Math 570

Final Exam. Due 11:00am on 12/16/2013.

Fall 2013

Choose four questions to answer.

You may use your notes, textbooks, and all information posted on the class website. Do not discuss problems with people other than the professor or look for information online.

DO ask clarification questions if you're unsure what a problem means.

1. Consider a language with a single unary function symbol f, a single unary predicate symbol P, and $=$. Let Σ be the sentences:

   - $\forall x f f x = x$,
   - $\forall x P x \leftrightarrow \neg P f x$
   - $\exists x_1 \exists x_2 x_1 \neq x_2$,
   - $\exists x_1 \exists x_2 \exists x_3 x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3$,
   - $\cdots$

   (The last two sentences and $\cdots$ collectively say there are infinitely many elements.) Prove that $Cn\Sigma$ is complete.

2. Consider the language of arithmetic $0, S, <, R$ where R is a unary predicate, and the model $\mathfrak{N}_R$ where $|\mathfrak{N}_R| = \mathbb{N}$, $0^{\mathfrak{N}_R} = 0$, $S^{\mathfrak{N}_R}(n) = n+1$, $<^{\mathfrak{N}_R} = \{(n,m) \mid n < m\}$, and $R^{\mathfrak{N}_R} = \{2i \mid i \in \mathbb{N}\}$. (In other words, $\mathfrak{N}_R$ is the standard model when R is interpreted as the even numbers.) Modify the proof of quantifier-elimination for $Th\mathfrak{N}_L$ to show that $Th\mathfrak{N}_R$ satisfies quantifier-elimination.

3. Recall our original sequence operation $\langle a_0, \ldots, a_n \rangle = \prod_{i \leq m} p_i^{a_i+1}$. Prove *carefully* that the function $f(\langle a_0, \ldots, a_n \rangle) = a_n$ (where $f(s) = 0$ if s does not code a sequence) is representable. You may use the representability of any function we proved representable in class, and any function whose representability is in the catalog in Enderton.

4. Prove that there are two non-isomorphic uncountable models of PA.

5. Prove the following: Given any formulas α, β, each having only the free variable x and y, we can find sentences σ, τ such that

   $$PA \vdash \sigma \leftrightarrow \alpha(\overline{\ulcorner \sigma \urcorner}, \overline{\ulcorner \tau \urcorner})$$

   and

   $$PA \vdash \tau \leftrightarrow \beta(\overline{\ulcorner \sigma \urcorner}, \overline{\ulcorner \tau \urcorner}).$$

   Hint: Let η be the formula functionally representing the function mapping $(\ulcorner \phi \urcorner, n, m)$ to $\ulcorner \phi(\overline{n}, \overline{m}) \urcorner$ and consider the formulas

   $$\forall z \forall w (\eta(x, x, y, z) \wedge \eta(y, x, y, w) \rightarrow \alpha(z, w))$$

   and

   $$\forall z \forall w (\eta(x, x, y, z) \wedge \eta(y, x, y, w) \rightarrow \beta(z, w)).$$

6. Assuming PA is consistent, find two sentences σ, τ such that $PA \nvdash \sigma$, $PA \nvdash \tau$, $PA \nvdash \neg\sigma$, $PA \nvdash \neg\tau$, but $PA \vdash \sigma \vee \tau$.