Homework #3 Solutions

6.2.19. $\frac{\partial}{\partial t}\int_{f(x)}^{g(x)} h(t)\,dt = h(g(x))\,g'(x) - h(f(x))\,f'(x)$

$h(t) = \sqrt{t}$, $g(x) = x^2+1$, $g'(x) = 2x$, $f(x) = 4$, $f'(x) = 0$

$\frac{\partial}{\partial t}\int_4^{x^2+1} \sqrt{t}\,dt = (\sqrt{x^2+1})(2x) - (\sqrt{4})(0) = 2x\sqrt{x^2+1}$

20. $g(x) = x^2-2$, $g'(x) = 2x$; $f(x) = 2$, $f'(x) = 0$, $h(u) = \sqrt{3+u}$

$\frac{\partial}{\partial t}\int_2^{x^2-2} \sqrt{3+u}\,du = (\sqrt{3+x^2-2})(2x) - (\sqrt{3+2})(0) = 2x\sqrt{x^2+1}$

57. $\int e^{2x}dx = \frac{1}{2}\int 2e^{2x}dx$ $\quad \frac{\partial}{\partial x}(e^{2x}) = 2e^{2x}$ (chain rule), so

$\int 2e^{2x}dx = e^{2x}+c$, so $\int e^{2x}dx = \frac{1}{2}\int 2e^{2x}dx = \frac{1}{2}e^{2x}+c$

58. $\int 2e^{3x}dx = 2\int e^{3x}dx = \frac{2}{3}\int 3e^{3x}dx$ $\quad \frac{\partial}{\partial x}e^{3x} = 3e^{3x}+c$, so

$\int 2e^{3x}dx = \frac{2}{3}\int 3e^{3x}dx = \frac{2}{3}e^{3x}+c$

91. Rewrite 3 as $e^{\ln(3)}$, so $\int 3^{-2x}dx = \int (e^{\ln(3)})^{-2x}dx = \int e^{-2\ln(3)x}dx$

$\int 3^{-2x}dx = \int e^{-2\ln(3)x}dx = \frac{-1}{2\ln(3)}\int(-2\ln(3))\,e^{-2\ln(3)x}\,dx$

Since $\frac{\partial}{\partial x}e^{-2\ln(3)x} = (-2\ln(3))e^{-2\ln(3)x}$, $\int(-2\ln(3))\,e^{-2\ln(3)x}dx = e^{-2\ln(3)x}+c$

$\int 3^{-2x}dx = \frac{-1}{2\ln(3)}\int(-2\ln(3))\,e^{-2\ln(3)x}dx = \frac{-1}{2\ln(3)}e^{-2\ln(3)x}+c$

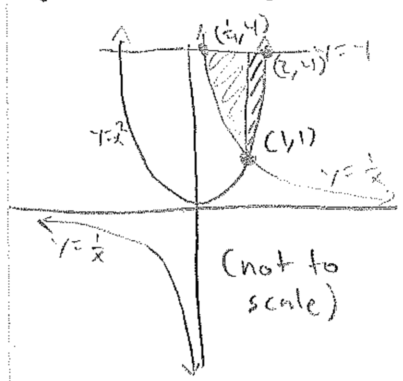
$= \frac{-e^{-\ln(9)x}}{\ln(9)}+c = \frac{-(\frac{1}{9})^x}{\ln(9)}+c$

You can also use the formula $\frac{\partial}{\partial x}a^x = \ln(a)\,a^x$, ie $\int a^x dx = \frac{a^x}{\ln a}+c$. I will use this method to solve #92, which can also be solved the way I solved #91.

92. One way to solve this is by rewriting 4 as $e^{\ln(4)}$, as in my solution to #91. Here is a faster way:

$$\int 4^{-x}\,dx = \int (4^{-1})^{x}\,dx = \int \tfrac{1}{4}^{x}\,dx = \frac{\frac{1}{4}^{x}}{\ln(\frac{1}{4})} + C = \frac{4^{-x}}{-\ln(4)} + C$$

6.3.7



First, solve for the intersection points
$x^2 = \frac{1}{x}$, $x^3=1$, $x=1$, so $(1,1)$ is an intersection point
$x^2=4$, $x=\pm 2$, so $(2,4)$ is the intersection point in the first quadrant.
$\frac{1}{x}=4$, $x=\frac{1}{4}$, so $(\frac{1}{4},4)$ is an intersection point.

Now, do one integral for each of the two regions.

$$\int_{\frac{1}{4}}^{1} 4-\frac{1}{x}\,dx + \int_{1}^{2} 4-x^2\,dx$$

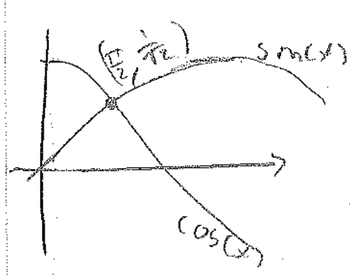
$$= \Big[4x - \ln|x|\Big]_{\frac{1}{4}}^{1} + \Big[4x - x^2\Big]_{1}^{2}$$

$$= 4(1) - \ln(1) - \left(4(\tfrac{1}{4}) - \ln(\tfrac{1}{4})\right) + 4(2) - (2)^2 - \left(4(1) - (1)^2\right)$$

$$= 4 - 0 - 1 + \ln(\tfrac{1}{4}) + 8 - 4 - 4 + 1$$

$$= 4 - \ln(4)$$

8.



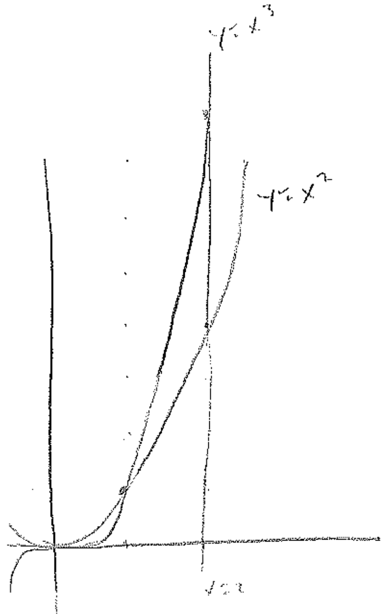
$$\int_0^{\frac{\pi}{4}} \cos x - \sin x\,dx$$

$$= \sin(x) + \cos(x)\Big]_0^{\frac{\pi}{4}}$$

$$= \sin(\tfrac{\pi}{4}) + \cos(\tfrac{\pi}{4}) - \sin(0) - \cos(0)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

11.



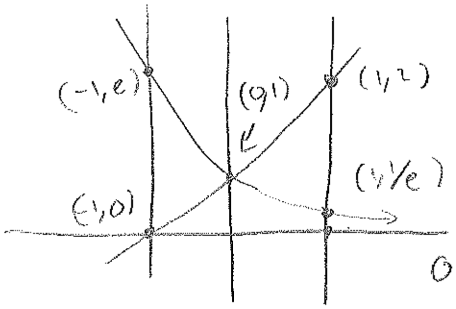
$$\int_0^1 x^2 - x^3 dx + \int_1^2 x^3 - x^2 dx$$

$$\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 + \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2$$

$$\left(\frac{1}{3} - \frac{1}{4}\right) - (0-0) + \left(\frac{16}{4} - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)$$

$$= \frac{2}{3} - \frac{1}{2} + 4 - \frac{8}{3} = -2 - \frac{1}{2} + 4 = \frac{3}{2}$$

12.



$$\int_{-1}^0 e^{-x} - (x+1)\, dx + \int_0^1 x+1-e^{-x}\, dx$$

First, note that $\int e^{-x} dx = -\int -e^{-x} dx = -e^{-x} + C$.
Once we know the antiderivative of e^{-x}, we can integrate.

$$\int_{-1}^0 e^{-x} - (x+1)\, dx + \int_0^1 x+1-e^{-x}\, dx$$

$$= \left[-e^{-x} - \frac{x^2}{2} - x\right]_{-1}^0 + \left[\frac{x^2}{2} + x + e^{-x}\right]_0^1$$

$$= (-1) - \left(-e - \frac{1}{2} + 1\right) + \left(\frac{1}{2} + 1 + \frac{1}{e}\right) - (1)$$

$$= -1 + e + \frac{1}{2} - 1 + \frac{1}{2} + 1 + \frac{1}{e} - 1$$

$$= e + \frac{1}{e} - 1$$

21. If $l(t)$ is the size of an organism at time t, then $\frac{dl}{dt}(t)$ is the growth rate at time t. So $\int_2^7 \frac{dl}{dt}\, dt = l(7) - l(2)$ by F.T.C. So $\int_2^7 \frac{dl}{dt}\, dt$ is the total change in size, or total growth, from time $t=2$ months to $t=7$ months

22. If $w(x)$ is the weight of an organism at age x, $\frac{dw}{dx}$ is the rate it gains weight, so by FTC, $\int_3^5 \frac{dw}{dx}\,dx = w(5)-w(3)$, which is the change in weight from year 3 to year 5, or the total amount of weight gained during this time-period.

25. Average value of f over $[a,b]$ is given by $\frac{1}{b-a}\int_a^b f(x)\,dx$

$$\frac{1}{2}\int_0^2 x^2-2\,dx = \frac{1}{2}\left[\frac{x^3}{3}-2x\right]_0^2 = \frac{1}{2}\left(\frac{8}{3}-4\right)-\frac{1}{2}(0) = -\frac{2}{3}$$

26. $\frac{1}{1-(-1)}\int_{-1}^{1} \sin(\pi t)\,dt = \frac{1}{2}(0) = 0$ because $\sin(\pi t)$ is an odd function and $[-1,1]$ is centered at 0.