Question 7.2.a.1

Derive a reduction formula for $\int \sin^n x\ dx$ *which expresses this integral in terms of* $\int \cos^2 x \sin^{n-2} x\ dx$.

Applying integration by parts to $\int \sin^n x\ dx$ with $u = \sin^{n-1} x$ and $dv = \sin x dx$, we have

$$\int \sin^n x\ dx = -\cos x \sin^{n-1} x + \int (n-1)\cos^2 x \sin^{n-2} x\ dx.$$

Question 7.2.a.2

Using the previous part and the substitution $\cos^2 x = 1 - \sin^2 x$, *give a formula for* $\int \sin^n x\ dx$ *in terms of* $\int \sin^{n-2} x\ dx$.

$$\begin{aligned}
\int \sin^n x\ dx &= -\cos x \sin^{n-1} x + \int (n-1)\cos^2 x \sin^{n-2} x\ dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x\ dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\ dx - (n-1)\int \sin^n x\ dx
\end{aligned}$$

and by pulling $\int \sin^n x\ dx$ to the left, we have

$$n\int \sin^n x\ dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\ dx$$

or

$$\int \sin^n x\ dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\ dx$$

Question 7.2.a.2

Find $\int \sin^6\ dx$ *using the formula from the previous part.*

$$\begin{aligned}
\int \sin^6\ dx &= -\frac{1}{6}\cos x \sin^5 x + \frac{5}{6}\int \sin^4 x\ dx \\
&= -\frac{1}{6}\cos x \sin^5 x + \frac{5}{6}\left[-\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\int \sin^2 x\ dx\right] \\
&= -\frac{1}{6}\cos x \sin^5 x + \frac{5}{6}\left[-\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\left[-\frac{1}{2}\cos x \sin x + \frac{1}{2}\int \sin^0 x\ dx\right]\right] \\
&= -\frac{1}{6}\cos x \sin^5 x + \frac{5}{6}\left[-\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\left[-\frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1\ dx\right]\right] \\
&= -\frac{1}{6}\cos x \sin^5 x + \frac{5}{6}\left[-\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\left[-\frac{1}{2}\cos x \sin x + \frac{x}{2}\right]\right]
\end{aligned}$$

Question 7.2.b

g is an unknown continuous function with the property that $g''' = g$. Find

$$\int g(\ln x)dx$$

Applying integration by parts with $u = g(\ln x)$ and $dv = dx$, we have

$$\begin{aligned}\int g(\ln x)dx &= x \cdot g(\ln x) - \int g'(\ln x)dx \\ &= x \cdot g(\ln x) - x \cdot g'(\ln x) + \int g''(\ln x)dx \\ &= x \cdot g(\ln x) - x \cdot g'(\ln x) + x \cdot g''(\ln x) - \int g'''(\ln x)dx\end{aligned}$$

and since $g''' = g$, we can pull the last term to the left hand side to get

$$2\int g(\ln x)dx = x \cdot g(\ln x) - x \cdot g'(\ln x) + x \cdot g''(\ln x)$$

and so

$$\int g(\ln x)dx = \frac{1}{2}\left[x \cdot g(\ln x) - x \cdot g'(\ln x) + x \cdot g''(\ln x)\right].$$

Question 7.3.d

Find r, s so that

$$\frac{1}{(x+r)(x+s)} = \frac{1}{x+r} - \frac{1}{x+s}$$

Multiply both sides by $(x+r)(x-s)$ to get

$$1 = x + s - x - r.$$

This gives two equations:

$$\begin{aligned}0 &= x - x \\ 1 &= s - r\end{aligned}$$

so any choice with $s = 1 + r$ will suffice. For example:

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}.$$

Questions from Section 7.4.1

Find $\int_0^\infty e^{x^2} dx$ *or indicate that it deserves.*

We know $\lim_{x\to\infty} e^{x^2} = \infty$. So there is an infinite amount of area under its curve.

Question 7.4.a

For which values of $p > 0$ *does*

$$\int_e^\infty \frac{1}{x \ln^p x} dx$$

converge.

$$\int_e^\infty \frac{1}{x \ln^p x} dx = \lim_{a\to\infty} \int_e^a \frac{1}{x \ln^p x} dx$$

and setting $u = \ln x$, $du = \frac{1}{x} dx$, this is equal to

$$\lim_{a\to\infty} \int_1^{\ln a} \frac{1}{u^p} du$$

If $p \neq 1$, this is equal to

$$\lim_{a\to\infty} \left. \frac{-(p-1)}{u^{p-1}} \right|_1^{\ln a} = \lim_{a\to\infty} \frac{-(p-1)}{(\ln a)^{p-1}} + p - 1.$$

When $p > 1$, $p - 1 > 0$, so since $\ln a \to \infty$, also $(\ln a)^{p-1} \to \infty$, and therefore $\frac{-(p-1)}{(\ln a)^{p-1}} \to 0$, so the limit approaches $p - 1$.

When $p < 1$, $p - 1 < 0$, so since $\ln a \to \infty$, $\frac{-(p-1)}{(\ln a)^{p-1}} \to \infty$, and therefore the limit diverges.

When $p = 1$, the original limit is equal to

$$\lim_{a\to\infty} \ln u|_1^{\ln a} = \lim_{a\to\infty} \ln \ln a - 0 = \lim_{a\to\infty} \ln \ln a = \infty.$$

So the integral converges when $p > 1$.

Question 7.4.b

Suppose that f *is a function which is continuous everywhere and that for some* a*,*

$$\int_{-\infty}^a f(x)dx \text{ and } \int_a^\infty f(x)dx$$

both exist. Prove that for any b,

$$\int_{-\infty}^{b} f(x)dx \text{ and } \int_{b}^{\infty} f(x)dx$$

both exist and

$$\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = \int_{-\infty}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx.$$

$$\begin{aligned}
\int_{-\infty}^{b} f(x)dx &= \lim_{c\to\infty} \int_{-c}^{b} f(x)dx \\
&= \lim_{c\to\infty} \left[\int_{-c}^{a} f(x)dx + \int_{a}^{b} f(x)dx\right] \\
&= \lim_{c\to\infty} \left[\int_{-c}^{a} f(x)dx\right] + \int_{a}^{b} f(x)dx \\
&= \int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx
\end{aligned}$$

and since both terms exist, this limit exists. The case for $\int_b^\infty f(x)dx$ is similar.

$$\begin{aligned}
\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx &= \left[\lim_{c\to\infty} \int_{-c}^{a} f(x)dx\right] + \left[\lim_{c\to\infty} \int_{a}^{c} f(x)dx\right] \\
&= \left[\lim_{c\to\infty} \int_{-c}^{b} f(x)dx + \int_{b}^{a} f(x)dx\right] + \left[\lim_{c\to\infty} \int_{b}^{c} f(x)dx + \int_{a}^{b} f(x)dx\right] \\
&= \left[\lim_{c\to\infty} \int_{-c}^{b} f(x)dx\right] + \left[\lim_{c\to\infty} \int_{b}^{c} f(x)dx\right] + \int_{b}^{a} f(x)dx + \int_{a}^{b} f(x)dx \\
&= \int_{-\infty}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx + \int_{b}^{a} f(x)dx - \int_{b}^{a} f(x)dx \\
&= \int_{-\infty}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx
\end{aligned}$$

Question 8.2.a.1

Identify, and classify as stable or unstable, the equilibria of:

$$\frac{dA}{dt} = (A-4)\ln(|A| + 1/2)$$

$g(A) = (A-4)\ln(|A|+1/2)$, so the equilibria are at $4, 1/2, -1/2$.

We can't take a single derivaive. We could find the derivative for $A > 0$ and for $A < 0$ separately, and deal with g' as a piecewise function, or we could just determine the values by inspection.

When A is slightly less than 4, say $A = 4 - \epsilon$, $g(A) = -\epsilon \ln(4 - \epsilon + 1/2) < 0$ while when $A = 4+\epsilon$, $g(A) = \epsilon \ln(4+\epsilon+1/2) > 0$, so 4 is an unstable equilibrium.

When A is near either $1/2$ or $-1/2$, $A-4$ will be negative. When A is slightly less than $1/2$, $\ln(|A| + 1/2) = \ln(A + 1/2) = \ln(1/2 - \epsilon + 1/2) = \ln(1 - \epsilon) < 0$, while when A is slightly greater than $1/2$, $\ln(|A| + 1/2) = \ln(1 + \epsilon) > 0$. Since $g(A)$ is a negative number times $\ln(|A| + 1/2)$, $1/2$ is a stable equilibrium.

When A is near $-1/2$, $A - 4$ is still negative. When A is slightly less than $-1/2$, $\ln(|A| + 1/2) = \ln(-A + 1/2) = \ln(1/2 + \epsilon + 1/2) = \ln(1 + \epsilon) > 0$, and similarly when A is slightly more than $-1/2$, $\ln(|A| + 1/2) < 0$. So $-1/2$ is unstable.

Question 8.2.a.2

You know that $A(t)$ satisfies the equation $\frac{dA}{dt} = (A-4)\ln(|A|+1/2)$ and $A(0) = 2$. What is $\lim_{t\to\infty} A(t)$?

We have already noticed that between $1/2$ and 4, $g(A)$ is negative, so A will approach the stable equilibrium below it: $\lim_{t\to\infty} A(t) = 1/2$.