THE MORICONE OF A CALABI–YAU SPACE FROM TORIC GEOMETRY

APPROVED:

Supervisor: ______________________

____

THE MORICONE OF A CALABI–YAU SPACE FROM TORIC GEOMETRY

by

VOLKER BRAUN

THESIS

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF ARTS

THE UNIVERSITY OF TEXAS AT AUSTIN

December, 1998

Table of Contents

List of Figures

Chapter 1

Introduction

Mirror symmetry is the phenomenon that string theory compactified on two topologically distinct Calabi–Yau manifolds leads to the same superconformal field theory. Most known examples for Calabi–Yau manifolds are constructed as hypersurfaces in toric varieties, and it turns out that this symmetry appears in toric geometry as exchanging the polyhedron with its dual.

Using this [1][2] showed how to obtain nonperturbative corrections from exact expressions on the mirror. With their methods we can calculate these Instantons from the Mori cone (the dual of the Kähler cone), which can be calculated directly from the geometric data.

However one has to distinguish between the Mori cone of the toric variety and of the hypersurface. While it is easy (and sufficient to obtain the instanton expansion) to calculate the Mori cone for the ambient space, there is no algorithm known for a generic Calabi–Yau hypersurface. Sheldon Katz [3] conjectured a way to calculate it as the intersection of the Mori cones of various ambient spaces, this is the approach i will follow.

Chapter 2

Kähler cone in Toric Geometry

The Kähler cone of a complex manifold X is defined as all $J \in H^{1,1}(X, \mathbb{R})$ that satisfy

$$\int_C J > 0, \qquad \int_S J \wedge J > 0, \quad \ldots, \int_X J \wedge \cdots \wedge J > 0 \tag{2.1}$$

where $C \subset X$ is any curve, $S \subset X$ any surface, etc. The name "cone" is justified because if J satisfies 2.1 so does λJ $\forall \lambda \in \mathbb{R}_{>0}$, i.e. it *is* a cone. By Poincaré duality we can identify the Kähler cone with the cone of ample divisors.

In general it is a difficult problem to determine the Kähler cone and there are examples where it is not a finitely generated cone. However if we restrict ourselves to toric varieties this bad behaviour does not occur and there is an easy way to calculate it. Since we are ultimately interested in Calabi–Yau manifolds, where the largest class of examples is realized as hypersurface in toric varieties, this restriction is justified.

So let me quickly review a few facts from toric geometry (or rather one possible way to look at it) to fix notation. A better introduction can be found in [4], for mathematical details see [5][6][7][8]. We start with a integral lattice $N \cong \mathbb{Z}^d$. Define a polyhedron in its real extension $N_\mathbb{R} \cong \mathbb{R}^d$ by $\nabla = \mathrm{conv}(\nabla(1)) \subset N_\mathbb{R}$ the convex hull of a finite number of integral points $\nabla(1) \subset$

N, such that the origin is the only internal point of ∇. Then the (rational, polyhedral and complete) fan[1] Σ is a collection of cones such that:

- The faces of each cone are themselves cones of the fan
- Any two cones of the fan intersect in a cone of the fan (possibly only the origin)
- The fan subdivides the cones over faces of ∇, i.e. $\sigma \cap \partial\nabla$ is contained in a face of ∇ for each cone $\sigma \in \Sigma$.
- The one-faces $\Sigma(1)$ are generated by integral points of ∇:

$$\sigma \cap \nabla \cap N \neq \emptyset \quad \forall \sigma \in \Sigma(1) \tag{2.2}$$

Assign to each one-cone $\sigma_1, \ldots, \sigma_n \in \Sigma(1)$ a variable x_i, and let $\nu_i^* = \sigma_i \cap \nabla \cap N$ be the corresponding integral point of ∇, let ν_0^* be the origin. Then we get the toric variety X_Σ by[2]

$$X_\Sigma = \frac{\mathbb{C}^n - \left\{ x_{i_1} \cdots x_{i_m} = 0 \,\middle|\, \left\langle \nu_{i_1}^*, \ldots, \nu_{i_m}^* \right\rangle \notin \Sigma \right\}}{\left(\mathbb{C}^*\right)^{n-d}} \tag{2.3}$$

where the torus action is defined by

$$(x_1, \ldots, x_n) \sim (\lambda^{q_1} x_1, \ldots, \lambda^{q_n} x_n) \qquad \forall \lambda \in \mathbb{C}^* \tag{2.4}$$

with the linear relations $\sum_{i=1}^n q_i \nu_i^* = 0$ defining the $q_1, \ldots, q_n \in \mathbb{Z}$. The vector space of linear relations among $\nu_1^*, \ldots, \nu_n^*$ is $n-d$–dimensional, compare eq. 2.3.

[1]The definition of a fan is more general, but this is sufficient for our purposes

[2]$\langle a_1, \ldots, a_m \rangle$ denotes — depending on the context — either the simplex $\{a_1, \ldots, a_m\}$ or the cone spanned by $\{a_1, \ldots, a_m\}$

In general the resulting toric variety will be singular. We are interested in Calabi–Yau hypersurfaces, so we want to resolve all singularities that are not points in X_Σ (which will be missed by a generic hypersurface). For the combinatorical data taking finer subdivisions of the fans corresponds to resolving singularities, and the integral points in the interior of codimension one faces (facets) correspond to singular points of the ambient space. We will be therefor be interested in the case where all points of $\nabla \cap N$ that are not interior to facets are rays of the fan, and Σ is simplicial (each cone is a cone over a simplex).

To each integral point ν_i^*, $i = 0 \dots n$ of ∇ one can associate a divisor D_i. One can see the exceptional divisors D_i, $i = 1 \dots n$ as the $x_i = 0$ hypersurface in eq. 2.3. The homology classes of divisors satisfy the linear equivalence

$$\left[\sum a_i D_i\right] = \left[\sum \left(a_i + \langle \nu_i^* | m \rangle\right) D_i\right] \tag{2.5}$$

for all $m \in \mathrm{Hom}(N, \mathbb{Z}) \stackrel{\mathrm{def}}{=} M \cong \mathbb{Z}^d$.

Now we are interested in constructing Calabi–Yau hypersurfaces in X_Σ. Define the dual polyhedron of ∇ by

$$\Delta = \left\{ m \in M_{\mathbb{R}} \;\middle|\; \langle \nu_i^* | m \rangle \geq -1 \; \forall \nu_i^* \in \nabla \right\} \subset M_{\mathbb{R}} \tag{2.6}$$

then the zero locus of the polynomial

$$p = \sum_{m \in \Delta \cap M} \prod_{i=1}^{n} x_i^{\langle \nu_i^* | m \rangle + 1} \tag{2.7}$$

is known to be a Calabi–Yau manifold if Δ is reflexive. Apart from my requirements for ∇, *reflexive* means that Δ is again a integral polyhedron.

Note that the dual of Δ is again ∇. Historically Batyrev started from Δ, so ∇ is called the dual polyhedron and N the dual lattice.

According to [9] the Kähler cone of X_Σ is isomorphic to the cone of convex piecewise linear functions on Σ, denoted $\mathrm{cpl}(\Sigma)$. Piecewise linear means that it is linear on each cone of the fan, convex means that its graph is the lower boundary of a convex polytope.

Definition 2.1 *$f : N_\mathbb{R} \to \mathbb{R}$ is a convex piecewise linear function, $f \in \mathrm{cpl}(\Sigma)$, if for all cones $\sigma \in \Sigma$ there exists a $m_\sigma \in M_\mathbb{R}$, $c_\sigma \in \mathbb{R}$ such that*

i) $f(x) = \langle m_\sigma | x \rangle + c_\sigma \quad \forall x \in \sigma$

ii) $f(x) > \langle m_\sigma | x \rangle + c_\sigma \quad \forall x \notin \sigma$

Note that this implies that the function is continuous everywhere, since the intersection of two cones of Σ is again a cone of Σ.

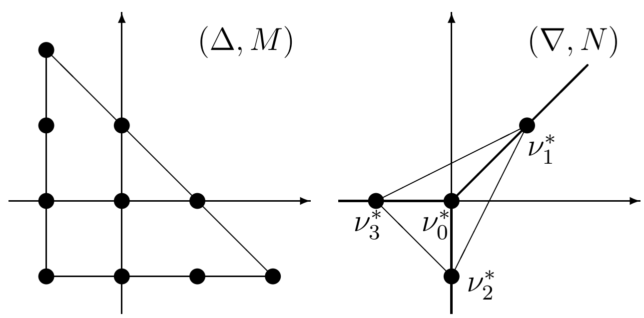

Figure 2.1: A simple example for Δ, ∇ and Σ. The maximal–dimensional cones are $\langle \nu_1^*, \nu_2^* \rangle$, $\langle \nu_1^*, \nu_3^* \rangle$, $\langle \nu_2^*, \nu_3^* \rangle$

In this easy example we can simply read off the condition for a piecewise linear function f to be convex. The graph f is a surface in $\mathbb{R}^3$, and one can easily see that the function is convex if the point $(\nu_0^*, f(\nu_0^*))$ is under (thinking of $N_{\mathbb{R}}$ as horizontal, the function value giving the height) the plane spanned by the three points $(\nu_i^*, f(\nu_i^*))$, $i = 1, 2, 3$. That is

$$f(\nu_0^*) < \frac{1}{3}\Big(f(\nu_1^*) + f(\nu_2^*) + f(\nu_2^*)\Big) \quad \Leftrightarrow \quad -3f(\nu_0^*) + f(\nu_1^*) + f(\nu_2^*) + f(\nu_2^*) > 0 \tag{2.8}$$

Then toric geometry translates this result into the Kähler cone $K(X_\Sigma)$ by taking the $f(\nu_i^*)$ to be the coefficients of D_i associated to the point ν_i^*, i.e. by linear equivalence:

$$\begin{aligned} K(X_\Sigma) &= \Big\{-\sum_{i=0}^{3} a_i D_i \,\Big|\, -3a_0 + a_1 + a_2 + a_3 > 0\Big\} = \\ &= \Big\{-(a_0 - \tfrac{1}{3}a_1 - \tfrac{1}{3}a_2 - \tfrac{1}{3}a_3)D_0 \,\Big|\, -3a_0 + a_1 + a_2 + a_3 > 0\Big\} = \\ &= \mathbb{R}_> D_0 \end{aligned} \tag{2.9}$$

We can immediately read off the Mori cone from 2.8, it is generated by the single coefficient vector

$$\ell = (-3, 1, 1, 1) \tag{2.10}$$

One can think of the Mori cone as linear functionals on the divisors, that is as a cone of curves. The components of any generator are the intersections with the corresponding divisor.

The goal of this work will be to repeat this analysis for a different (and by far bigger) Calabi–Yau, given by a reflexive dual polyhedron ∇ with 18 integral points in 5 dimensions. This is an example for a class of Calabi–Yau fourfolds previously discussed by [10] where the dual polyhedron is relatively small (compared to the polyhedron Δ which contains 2861 integral points).

There are 4 points in the interior of codimension one faces. The remaining 14 points are labelled $\nu_0^*, \ldots, \nu_{13}^*$. This specific polyhedron is interesting because there are various [11] fibrations visible as projections and slices, for example the elliptic fibration $X = (\mathcal{E}, B^X)$ as the slice $(0,0,0,z_4,z_5)$ which is the dual polyhedron of $\mathbb{P}^{(1,2,3)}[6]$. The same polyhedron also appears as the projection onto the last two coordinates $(\widehat{z_1}, \widehat{z_2}, \widehat{z_3}, z_4, z_5)$, so the mirror manifold $\tilde{X} = (\tilde{\mathcal{E}}, B^{\tilde{X}})$ is also an elliptic fibration.

∇
(z_1,z_2,z_3,z_4,z_5)
(-1, 0, 0, 2, 3)
(0,-1, 0, 2, 3)
(0, 0,-1, 2, 3)
(0, 0,-1, 1, 2)
(0, 0, 0,-1, 0)
(0, 0, 0, 0,-1)
(0, 0, 0, 0, 0)
(0, 0, 0, 0, 1)
(0, 0, 0, 1, 1)
(0, 0, 0, 1, 2)
(0, 0, 0, 2, 3)
(0, 0, 1, 1, 2)
(0, 0, 1, 2, 3)
(0, 0, 2, 2, 3)
(0, 0, 1, 1, 1)
(0, 1, 2, 2, 3)
(0, 1, 3, 2, 3)
(1, 2, 4, 2, 3)

$$
\begin{aligned}
\nu_0^* &= (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0)\\
\nu_1^* &= (-1,\ \ 0,\ \ 0,\ \ 2,\ \ 3)\\
\nu_2^* &= (\ \ 0,-1,\ \ 0,\ \ 2,\ \ 3)\\
\nu_3^* &= (\ \ 0,\ \ 0,-1,\ \ 2,\ \ 3)\\
\nu_4^* &= (\ \ 0,\ \ 0,-1,\ \ 1,\ \ 2)\\
\nu_5^* &= (\ \ 0,\ \ 0,\ \ 0,-1,\ \ 0)\\
\nu_6^* &= (\ \ 0,\ \ 0,\ \ 0,\ \ 0,-1)\\
\nu_7^* &= (\ \ 0,\ \ 0,\ \ 0,\ \ 2,\ \ 3)\\
\nu_8^* &= (\ \ 0,\ \ 0,\ \ 1,\ \ 2,\ \ 3)\\
\nu_9^* &= (\ \ 0,\ \ 0,\ \ 2,\ \ 2,\ \ 3)\\
\nu_{10}^* &= (\ \ 0,\ \ 0,\ \ 1,\ \ 1,\ \ 1)\\
\nu_{11}^* &= (\ \ 0,\ \ 1,\ \ 2,\ \ 2,\ \ 3)\\
\nu_{12}^* &= (\ \ 0,\ \ 1,\ \ 3,\ \ 2,\ \ 3)\\
\nu_{13}^* &= (\ \ 1,\ \ 2,\ \ 4,\ \ 2,\ \ 3)
\end{aligned}
\tag{2.11}
$$

Chapter 3

Triangulations

The first step is to determine all fans. I do not know any algorithm to do that directly, however there is a way to systematically find all regular (see below) triangulations of a point set [12]. Since a triangulation is more general than the triangulation that follows from intersecting the dual polyhedron with the fan we will need to exclude "bad" triangulations.

Definition 3.1 *A triangulation is called a star triangulation if all maximal–dimensional simplices contain a common point. That is if T is the corresponding simplicial complex then there exists a vertex $v \in T$ such that $\overline{\mathrm{St}(v)} = T$*

In our case there is exactly one point (the origin) inside the convex hull of the point set. From the definition one can easily see that the facets of the maximal–dimensional simplices of the triangulation that do not contain the origin subdivide the facets of the convex hull, so there is a 1–1 correspondence between the the fans Σ that subdivide the cones over facets of the dual polyhedron, and star triangulations T_Σ.

Every such fan defines a toric variety, but for mirror symmetry we are interested in those that subdivide the polyhedron as far as possible.

Definition 3.2 *A triangulation of a points set A is called maximal if every point in A is a vertex of the triangulation.*

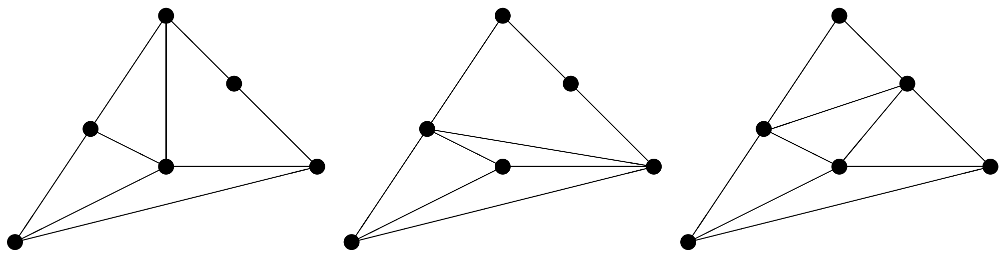

Figure 3.1: Different triangulations, from left to right: star and not maximal, nonstar and not maximal, nonstar and maximal

So we want to calculate all maximal star triangulations of the points of the dual polyhedron that are not in codimension one faces. We will also demand that the Kähler cone is not empty. This is equivalent to demanding regularity:

Definition 3.3 *A triangulation of a point set* $A = \{a_1, \ldots, a_k\}$, $a_i \in \mathbb{R}^d$ *is regular if it can be obtained by the lower convex hull of a* $d+1$*–dimensional polyhedron. That is choose* $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$*, and let* $W = \mathrm{conv}((a_1, \lambda_1), \ldots, (a_k, \lambda_k)) \subset \mathbb{R}^{d+1}$*. The triangulation is regular if for every maximal simplex* $\left\langle a_{i_1}, \ldots, a_{i_{d+1}} \right\rangle$ *there is a facet*

$$\left\langle (a_{i_1}, \lambda_{i_1}), \ldots, (a_{i_{d+1}}, \lambda_{i_{d+1}}) \right\rangle \supset W \tag{3.1}$$

such that the outer facet normal $\vec{n}$ *is pointing downward, i.e.* $(\vec{n})_{d+1} < 0$

Nonregular triangulations do exist, one example can be seen in Fig. 3.2. By adding a everywhere linear function one can assume that the value of a piecewise linear function is constant on the inner triangle. Then following the outer points clockwise, the function value must always be bigger that the

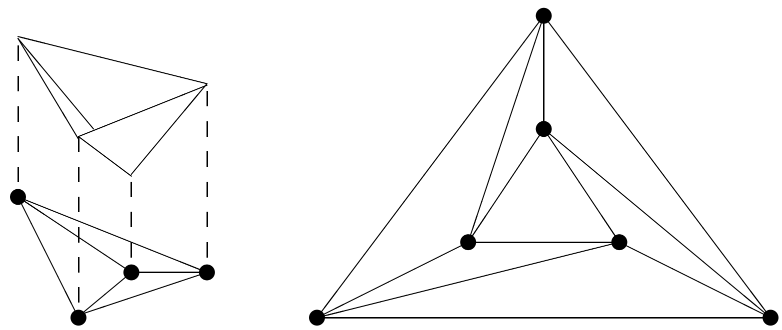

Figure 3.2: A regular (left) and a nonregular (right) triangulation

previous value such that the adjacent trapezoid is triangulated in the given way, leading to a contradiction. A star triangulation can also be nonregular, e.g. if the above nonregular triangulation appears in a face.

We will define a operation on the triangulations to get new triangulations out of known ones. For that we need the *circuits* (also known as primitive relations) of the point set put on a generic hyperplane (not through the origin). If the original point set is $\{a_1, \ldots, a_k\}$ we will use $\{\bar{a}_1, \ldots, \bar{a}_k\} = \{(1, a_1), \ldots, (1, a_k)\}$.

Definition 3.4 *A circuit $Z \subset A$ of the set of points $A = \{\bar{a}_1, \ldots, \bar{a}_k\}$ is a linear dependent subset such that every proper subset is linear independent. There is a (up to a constant) unique linear relation $\sum_i \lambda_i \bar{a}_i = 0$ between the points in Z, which divides it in the disjoint sets $Z_+ = \{\bar{a}_i | \lambda_i > 0\}$ and $Z_- = \{\bar{a}_i | \lambda_i < 0\}$. This is known as the circuits of the oriented matroid represented by A, see e.g. [13].*

For example let

$$\begin{aligned} A &= \{\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5\} \stackrel{\text{def}}{=} \\ &\stackrel{\text{def}}{=} \{(1,0,0,0),(1,0,2,0),(1,0,0,2),(1,0,2,2),(1,1,1,1)\} \end{aligned} \tag{3.2}$$

There is only one circuit $Z = Z_+ \cup Z_-$ with $Z_- = \{a_1, a_4\}$, $Z_+ = \{a_2, a_3\}$. Given a circuit there are always two triangulations of $\text{conv}(Z)$, called $t_\pm(Z) = \{Z - \{p\} | p \in Z_\pm\}$

Suppose A it is triangulated by the maximal–dimensional simplices $\{\langle a_1, a_2, a_4, a_5\rangle, \langle a_1, a_3, a_4, a_5\rangle\}$. Note that this is $t_+(Z)$ where a_5 is added to each simplex. The obvious operation on this triangulation given the circuit Z is to replace the subsets $t_+(Z)$ by $t_-(Z)$.

Or in general if for a triangulation T of A

$$\exists\, B \subset A - Z : \qquad \Big\{B \cup \tau | \tau \in t_+(Z)\Big\} \subset T \tag{3.3}$$

then we want to replace all such $\{B \cup \tau | \tau \in t_+(Z)\}$ by $\{B \cup \tau | \tau \in t_-(Z)\}$. If there is more than one such $B \subset A - Z$ we have to do this for all of them, or the result is no simplicial complex. And if there is a $B \subset A - Z$ such that $B \cup \tau \in T$ for one $\tau \in t_+(Z)$ but not for all then we cannot perform this operation.

Definition 3.5 *The triangulation T is supported on the circuit $Z \subset A$ if*

i)$t_+(Z) \subset T$

ii)If $\sigma \supset \tau$ for some maximal–dimensional $\sigma \in T$, $\tau \in t_+(Z)$ then

$$\Big\{(\sigma - \tau) \cup \tau' | \tau' \in t_+(Z)\Big\} \subset T \tag{3.4}$$

Definition 3.6 *If the circuit $Z \subset A$ supports the triangulation T, then there is another triangulation* $\mathrm{flip}_Z(T)$*, the flip of T, obtained by replacing all simplices of the form $B \cup \tau$, $\tau \in t_+(Z)$ by $B \cup \tau$, $\tau \in t_-(Z)$*

Then we use a Theorem by Gel'fand, Kapranov and Zelevinsky that all regular triangulations are connected by flips. There is only a finite number of flips one can perform on each triangulation (and of course the total number of triangulations of a finite point set is finite), so one can get all regular triangulations from a single regular triangulation (which is easy to find).

Now we can apply this algorithm to the integral points 2.11 of the dual polyhedron that are not in codimension one faces. For this special point set Ξ i will call the maximal triangulations the *simplified* triangulations, and the fans from maximal star triangulations the *simplified* fans.

Puntos [12] is a implementation of this algorithm in Maple. I rewrote the main loop in C++ and performed some minor optimisations. The modified version triangulates the points 2.11 in less than 30 minutes on a modern PC. I find 15176 triangulations, of which 2752 are maximal. 990 of all found and 165 of the maximal triangulations are star triangulations.

Using *Schubert* we can calculate the intersection numbers for each hypersurface in the 165 simplified fans. There are 7 distinct possibilities for the 330 intersection numbers. As we will see later the hypersurfaces are in fact isomorphic in each of the 7 classes. We are particularly interested in a special class of 20 Fans that contains 4 special triangulations that project nicely (see [11]).

Chapter 4

Calculating the Mori cone of X_Σ

Again let $\bar{\nu}_i^* = (1, \nu_i^*)$, $i = 0 \dots n$. Then we can rewrite the condition for $f \in \mathrm{cpl}(\Sigma)$ as

$$\forall \sigma \in \Sigma \quad \exists \overline{m}_\sigma = (c_\sigma, m_\sigma) \in \mathbb{R} \times M_{\mathbb{R}} : \qquad \begin{cases} f(x) = \langle \overline{m}_\sigma | \bar{x} \rangle & \forall x \in \sigma \\ f(x) > \langle \overline{m}_\sigma | \bar{x} \rangle & \forall x \notin \sigma \end{cases} \tag{4.1}$$

There are two useful ways to describe $f \in \mathrm{cpl}(\Sigma)$, either specify the $f(\nu_i^*) \stackrel{\text{def}}{=} u_i$ or specify a dual vector $\overline{m}_\sigma$ for each maximum–dimensional cone $\sigma \in \Sigma$. The first way guarantees piecewise linearity, and we need the second form to check convexity. For any such $\sigma = \left\langle \nu_{i_1}^*, \dots, \nu_{i_d}^* \right\rangle$ finding $\overline{m}_\sigma$ from given $u_0, \dots, u_n$ amounts to solving the linear system of equations

$$\begin{pmatrix} u_0 \\ u_{i_1} \\ \vdots \\ u_{i_d} \end{pmatrix} = \begin{pmatrix} \bar{\nu}_0^* \\ \bar{\nu}_{i_1}^* \\ \vdots \\ \bar{\nu}_{i_d}^* \end{pmatrix} \overline{m}_\sigma \tag{4.2}$$

Inserting the solution $\overline{m}_\sigma = \overline{m}_\sigma(u_0, \dots, u_n)$ in 4.1 gives a system of $n - d$ linear inequalities for each maximal–dimensional cone. In our example there are simplified triangulations with up to 58 maximal–dimensional simplices, which leads to a system of 464 linear inequalities. This system is highly degenerate, and can in all 165 cases be reduced to a system of 8 to 10 inequalities. Each Mori cone is nonempty (all 165 triangulations are regular) and $8 = n - d$ dimensional.

I use *cdd* [14] to find the essential inequalities. *cdd* implements the double description method, which is an algorithm to convert from the hyperplane representation

$$P = \left\{ Ax > b \,\middle|\, x \in \mathbb{R}^m \right\} \tag{4.3}$$

of any polytope P to the vertex/ray representation

$$P = \operatorname{conv}(\{v_1, \ldots, v_\nu\}) + \sum_{i=1}^{\mu} \mathbb{R} r_i \tag{4.4}$$

efficiently for degenerate systems of linear inequalities. I have written [15] Maple-functions to generate the input files for *cdd* and on top of that a program to calculate the Mori cone of X_Σ.

For example for the triangulation T where the maximal dimensional simplices are

$$
\begin{array}{llll}
\langle \nu_1^*,\nu_3^*,\nu_4^*,\nu_6^*,\nu_{11}^* \rangle & \langle \nu_1^*,\nu_3^*,\nu_4^*,\nu_7^*,\nu_{11}^* \rangle & \langle \nu_1^*,\nu_3^*,\nu_6^*,\nu_7^*,\nu_{11}^* \rangle & \langle \nu_1^*,\nu_5^*,\nu_6^*,\nu_{11}^*,\nu_{12}^* \rangle \\
\langle \nu_1^*,\nu_5^*,\nu_6^*,\nu_{10}^*,\nu_{12}^* \rangle & \langle \nu_1^*,\nu_4^*,\nu_5^*,\nu_6^*,\nu_{11}^* \rangle & \langle \nu_1^*,\nu_5^*,\nu_7^*,\nu_8^*,\nu_{12}^* \rangle & \langle \nu_1^*,\nu_6^*,\nu_7^*,\nu_{11}^*,\nu_{12}^* \rangle \\
\langle \nu_1^*,\nu_5^*,\nu_9^*,\nu_{10}^*,\nu_{12}^* \rangle & \langle \nu_1^*,\nu_5^*,\nu_8^*,\nu_9^*,\nu_{12}^* \rangle & \langle \nu_1^*,\nu_8^*,\nu_9^*,\nu_{10}^*,\nu_{12}^* \rangle & \langle \nu_2^*,\nu_4^*,\nu_5^*,\nu_7^*,\nu_{13}^* \rangle \\
\langle \nu_2^*,\nu_3^*,\nu_6^*,\nu_7^*,\nu_{13}^* \rangle & \langle \nu_2^*,\nu_3^*,\nu_4^*,\nu_7^*,\nu_{13}^* \rangle & \langle \nu_2^*,\nu_5^*,\nu_9^*,\nu_{10}^*,\nu_{13}^* \rangle & \langle \nu_2^*,\nu_5^*,\nu_7^*,\nu_8^*,\nu_{13}^* \rangle \\
\langle \nu_2^*,\nu_5^*,\nu_6^*,\nu_{10}^*,\nu_{13}^* \rangle & \langle \nu_2^*,\nu_8^*,\nu_9^*,\nu_{10}^*,\nu_{13}^* \rangle & \langle \nu_3^*,\nu_4^*,\nu_6^*,\nu_{11}^*,\nu_{13}^* \rangle & \langle \nu_3^*,\nu_4^*,\nu_7^*,\nu_{11}^*,\nu_{13}^* \rangle \\
\langle \nu_3^*,\nu_6^*,\nu_7^*,\nu_{11}^*,\nu_{13}^* \rangle & \langle \nu_4^*,\nu_5^*,\nu_6^*,\nu_{11}^*,\nu_{13}^* \rangle & \langle \nu_2^*,\nu_5^*,\nu_8^*,\nu_9^*,\nu_{13}^* \rangle & \langle \nu_1^*,\nu_2^*,\nu_4^*,\nu_5^*,\nu_7^* \rangle \\
\langle \nu_1^*,\nu_2^*,\nu_5^*,\nu_6^*,\nu_{10}^* \rangle & \langle \nu_1^*,\nu_2^*,\nu_5^*,\nu_7^*,\nu_8^* \rangle & \langle \nu_2^*,\nu_3^*,\nu_4^*,\nu_6^*,\nu_{13}^* \rangle & \langle \nu_1^*,\nu_2^*,\nu_3^*,\nu_4^*,\nu_7^* \rangle \\
\langle \nu_1^*,\nu_2^*,\nu_3^*,\nu_6^*,\nu_7^* \rangle & \langle \nu_2^*,\nu_4^*,\nu_5^*,\nu_6^*,\nu_{13}^* \rangle & \langle \nu_1^*,\nu_2^*,\nu_4^*,\nu_5^*,\nu_6^* \rangle & \langle \nu_1^*,\nu_2^*,\nu_3^*,\nu_4^*,\nu_6^* \rangle \\
\langle \nu_1^*,\nu_4^*,\nu_5^*,\nu_{11}^*,\nu_{12}^* \rangle & \langle \nu_1^*,\nu_4^*,\nu_7^*,\nu_{11}^*,\nu_{12}^* \rangle & \langle \nu_1^*,\nu_2^*,\nu_5^*,\nu_8^*,\nu_9^* \rangle & \langle \nu_1^*,\nu_2^*,\nu_5^*,\nu_9^*,\nu_{10}^* \rangle \\
\langle \nu_1^*,\nu_2^*,\nu_8^*,\nu_9^*,\nu_{10}^* \rangle & \langle \nu_4^*,\nu_7^*,\nu_{11}^*,\nu_{12}^*,\nu_{13}^* \rangle & \langle \nu_6^*,\nu_7^*,\nu_{10}^*,\nu_{12}^*,\nu_{13}^* \rangle & \langle \nu_7^*,\nu_8^*,\nu_{10}^*,\nu_{12}^*,\nu_{13}^* \rangle \\
\langle \nu_1^*,\nu_4^*,\nu_5^*,\nu_7^*,\nu_{12}^* \rangle & \langle \nu_4^*,\nu_5^*,\nu_7^*,\nu_{12}^*,\nu_{13}^* \rangle & \langle \nu_4^*,\nu_5^*,\nu_{11}^*,\nu_{12}^*,\nu_{13}^* \rangle & \langle \nu_1^*,\nu_2^*,\nu_7^*,\nu_8^*,\nu_{10}^* \rangle \\
\langle \nu_1^*,\nu_6^*,\nu_7^*,\nu_{10}^*,\nu_{12}^* \rangle & \langle \nu_1^*,\nu_7^*,\nu_8^*,\nu_{10}^*,\nu_{12}^* \rangle & \langle \nu_2^*,\nu_6^*,\nu_7^*,\nu_{10}^*,\nu_{13}^* \rangle & \langle \nu_2^*,\nu_7^*,\nu_8^*,\nu_{10}^*,\nu_{13}^* \rangle \\
\langle \nu_6^*,\nu_7^*,\nu_{11}^*,\nu_{12}^*,\nu_{13}^* \rangle & \langle \nu_8^*,\nu_9^*,\nu_{10}^*,\nu_{12}^*,\nu_{13}^* \rangle & \langle \nu_1^*,\nu_2^*,\nu_6^*,\nu_7^*,\nu_{10}^* \rangle & \langle \nu_5^*,\nu_6^*,\nu_{10}^*,\nu_{12}^*,\nu_{13}^* \rangle \\
\langle \nu_5^*,\nu_6^*,\nu_{11}^*,\nu_{12}^*,\nu_{13}^* \rangle & \langle \nu_5^*,\nu_7^*,\nu_8^*,\nu_{12}^*,\nu_{13}^* \rangle & \langle \nu_5^*,\nu_8^*,\nu_9^*,\nu_{12}^*,\nu_{13}^* \rangle & \langle \nu_5^*,\nu_9^*,\nu_{10}^*,\nu_{12}^*,\nu_{13}^* \rangle
\end{array}
\tag{4.5}
$$

the Mori cone is generated by

$$
\begin{array}{rcl}
(\;\; 0,\; 0,\; 0,\; 1,\; 0,\; 0,\; 0,-1,\; 0,\; 0,\; 0,-1,\; 1,\; 0) & = & a^{(1)} \\
(\;\; 0,\; 0,\; 0,\; 0,\; 0,\; 0,\; 0,\; 1,-2,\; 1,\; 0,\; 0,\; 0,\; 0) & = & a^{(2)} \\
(\;\; 0,\; 0,\; 0,\; 0,\; 0,\; 0,-1,-1,\; 0,\; 0,\; 2,\; 2,-2,\; 0) & = & a^{(3)} \\
(\;\; 0,\; 0,\; 0,\; 0,-3,\; 1,\; 0,\; 2,\; 0,\; 0,\; 0,\; 3,-3,\; 0) & = & a^{(4)} \\
(-2,\; 0,\; 0,-2,\; 2,\; 0,\; 1,\; 1,\; 0,\; 0,\; 0,\; 0,\; 0,\; 0) & = & a^{(5)} \\
(-3,\; 0,\; 0,\; 0,\; 0,\; 1,\; 0,\; 0,\; 1,-2,\; 3,\; 0,\; 0,\; 0) & = & a^{(6)} \\
(\;\; 0,\; 0,\; 0,\; 0,\; 3,-1,\; 0,-5,\; 3,\; 0,\; 0,\; 0,\; 0,\; 0) & = & a^{(7)} \\
(\;\; 0,\; 0,\; 0,\; 0,\; 0,\; 0,\; 1,-1,\; 2,\; 0,-2,\; 0,\; 0,\; 0) & = & a^{(8)} \\
(\;\; 0,\; 0,\; 1,\; 0,\; 0,\; 0,\; 0,\; 0,-1,-1,\; 0,\; 0,\; 1,\; 0) & = & a^{(9)} \\
(\;\; 0,\; 1,\; 0,\; 0,\; 0,\; 0,\; 0,\; 0,\; 0,\; 0,\; 0,-2,\; 0,\; 1) & = & a^{(10)}
\end{array}
\qquad (4.6)
$$

One can check that $a^1, \ldots, a^{10}$ span a 8 dimensional hyperplane, so one can choose a basis for that hyperplane by picking 8 entries among the 14. One possible choice is to pick $(a_8, a_4, a_5, a_9, a_{10}, a_{12}, a_{13}, a_{14})$, this corresponds to the divisors

$$(D_7, D_3, D_4, D_8, D_9, D_{11}, D_{12}, D_{13}) = (B, C_1, C_2, E_1, E_2, F, G, Y) \qquad (4.7)$$

in the notation of [11].

But we are interested in the Mori cone of the Calabi–Yau hypersurface, and this gives us the Mori cone of the ambient space. Since not all curves $C \in X_\Sigma$ are contained in the hypersurface V, we see from 2.1 that the Kähler cone $K(X_\Sigma) \supset K(V)$. That is the Mori cone is too big. The idea is to find all Mori cones from isomorphic hypersurfaces, and calculate their intersection.

Chapter 5

Calculating the Mori cone of the Calabi–Yau hypersurface

In this section we try to find isomorphic Calabi–Yau hypersurfaces to a given $V \subset X_\Sigma$. The intersection of the individual Mori cones will give us a smaller Mori cone, which turned out to be the real Mori cone of V in previous examples.

Assume T is a simplified star triangulation supported by a circuit Z, such that $T' \stackrel{\text{def}}{=} \mathrm{flip}_Z(T)$ is again a simplified star triangulation ($\Rightarrow \bar{\nu}_0^* \notin Z$). Then if Σ, Σ' are the corresponding simplified fans we write $\Sigma' = \mathrm{flip}_Z(\Sigma)$. On the toric side this transformation is a flop,of that is the blowdown of a subvariety $\cap_{\bar{\nu}_i^* \in Z_-} D_i$ of X_Σ (each simplex of $t_+(Z)$ contains Z_-) followed by the blowup of the corresponding singularity to $\cap_{\bar{\nu}_i^* \in Z_+} D'_i$ of $X_{\Sigma'}$. A sufficient condition for this not to change the Calabi–Yau hypersurface V is that only points disjoint of V are changed. We call the flop and the corresponding flip trivial.

Definition 5.1 flip_Z *is called a trivial flip of a triangulation T supported on Z if*

$$\cap_{\bar{\nu}_i^* \in Z_-} D_i \cap V = \emptyset \tag{5.1}$$

The problem is to determine all trivial flips for a given triangulation. There is a well-known object, the Stanley–Reisner ideal of the triangulation.

It is generated by all sets of points ν_i^* that are not on a common simplex:

$$SR_T = \Big\langle\ \Big\{ \nu_{i_1}^* \cdots \nu_{i_m}^* \ \Big|\ \Big\langle \nu_{i_1}^*, \dots, \nu_{i_m}^* \Big\rangle \notin T \Big\}\ \Big\rangle \subset \mathbb{C}[\nu_0^*, \dots, \nu_n^*] \tag{5.2}$$

Since we are interested in star triangulations we can remove the interior point ν_0^*, i.e. set the corresponding variable in the ideal to 0. From eq. 2.3 then follows that $x_{i_1} = \ldots = x_{i_m} = 0$ is excluded in X_Σ if $\Big\langle \nu_{i_1}^*, \dots, \nu_{i_m}^* \Big\rangle \notin T$. But we identified $x_i = 0$ with the divisor D_i, so $D_{i_1} \cap \cdots \cap D_{i_m} = \emptyset$.

This allows us to define the toric version of the Stanley–Reisner ideal SR_Σ, whose monomials are the divisors that do not intersect at all (so especially not on V):

$$SR_\Sigma = \Big\langle\ \Big\{ D_{i_1} \cdots D_{i_m} \ \Big|\ \Big\langle \nu_{i_1}^*, \dots, \nu_{i_m}^* \Big\rangle \notin \Sigma - \{\langle \nu_0^* \rangle\} \Big\}\ \Big\rangle \subset \mathbb{C}[D_1, \dots, D_k] \tag{5.3}$$

But this is not sufficient to identify trivial flips. If a circuit Z is supported then $t_+(Z)$ are subsimplices, but each simplex of $t_+(Z)$ contains by definition Z_-. Thus $Z_- \notin SR_T$.

We need to identify divisors that do intersect, but not on the hypersurface. By inspecting eq. 2.7 we see that if $x_i = 0$ then only monomials that correspond to facets of ∇ (dual to vertices of Δ) containing ν_i^* are nonzero. If there is a intersection $D_{i_1} \cap \ldots \cap D_{i_r} \leftrightarrow x_{i_1} = \ldots = x_{i_r} = 0$ such that only one monomial $p \sim x_{j_1}^{a_1} \cdots x_{j_s}^{a_s}$ survives then one of $x_{j_1}, \dots, x_{j_s}$ must be zero for $D_{i_1} \cap \ldots \cap D_{i_r} \cap V \neq \emptyset$ (if the coefficients in the original polynomial were generic). But this may be impossible if

$$D_{i_1} \cdots D_{i_s} D_j \in SR_\Sigma \qquad \forall j = j_1, \dots, j_s \tag{5.4}$$

So for each Fan one needs to calculate the Stanley–Reisner ideal and perform the trivial flips. For our polyhedron (eq. 2.11) the vertices of Δ and corresponding facets of ∇ are

$$
\begin{array}{lcl}
\nu_1 = (-14, \ \ 2, \ 2, -1, \ 1) & \leftrightarrow & \langle \nu_2^*, \nu_3^*, \nu_4^*, \nu_6^*, \nu_{13}^* \rangle \\
\nu_2 = (\ \ 0, \ \ 3, -3, \ 1, \ 1) & \leftrightarrow & \langle \nu_5^*, \nu_6^*, \nu_9^*, \nu_{10}^*, \nu_{12}^*, \nu_{13}^* \rangle \\
\nu_3 = (\ \ 6, \ \ 6, -3, \ 1, \ 1) & \leftrightarrow & \langle \nu_1^*, \nu_2^*, \nu_5^*, \nu_6^*, \nu_9^*, \nu_{10}^* \rangle \\
\nu_4 = (\ \ 0, \ \ 0, \ 0, \ 1, -1) & \leftrightarrow & \langle \nu_1^*, \nu_2^*, \nu_3^*, \nu_4^*, \nu_5^*, \nu_7^*, \nu_8^*, \nu_9^*, \nu_{11}^*, \nu_{12}^*, \nu_{13}^* \rangle \\
\nu_5 = (\ \ 0, \ \ 0, \ 0, -2, \ 1) & \leftrightarrow & \langle \nu_1^*, \nu_2^*, \nu_3^*, \nu_6^*, \nu_7^*, \nu_8^*, \nu_9^*, \nu_{10}^*, \nu_{11}^*, \nu_{12}^*, \nu_{13}^* \rangle \\
\nu_6 = (\ \ 6, -6, \ 0, \ 1, \ 1) & \leftrightarrow & \langle \nu_1^*, \nu_5^*, \nu_6^*, \nu_{11}^*, \nu_{12}^*, \nu_{13}^* \rangle \\
\nu_7 = (\ \ 6, \ \ 3, -3, \ 1, \ 1) & \leftrightarrow & \langle \nu_1^*, \nu_5^*, \nu_6^*, \nu_9^*, \nu_{10}^*, \nu_{12}^* \rangle \\
\nu_8 = (-6, \ \ 6, -3, \ 1, \ 1) & \leftrightarrow & \langle \nu_2^*, \nu_5^*, \nu_6^*, \nu_9^*, \nu_{10}^*, \nu_{13}^* \rangle \\
\nu_9 = (\ \ 6, \ \ 6, \ 4, \ 1, \ 1) & \leftrightarrow & \langle \nu_1^*, \nu_2^*, \nu_4^*, \nu_5^*, \nu_6^* \rangle \\
\nu_{10} = (\ \ 2, \ \ 2, \ 2, -1, \ 1) & \leftrightarrow & \langle \nu_1^*, \nu_2^*, \nu_3^*, \nu_4^*, \nu_6^* \rangle \\
\nu_{11} = (\ \ 6, -14, \ 4, \ 1, \ 1) & \leftrightarrow & \langle \nu_1^*, \nu_4^*, \nu_5^*, \nu_6^*, \nu_{11}^*, \nu_{13}^* \rangle \\
\nu_{12} = (\ \ 2, -6, \ 2, -1, \ 1) & \leftrightarrow & \langle \nu_1^*, \nu_3^*, \nu_4^*, \nu_6^*, \nu_{11}^*, \nu_{13}^* \rangle \\
\nu_{13} = (-34, \ \ 6, \ 4, \ 1, \ 1) & \leftrightarrow & \langle \nu_2^*, \nu_4^*, \nu_5^*, \nu_6^*, \nu_{13}^* \rangle
\end{array}
$$

For example take the circuit

$$\bar{\nu}_6^* + 2\bar{\nu}_8^* - \bar{\nu}_7^* - 2\bar{\nu}_{10}^* = 0 \qquad \Rightarrow \qquad Z = Z_+ \cup Z_- = \{\bar{\nu}_6^*, \bar{\nu}_8^*\} \cup \{\bar{\nu}_7^*, \bar{\nu}_{10}^*\} \quad (5.5)$$

This circuit is actually supported on the triangulation 4.5. The Stanley–Reisner ideal is

$$
\begin{aligned}
SR_\Sigma \ = \ \Big\langle & D_1D_{13}, D_2D_{11}, D_2D_{12}, D_3D_5, D_3D_8, D_3D_9, D_3D_{10}, D_3D_{12}, D_4D_8, \\
&, D_4D_9, D_4D_{10}, D_6D_8, D_6D_9, D_7D_9, D_8D_{11}, D_9D_{11}, D_{10}D_{11}, D_4D_6D_7, \\
&, D_4D_6D_{12}, D_5D_6D_7, D_5D_7D_{10}, D_5D_7D_{11}, D_5D_8D_{10}\Big\rangle
\end{aligned} \qquad (5.6)
$$

Note that the points $Z_- = \{\bar{\nu}_7^*, \bar{\nu}_{10}^*\}$ are both only on the facet of ∇ dual to ν_5. So on the intersection $D_7 \cap D_{10}$ (that is $x_7 = x_{10} = 0$), the defining polynomial for V reduces to a monomial in x_4, x_5. But $D_4 D_{10} \in SR_\Sigma \Rightarrow x_4 \neq 0$ and $D_5 D_7 D_{10} \in SR_\Sigma \Rightarrow x_5 \neq 0$. The circuit 5.5 therefor leads to a trivial flip. Applying the same reasoning to all circuits on can find 4 trivial flips, which are in addition to 5.5:

$$2\bar{\nu}_{10}^* + 2\bar{\nu}_{11}^* - \bar{\nu}_6^* - \bar{\nu}_7^* - 2\bar{\nu}_{12}^* = 0 \tag{5.7}$$

$$3\bar{\nu}_4^* + 3\bar{\nu}_8^* - \bar{\nu}_5^* - 5\bar{\nu}_7^* = 0 \tag{5.8}$$

$$\bar{\nu}_5^* + 2\bar{\nu}_7^* + 3\bar{\nu}_{11}^* - 3\bar{\nu}_4^* - 3\bar{\nu}_{12}^* = 0 \tag{5.9}$$

By trivial flips i can generate all 20 triangulations with the same intersection numbers from a single triangulation. The intersection of their Mori cones is

$$\begin{array}{rcl}
(\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,-1,-1,\ 0,\ 0,\ 1,\ 0) = \ell^{(1)} & = & D_8 D_9 D_{13} \\
(-2,\ 0,\ 0,-2,\ 2,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0) = \ell^{(2)} & = & D_3 D_{11} D_{13} \\
(-2,\ 0,\ 0,\ 1,-1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0) = \ell^{(3)} & = & \frac{1}{2} D_4 D_{11} D_{13} \\
(\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,-2,\ 1,\ 0,\ 0,\ 0,\ 0) = \ell^{(4)} & = & D_8 D_{11} D_{13} \\
(-3,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,-2,\ 3,\ 0,\ 0,\ 0) = \ell^{(5)} & = & D_9 D_{11} D_{13} \\
(\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,-2,\ 0,\ 0,\ 0) = \ell^{(6)} & = & \frac{1}{3} D_{10} D_{11} D_{13} \\
(\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,-1,\ 1,\ 0,\ 0,\ 1,-1,\ 0) = \ell^{(7)} & = & D_7 D_{11} D_{13} \\
(\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,-1,\ 0,\ 0,\ 0,-1,\ 1,\ 0) = \ell^{(8)} & = & D_7 D_{12} D_{13} \\
(\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,-2,\ 0,\ 1) = \ell^{(9)} & = & D_7 D_{11} D_{12} \\
(-1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,-1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0) = \ell^{(10)} & = & ?
\end{array} \tag{5.10}$$

We can find curves corresponding to $\ell^{(1)}, \ldots, \ell^{(9)}$ so these generators must be contained in the true Mori cone. Since they are already edges they cannot be in the interior, so they must be edges of the true Mori cones.

However 5.10 cannot be the true Mori cone since the curve ℓ^{10} is not in the hypersurface: The curve is contained in D_7 since $D_7 \cdot \ell^{10} = (\ell^{10})_8 = -1$, but it also intersects D_4 and D_{10}. One can check — using similar arguments as above — that $D_4 \cap D_7 \cap D_{10} \cap V = \emptyset$ in all 20 varieties, and therefor $\ell^{10} \not\subset V$. For example if the triangulation is 4.5 we have already seen that $D_7 \cap D_{10} \cap V = \emptyset$.

Chapter 6

Conclusion

This calculation is based on a bigger polyhedron than any similar calculation i know of. We find the first example where the intersection of the Mori cones for all simplified fans is not simplicial, and not the Mori cone of the Calabi–Yau manifold.

For a threefold we can use this to generate the initial data necessary for *Instanton* automatically from the dual polyhedron.

BIBLIOGRAPHY

[1] S. Hosono, A. Klemm, S. Theisen, S.–T. Yau, *Mirror Symmetry and Applications to Calabi–Yau Hypersurfaces*, Commun. Math. Phys. **167** (1995) 301—350, hep-th/9308122

[2] S. Hosono, A. Klemm, S. Theisen, S.–T. Yau, *Mirror Symmetry, Mirror Map and Applications to Complete Intersections in Calabi–Yau Spaces*, Nucl. Phys. B **433** (1995) 501—552, hep-th/9406055

[3] Sheldon Katz, private communication

[4] H. Skarke, *String Dualities and Toric Geometry: An Introduction*, hep-th/9806059

[5] D. Cox, *Recent Developments in Toric Geometry*, alg-geom/9606016

[6] D. Cox, *The Homogenous Coordiante Ring of a Toric Variety*, alg-geom/9210008

[7] V. Batyrev, D. Cox, *On the Hodge Structure of Projective Hypersurfaces in Toric Varieties*, alg-geom/9306011

[8] V. Batyrev, *Dual polyhedra and mirror symmetry*, J. Alg. Geom **3** (1994) 493—535

[9] T. Oda, *Convex Bodies and Algebraic Geometry*, Springer–Verlag (1988)

[10] A. Klemm, B. Lian, S.–S. Roan, S.–T. Yau, *Calabi–Yau fourfolds for M– and F–Theory compactifications*, hep-th/9701023

[11] P. Candelas, *Notes on Duality Between $N = 1$ Theories*, unpublished

[12] J. de Loera, *Puntos*, http://www.math.okstate.edu/~katz/CY

[13] A. Bachem, K. Walter, *Linear Programming Duality: An introduction to oriented Matroids*, Springer–Verlag (1992)

[14] K. Fukuda, *cdd*, http://www.ifor.math.ethz.ch/ifor/staff/fukuda/cddman/cddman.html

[15] V. Braun, *Useful Programs for Toric Geometry*, http://zippy.ph.utexas.edu/~vrbraun

VITA

Volker Friedrich Braun was born in Bamberg, Germany on July 10, 1975, the eldest son of Joachim Braun and Roswitha Braun. From 1981 to 1985 he attended the Grundschule (elementary school), then the Dientzenhofer Gymansium (Highschool/College), Bamberg, Germany. He completed the Gymnasium with the Abitur in 1994, and entered the University of Wüerzburg with the Wintersemester 1994/95. In the Summer of 1996 he passed the Vordiplom. In August 1997 he moved to Texas and entered The Graduate School at The University of Texas.

Permanent address: Spielleite 4
96170 Priesendorf
Germany

This thesis was typeset[1] with LaTeX by the author.

[1] LaTeX document preparation system was developed by Leslie Lamport as a special version of Donald Knuth's TeX program for computer typesetting. TeX is a trademark of the American Mathematical Society. The LaTeX macro package for The University of Texas at Austin thesis format was written by Khe-Sing The.