Abstracts

Computation of Some K-groups

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1. Introduction

By now a well-established result is that the D-brane charges in string theory are precisely the K-theory group of the space-time, see [1]. Hence, computing certain K-groups has immediate physical interest. For example, cancellation of the total D-brane charge for compact directions places additional restrictions on allowed compactifications, which eliminates some torus orientifold constructions.

In this talk, I will review the computation of the twisted K-theory that is relevant for $N = 1$ supersymmetric Wess-Zumino-Witten models. I solved the case for compact, simple, simply connected Lie groups in [2]. As a non-simply connected example, I will present $SO(3)$ in Section 3. The latter is joint work with Sakura Schäfer-Nameki [3]

2. Twisted K-theory for Lie Groups

In the following, let G always be a compact, simple, simply connected Lie group, together with a gerbe on G with characteristic class

$$t \in H^3\big(G; \mathbb{Z}\big)\,. \tag{1}$$

The corresponding Grothendieck group of twisted vector bundles on G is the twisted K-theory ${}^tK(G)$. It is a generalized (twisted) cohomology theory. To compute the K-groups, we relate it to equivariant twisted K-theory by rewriting

$$ {}^tK^*(G) = {}^tK^*_G(G^{\mathrm{Tr}} \times G^{\mathrm{L}}) = {}^tK^*_G(G^{\mathrm{Ad}} \times G^{\mathrm{L}})\,, \tag{2}$$

where the superscripts refer to the **T**rivial, **L**eft, and **Ad**joint action of G on itself. The first equality is obvious, the second follows from the G-isomorphism $G^{\mathrm{Tr}} \times G^{\mathrm{L}} = G^{\mathrm{Ad}} \times G^{\mathrm{L}}$ through conjugation. To compute the K-theory of the product, we use a certain equivariant Künneth theorem which follows from [4]:

Theorem 1 (Equivariant Künneth Theorem)**.** *Let G be a compact, simple, simply connected Lie group. Let X be a G-space with twist class, let Y be a G-space. Then there is a spectral sequence*

$$E_2^{-p,*} = Tor^p_{RG}\Big({}^tK^*_G(X),\ K^*_G(Y)\Big) \Rightarrow {}^tK^{p+*}_G(X)\,. \tag{3}$$

The point of doing so is that we can now apply the theorem of Freed-Hopkins-Teleman [5], which identifies the twisted equivariant K-theory with the Verlinde algebra at level $k = t - \check{h}$,

$$ {}^tK^*_G(G^{\mathrm{Ad}}) = RG/I_k\,. \tag{4}$$

Hence, it remains to compute

$$Tor^p_{RG}\left({}^tK^*_G(G^{\mathrm{Ad}}),\ K^*_G(G^{\mathrm{L}})\right) = Tor^p_{RG}\left(RG/I_k,\ \mathbb{Z}\right). \tag{5}$$

A widely believed fact is that the Verlinde algebra is a complete intersection, and hence there exists a Koszul resolution. Although not strictly proven, this was checked for a large number of cases in [6]. Henceforth, I assume that there exists a regular sequence $y_1, \ldots, y_n$, $n = rk(G)$. A bit of homological algebra yields

$$Tor^p_{RG}\left(RG/I_k,\ \mathbb{Z}\right) = Tor^p_{RG}\left(RG/\langle y_1, \ldots, y_n\rangle,\ \mathbb{Z}\right) = \bigoplus_{2^{n-1}} \mathbb{Z}_{\gcd(y_1,\ldots,y_n)}. \tag{6}$$

Finally, what about higher differentials and extension ambiguities? The dual K-homology spectral sequence is a spectral sequence of algebras under the Pontryagin product. One can use this to show that there are no further differentials, and that all extension ambiguities are trivial. Hence,

$${}^tK^*(G) = \bigoplus_{2^{n-1}} \mathbb{Z}_{\gcd(y_1,\ldots,y_n)}. \tag{7}$$

3. $SO(3)$ Wess-Zumino-Witten Model

As an example of a non-simply connected Lie group, let us consider $SO(3)$. This Wess-Zumino-Witten (WZW) model was treated from the boundary conformal field theory side in [7], where it was found that the D-brane charge groups is either $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$ depending on whether $\kappa \stackrel{\text{def}}{=} k+1$ is odd or even. Interestingly, the charge groups do not grow with the level in this example. This is in contradiction to the usual Atiyah-Hirzebruch spectral sequence, which predicts ${}^kK^*\big(SO(3)\big) = \mathbb{Z}_2 \oplus \mathbb{Z}_k$. Our resolution to this paradox is that D-brane charges in the $SO(3)$ WZW model, that is the bosonic $SO(3)$ supersymmetrized with free fermions, correspond to another twisted K-theory. Recall that the possible twists of K-theory actually contain

$$H^1\Big(SO(3);\,\mathbb{Z}_2\Big) \oplus H^3\Big(SO(3);\,\mathbb{Z}\Big) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}. \tag{8}$$

The WZW model of [7] corresponds to the $(-,\kappa)$ twisted K-theory! We can easily estimate the resulting K-groups from a twisted Atiyah-Hirzebruch spectral sequence

$$E_2 = {}^-H^p\Big(SO(3);\,K^q(\mathrm{pt.})\Big) \Rightarrow {}^{(-,\kappa)}K^{p+q}\big(SO(3)\big). \tag{9}$$

to be either $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$, depending on an extension ambiguity.

To resolve this ambiguity, we again rewrite the K-groups as certain equivariant K-groups. But since the Künneth theorem fails for non-simply connected groups, we chose to work $SU(2)$ equivariant, and obtain

$${}^tK^*\big(SO(3)\big) = {}^tK^*_{SU(2)}\Big(SO(3)^{\mathrm{Ad}} \times SU(2)^{\mathrm{L}}\Big) \tag{10}$$

We found the twisted equivariant K-groups ${}^{t}K^{*}_{SU(2)}\big(SO(3)^{\mathrm{Ad}}\big)$ by a Mayer-Vietoris sequence for a certain cell decomposition, whose details I am going to skip. The result is that

$$
\begin{aligned}
{}^{(-,\kappa)}K^{0}_{SU(2)}\Big(SO(3)\Big) &= 0 \\
{}^{(-,\kappa\ \mathrm{odd})}K^{1}_{SU(2)}\Big(SO(3)\Big) &= \mathbb{Z}[\Lambda,\sigma]\Big/\Big\langle {\scriptstyle \Lambda(\sigma-1),\,\sigma^2-1,\,p_\kappa(\Lambda)} \Big\rangle \\
{}^{(-,\kappa\ \mathrm{even})}K^{1}_{SU(2)}\Big(SO(3)\Big) &= \mathbb{Z}[\Lambda,\sigma]\Big/\Big\langle {\scriptstyle \Lambda(\sigma-1),\,\sigma^2-1,\,p_\kappa(\Lambda)+(-1)^{\frac{\kappa}{2}}(1+\sigma)} \Big\rangle
\end{aligned}
\tag{11}
$$

as $RSU(2) = \mathbb{Z}[\Lambda]$ modules, where p_κ are certain degree κ polynomials. A bit of homological algebra then shows that only the Tor^0 in the equivariant Künneth theorem is nonvanishing, and moreover that

$$
{}^{(-,\kappa)}K^{*}\Big(SO(3)\Big) = E_2^{0,*} = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \kappa \text{ odd} \\ \mathbb{Z}_4 & \kappa \text{ even,} \end{cases}
\tag{12}
$$

as predicted by the boundary conformal field theory.

References

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