for $\frac{\pi}{2} < x < \frac{3\pi}{2}$. Moreover from the symmetry analysis, we can conclude that:

when $0 < a < \pi$, $\int_0^a \cos x dx > 0$;

when $\pi < a < 2\pi$, $\int_0^a \cos x dx < 0$.

Then combining the results above together implies that $a = \frac{\pi}{2}$ maximizes the integral.

82. See Figure 8. We see from the graph of $f(x) = \sin x$ that the graph of $f(x)$ is positive for $0 < x < \pi$ and negative for $\pi < x < 2\pi$. Moreover from the symmetry analysis, we can conclude that:

when $0 < a < 2\pi$, $\int_0^a \sin x dx > 0$ and $\int_0^{2\pi} \sin x dx = 0$.

Then there is only one value $a = 2\pi$ in $(0, 2\pi]$ such that

$$\int_0^a \sin x dx = 0.$$

Section 6.2

8. Remark: There is a problem in this question.

First note that $f(x) = \sqrt{2 + \csc^2 x}$ is continuous for $0 < x < \pi$, but $f(x)$ is not well defined at $x = 0$. So if we still want to apply FTC, we have to change the lower limit of integration from 0 to some small positive number $\varepsilon > 0$.

Thus the result should be that for $\varepsilon \leq x < \pi$,

$$\frac{dy}{dx} = \frac{d}{dx} \int_{\varepsilon}^{x} \sqrt{2 + \csc^2 u} du = \sqrt{2 + \csc^2 x}$$

9. First note that $f(x) = xe^{4x}$ is continuous everywhere. Then from FTC, we have

$$\frac{dy}{dx} = \frac{d}{dx} \int_3^x ue^{4u} du = xe^{4x}$$

Additional*: Prove Leibniz's rule in the case where f is continuous everywhere.

Leibniz's Rule: *If $g(x)$ and $h(x)$ are differentiable functions and $f(u)$ is continuous for u between $g(x)$ and $h(x)$, then*

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(u)du = f[h(x)]h'(x) - f[g(x)]g'(x)$$

Proof. First note that $f(u)$ is continuous everywhere, then write $F(x) = \int_0^x f(u)du$, and thus from FTC,

$$F'(x) = f(x).$$

Next by some basic properties of integrals and chain rule,

$$\begin{aligned} \frac{d}{dx}\int_{g(x)}^{h(x)} f(u)du &= \frac{d}{dx}[\int_0^{h(x)} f(u)du + \int_{g(x)}^{0} f(u)du] \\ &= \frac{d}{dx}\{F[h(x)] - F[g(x)]\} \\ &= F'[h(x)]h'(x) - F'[g(x)]g'(x) \\ &= f[h(x)]h'(x) - f[g(x)]g'(x) \end{aligned}$$

Figure 1

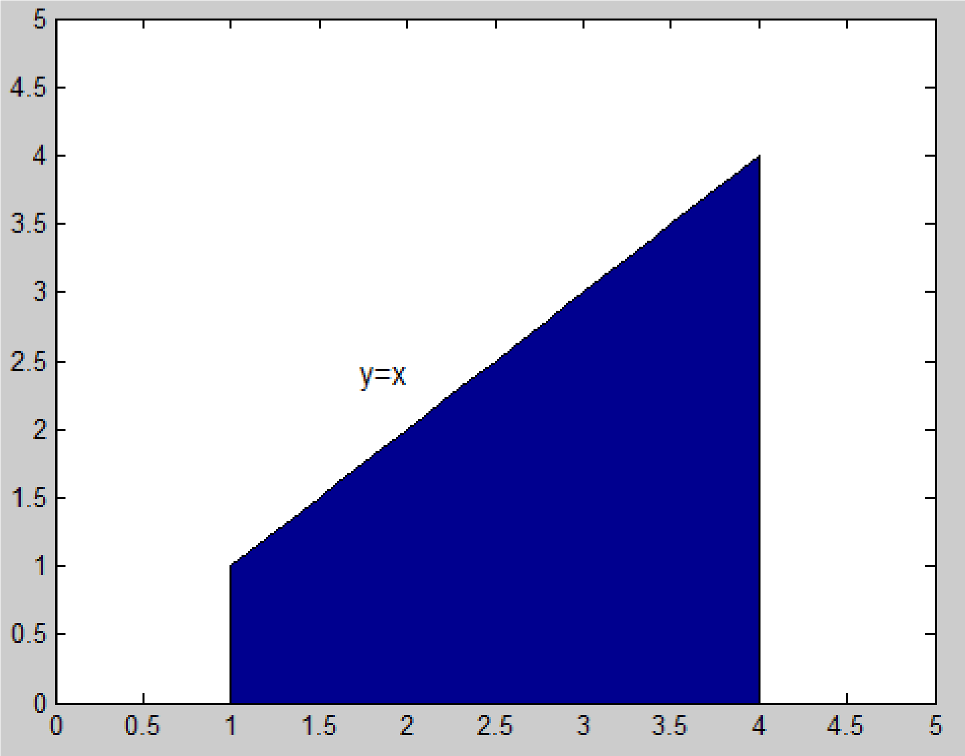

Figure 2

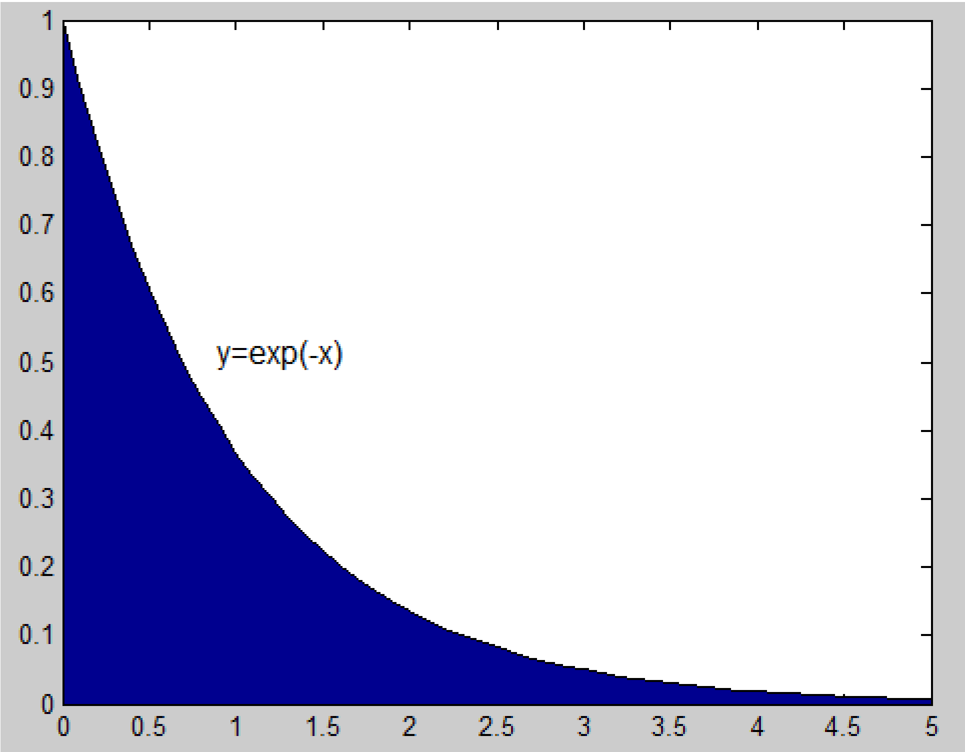

Figure 3

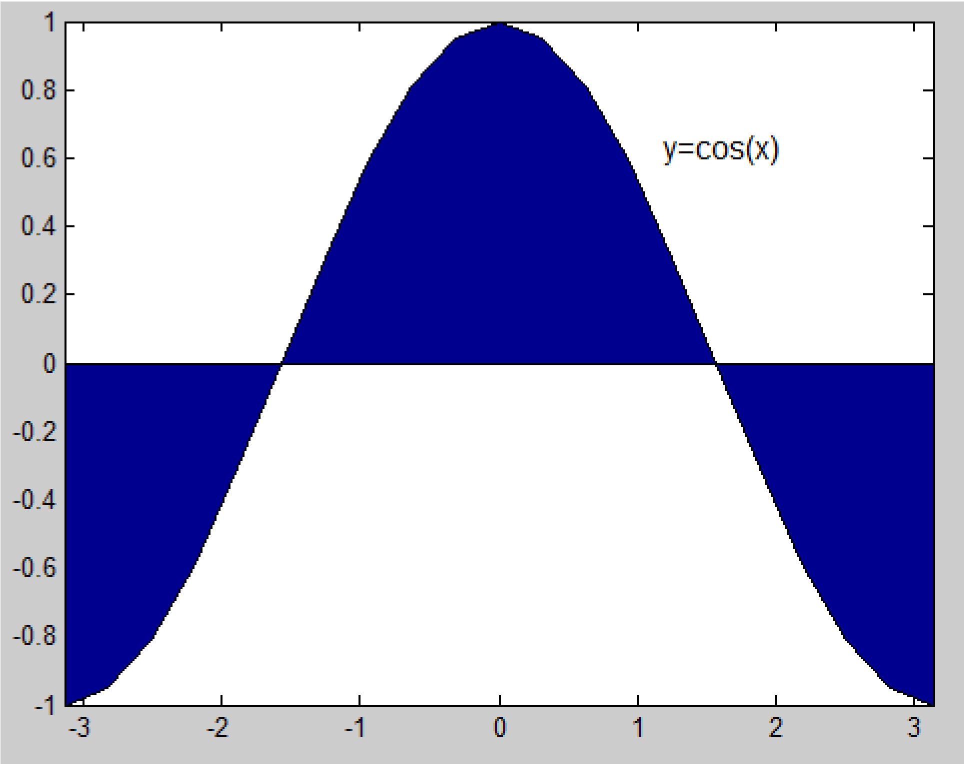

Figure 4

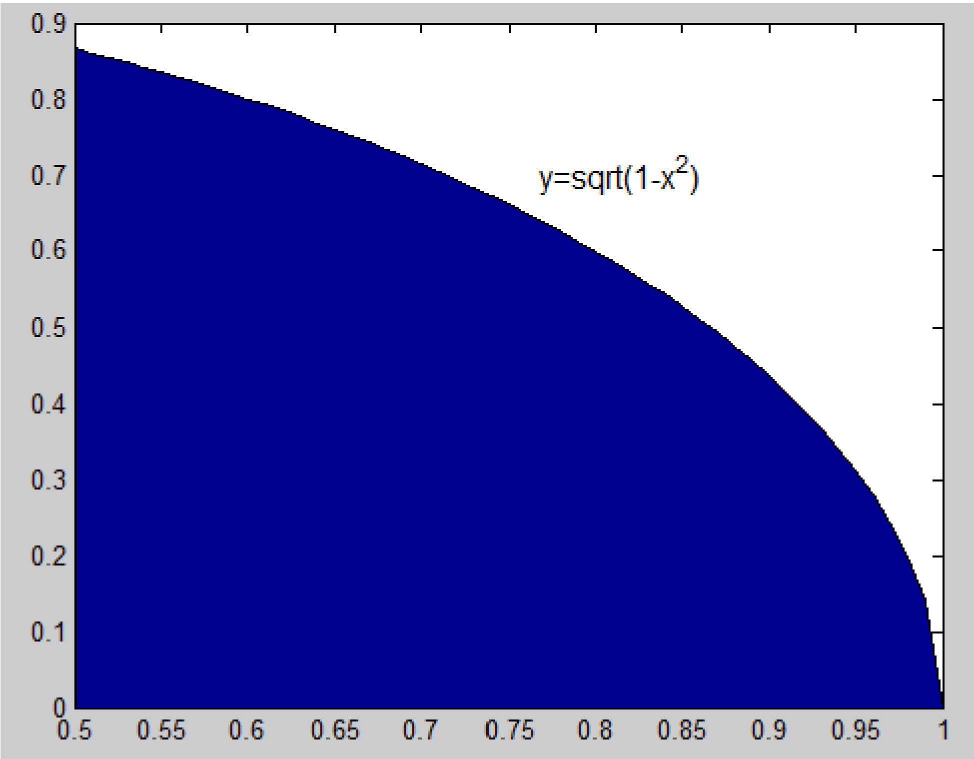

Figure 5

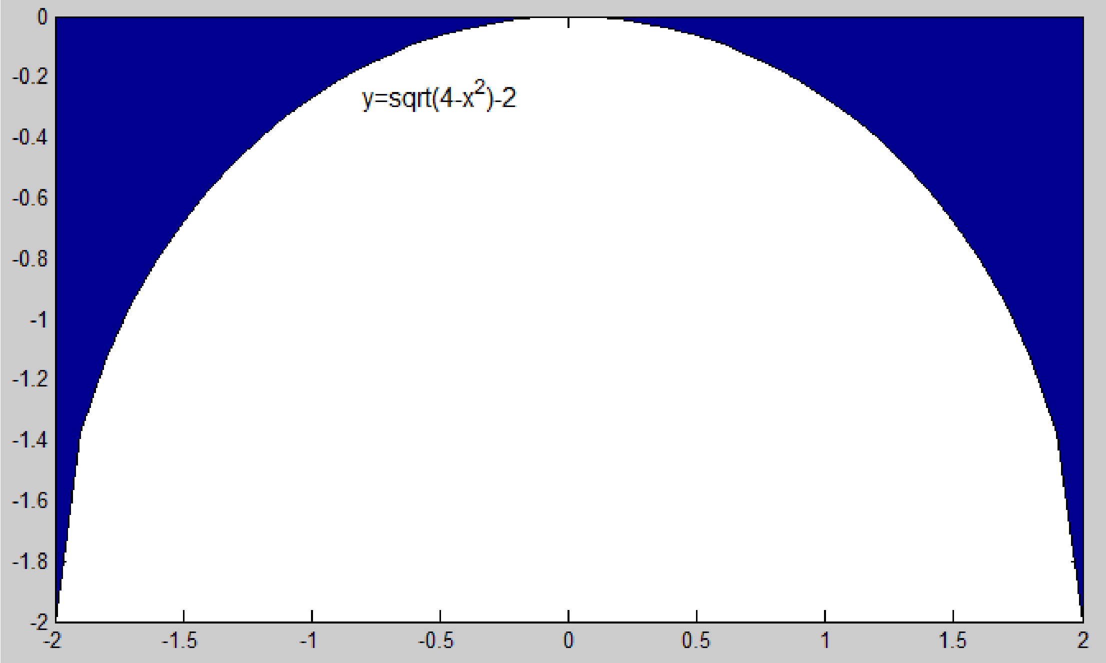

Figure 6

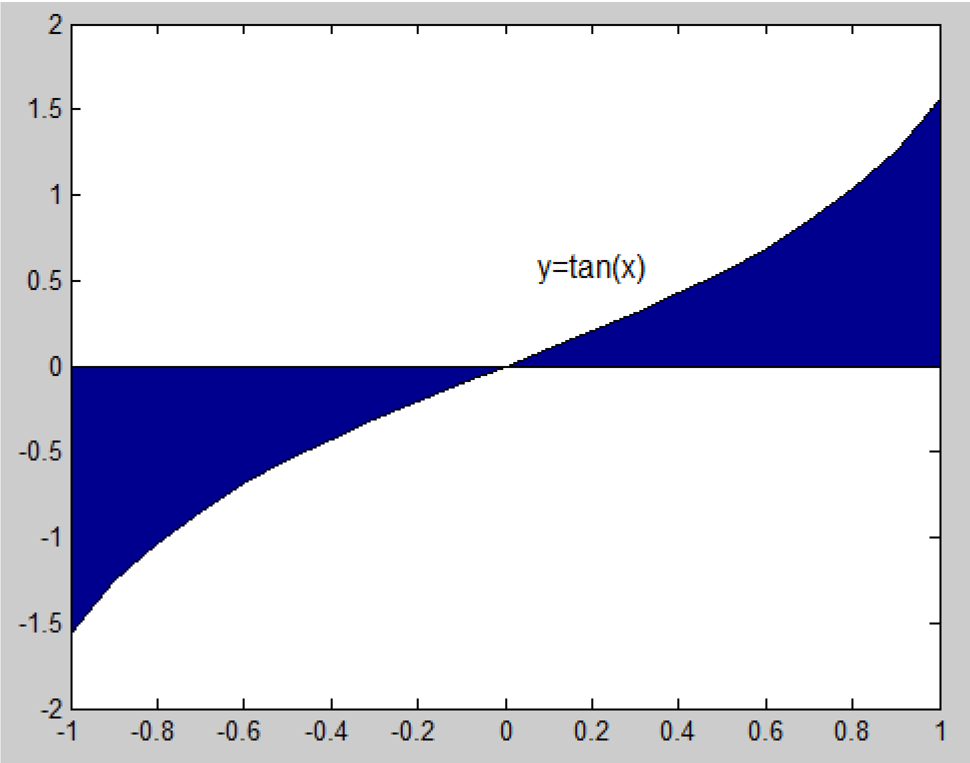

Figure 7

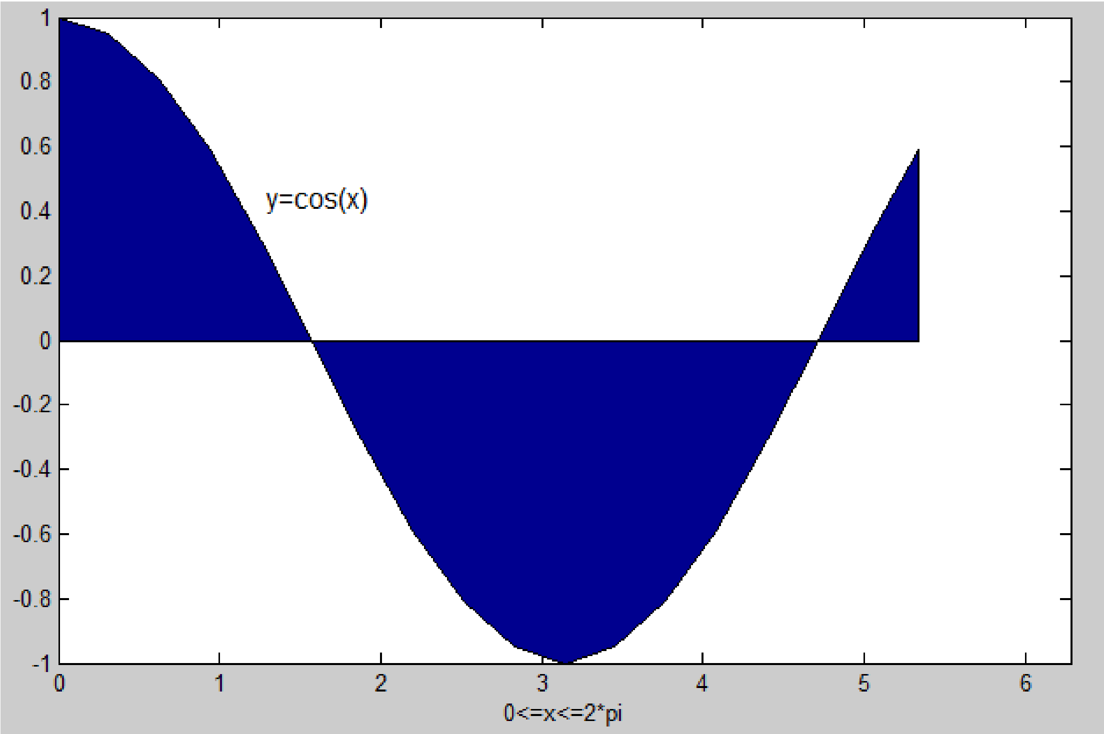

Figure 8