Minimal Models and K-theory

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There are only certain values of the central charge which can occur in a $N = 2$ unitary CFT, they must be $c \in \{\frac{3k}{k+2}, k \in \mathbb{Z}_{\geq}\} \cup [3, \infty)$. The discrete values $c < 3$ are rational with respect to the $N = 2$ chiral algebra, the so-called minimal models[1]. There are 3 widely used constructions for these CFTs.

- The coset $\dfrac{\mathfrak{su}(2)_k \oplus \mathfrak{u}(1)_2}{\mathfrak{u}(1)_{k+2}}$ with diagonal modular invariant.
- MM_k, the $W = x^{k+2} + y^2$ LG model.
- $MM_k/\mathbb{Z}_2$, the $W = x^{k+2}$ LG model.

The D-brane charges for the coset model are well understood. They can either be determined by renormalization group flow, or as equivariant K-groups for twisted complex K-theory [5, 6, 1], where the twist class is related to the level k.

$$ {}^tK^0_{U(1)}\Big(SU(2)\Big)^{\mathbb{Z}_2} = 0\,, \quad {}^tK^1_{U(1)}\Big(SU(2)\Big)^{\mathbb{Z}_2} = \mathbb{Z}^{k+1}\,. \tag{1} $$

Especially, they are torsion free abelian groups. Now recently Hori [2] argued that the $W = x^{k+2}$ Landau-Ginsburg (LG) model allows for non-vanishing charges for the B type D-branes. We can verify that by an honest K-theory computation. In summary, there are the following D-brane charge groups.

[1] In the following, I will only consider the A type minimal models for simplicity. The D and E type can be treated similarly.

Model	coset	MM_k	$MM_k/\mathbb{Z}_2$
K(A-branes)	$\mathbb{Z}^{k+1}$	$\mathbb{Z}^{k+1}$	$\mathbb{Z}^{k+1}$
K(B-branes)	0	0	$\mathbb{Z}_{k+2}$

Obviously, the MM_k and $MM_k/\mathbb{Z}_2$ are different CFTs, and it is well known that one can be obtained from the other as $\mathbb{Z}_2$ orbifold.

The LG B-branes can be understood as follows. If one were to use the usual LG action on worldsheets with boundary, then the supersymmetry variation is not zero but a boundary term. To cancel this and obtain an $\mathcal{N} = 2$ SCFT one must add a boundary action. A popular ansatz [3] contains a choice of matrix factorization

$$\phi_0, \phi_1 \in \mathrm{Mat}(n, R) : \quad W \cdot \mathbf{1}_n = \phi_0\phi_1 = \phi_1\phi_0 \,, \tag{2}$$

where $R \stackrel{\text{def}}{=} \mathbb{C}[\bar{z}]$ is the polynomial ring in the LG fields. Different factorizations yield different boundary theories, and hence describe different D-branes. This can be formalized to a category **MF** whose objects are 2-periodic complexes

$$\cdots \xrightarrow{\phi_0} (R/W)^n \xrightarrow{\phi_1} (R/W)^n \xrightarrow{\phi_0} \cdots \,, \tag{3}$$

and maps are ordinary chain maps. In analogy with the usual B-model on a Calabi-Yau manifold X, **MF** plays the role of $\mathrm{Coh}(X)$. This category needs to be extended to a triangulated category, and Kontsevich proposed the category **DB**. It can be obtained as the homotopy category of **MF**, or as the stable category associated to **MF**. This category then plays the role analogous to the derived category $D(\mathrm{Coh}\, X)$. Orlov [4] showed that $\mathbf{DB} \simeq D_{\text{sg}}\big(\{W = 0\}\big)$, which gives a nice geometrical interpretation as sheaves on the singularity.

One can [7] identify **MF** with the category of Cohen-Macaulay modules over R/W, which gives a computationally useful way to understand the matrix factorizations. Here one must mod out the trivial matrix factorizations and the trivial module R/W, but we will ignore this subtlety in the following. The Auslander-Reiten (AR) quivers of the module categories are known. Especially, the AR quiver for $W = x^n + y^2$ is the $\mathbb{Z}_2$ orbifolds of the $W = x^n$ AR quiver. The $\mathbb{Z}_2$ action fixes one of the modules if n is even, and acts freely if n is odd. In the former case one has to add an extra module, corresponding to a twisted sector. In any case, one can easily compute the Grothendieck group of the module category, which I already listed in the beginning.

The true importance of the minimal models is that they serve as building blocks for string theory compactifications. For this, one has to construct a suitable $c = 9$ SCFT and then impose the GSO projection. This can be archived by tensoring minimal models, a construction is known as Gepner models. For example, the $(k = 3)^5$ Gepner model corresponds to the Fermat quintic. We can check that

$$ {}^tK^i_{U(1)^5\times\mathbb{Z}_5}\Big(SU(2)^5\Big)^{(\mathbb{Z}_2)^5} \otimes_{\mathbb{Z}} \mathbb{C} = K^i(\text{Quintic}) \otimes_{\mathbb{Z}} \mathbb{C} = \begin{cases} \mathbb{C}^{204} & i = 1 \\ \mathbb{C}^4 & i = 0\,. \end{cases} \tag{4} $$

Of course, such an identity should be lifted to an equivalence of derived categories.

References

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