# K–Theory and Exceptional Holonomy in String Theory

D I S S E R T A T I O N

zur Erlangung des akademischen Grades
doctor rerum naturalium
(Dr. rer. nat.)
im Fach Physik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät I
der Humboldt-Universität zu Berlin

von
Herrn Volker Friedrich Braun
geboren am 10. Juli 1975 in Bamberg

Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. J. Mlynek

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät I:
Prof. Dr. B. Ronacher

Gutachter:

1. Prof. Dr. A. Klemm
2. Prof. Dr. H. Kurke
3. Prof. Dr. D. Lüst

| | |
|---|---|
| eingereicht am: | 24. April 2002 |
| Tag der mündlichen Prüfung: | 22. Juli 2002 |

## Abstract

In this thesis I consider various aspects of string theory compactifications, especially for nontrivial internal manifolds.

The first part is dedicated to the application of K–theory to the study of D–branes. It is the generalized cohomology theory which classifies the possible charges on a given spacetime. A natural question is whether there is any difference between K–theory and the usual description via (de Rahm) cohomology/homology. For this I present a Calabi–Yau manifold which illustrates this difference.

Instead of compactifying on a complicated smooth manifold one can also consider orbifolds of simple manifolds to get interesting compactifications. These are described by equivariant K–theory. To be able to compare this with the physical prediction I calculate all $KO_{\mathbb{Z}_2}(\mathbb{R}^{p,q})$. Furthermore one can consider orientifolds, which suggests the definition of new K–theories. I investigate simple properties of these.

In the second part I present compactifications on $G_2$ and Spin(7) manifolds and their description as Gepner models. The SCFT and the geometric description disagree. An explanation for this phenomenon is offered.

**Keywords:**
D–branes, K–theory, Gepner models, Exceptional holonomy

**Abstract**

In dieser Arbeit beschreibe ich verschiedene Aspekte der Kompaktifizierung der String Theorie, insbesondere auf nichttrivialen Mannigfaltigkeiten.

Im ersten Teil betrachte ich K–Theorie und ihre Anwendung in der Untersuchung von D–Branen. Es handelt sich um eine verallgemeinerte Kohomologietheorie welche die möglichen Ladungen für eine gegebene Raumzeitmannigfaltigkeit klassifiziert. Eine natürliche Fragestellung ist inwiefern sich diese Beschreibung von der üblichen mit (de Rahm) Kohomologie/Homologie unterscheidet. Hierzu gebe ich eine Calabi–Yau Mannigfaltigkeit an die den Unterschied illustriert.

Anstatt der Kompaktifizierung auf einer komplizierten glatten Mannigfaltigkeit kann man auch Orbifolds von einfachen Mannigfaltigkeiten studieren um interessante Kompaktifizierungen zu erhalten. Dies wird mit äquivarianter K–Theorie beschrieben. Um dies mit physikalischen vorhersagen zu vergleichen berechne ich alle $KO_{\mathbb{Z}_2}(\mathbb{R}^{p,q})$. Darüberhinaus kann man Orientifolds betrachten, diese führen auf die Definition von neuen K–Theorien. Ich beschreibe einfache Eigenschaften dieser Theorien.

Im zweiten Teil präsentiere ich Kompaktifizierungen auf $G_2$ und Spin(7) Mannigfaltigkeiten und ihre Beschreibung als Gepner Modelle. Die SCFT und die geometrische Beschreibung unterscheiden sich, und ich gebe eine Erklärung für dieses Phänomen.

**Sclagwörter:**
D–Branen, K–Theorie, Gepner Modelle, Exeptionelle Holonomie

# Contents

# List of Figures

# List of Tables

# Part I

# K–theory and D–branes

# Chapter 1

# Introduction

## 1.1 Superstring theory

String theory is currently the best candidate for a unified theory of gravity and all fundamental interactions. Now originally this was considered to be the quantum theory of a string (a one dimensional object) moving through spacetime, with interactions coming from splitting and joining strings (see figure 1.1). However today we know from dualities that string theory not only

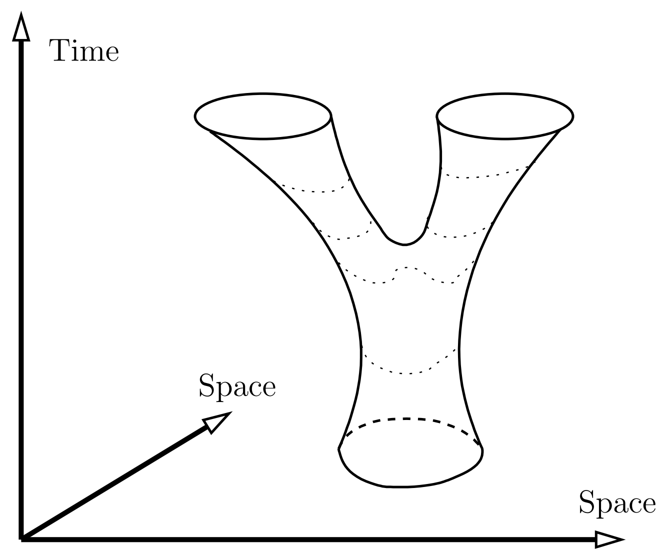


Figure 1.1: String propagating through spacetime

contains the 2 dimensional string worldsheets, but various extended objects of different dimensions. Unfortunately we are still unable to quantize the full theory (with all extended objects) from first principles, but rather rely on investigating how the strings interact with other objects.

Specifically I will be interested in D–branes, that is the boundary conditions of open strings. So in this sense a D-$p$–brane is a fixed submanifold $Z$ (with $p$ spacial and one time direction) of the spacetime $X$; The objects of the string theory are then maps $f : \Sigma \to X$ (with $\Sigma$ a Riemann surface with boundary) such that $f(\partial\Sigma) \subset Z$.

For concreteness consider Type IIB string theory in the presence of a stack of $n$ D–branes in a 10 dimensional spacetime $X$. This is a theory of open and closed strings. The massless bosonic spectrum is in table 1.1. Now we certainly want a nontrivial Riemannian manifold as spacetime, not

| Origin | Field | Name | Type |
|---|---|---|---|
| NS–NS sector | $\Phi$ | Dilaton | scalar |
| | $G_{\mu\nu}$ | Graviton | symmetric 2 tensor |
| | $B_{\mu\nu}$ | B–field | 2-form |
| R–R sector | $C^{(0)}$ | | 0-form |
| | $C^{(2)}$ | | 2-form |
| | $C^{(4)}$ | | 4-form |
| | $C^{(6)}$ | | 6-form |
| | $C^{(8)}$ | | 8-form |
| open string | $A_\mu$ | Gauge field on the brane | $u(n)$ valued 1-form |

Table 1.1: Massless bosonic spectrum in IIB theory

just flat $\mathbb{R}^{10}$. So we demand that at least the metric $G_{\mu\nu}$ has a nontrivial background value. We also want nontrivial $A_\mu$, then the obvious guess is to allow $G_{\mu\nu} \neq 0$, $A_\mu \neq 0$ and demand that all other fields vanish. Suppose you are given such fields then one can consider string theory in this background.

There are various ways to investigate the string theory. Here I will consider the *nonlinear sigma model* approach. This amounts to the following action for a $f \in \mathrm{Map}(\Sigma, \partial\Sigma; X, Z)$ (for simplicity consider $A = A_\mu \mathrm{d}x^\mu$ scalar valued):

$$\begin{aligned} S[f] &= \int_\Sigma \mathrm{d}^2 Vol(G) + \int_{\partial\Sigma} A + S_{\text{fermions}} \\ &= \int_\Sigma \mathrm{d}^2 Vol(G) + \int_\Sigma \mathrm{d}A + S_{\text{fermions}} \end{aligned} \tag{1.1}$$

Here and in the following I will not distinguish between forms on $X$ or the brane $Z$ and their pullback to the world sheet via the map $f : \Sigma \to X$. Moreover I will restrict myself for simplicity to the bosonic part of the action.

The action in eq. 1.1 is invariant under $A \mapsto A + \mathrm{d}\Lambda$:

$$S_{A+\mathrm{d}\Lambda}[f] = S_A[f] + \int_\Sigma \mathrm{d}^2\Lambda = S_A[f] \tag{1.2}$$

so we identify $A$ with a gauge field, that is a connection on a $U(n)$ gauge bundle.

So really D–branes are specified by

1. a submanifold $Z \subset X$.

2. a $U(n)$ gauge bundle on $Z$.

When can one deform one set of D–branes into another? Obviously we expect that "continuous deformation" (homotopy) preserves the basic properties of the D–brane. However homotopy alone is not enough to classify physically different D–branes, as we will see in the next sections.

### D–branes and R–R charge

D–branes can be BPS solutions, that is partially preserve supersymmetry. The prime example are two parallel D-$p$–branes ($p$ odd for Type IIB) in flat $\mathbb{R}^{10}$. As a consequence of the BPS property the setup is stable, that is there is no force between the branes.

In string theory of course one has to calculate the force between the branes by analyzing the amplitude for a string being exchanged between the two branes, see figure 1.2. Now intuitively there has to be an attractive force between the branes, since everything must gravitationally attract every other object. So for the net force to be zero there must be another interaction that cancels the gravitational force.

In the calculation of the amplitudes the graviton contributes as one of the NS–NS sector modes. Their contribution is just canceled by the R–R sector modes. So the D–branes must carry charges for the R–R sector fields, and the repulsion of these charges is precisely what cancels the gravitational attraction.

But if the D–brane is characterized by the property that it is the source for the R–R sector fields then should not the different charges correspond to the cohomology classes of the field strengths $dC^{(p)}$? Certainly we should be able to "deform" setups with the same quantum numbers (the same charges) into another. This seems to be a very different picture of D–branes than submanifolds + gauge bundles. These two seemingly different points of view will be reconciled later by K–theory.

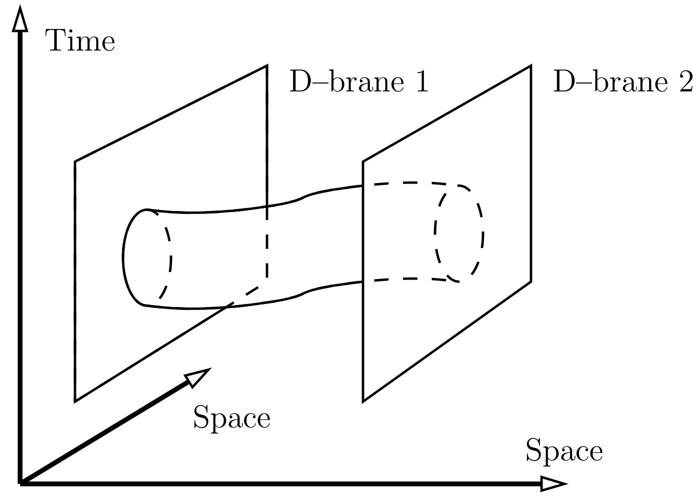


Figure 1.2: String exchange of two D–branes

## 1.2 Sen's Conjecture

So we know that D–branes are objects with conserved quantum numbers (R–R charges). The question is still what are all possible charges, and of course without a fundamental description there cannot be a "proof" of what the correct description is. However there is a nice description (see [55, 56]) that incorporates all the features above, and which will therefore the basis for everything that follows:

**Conjecture 1 (Sen).** *Every D-p–brane is the decay product of D9–, $\overline{D9}$–branes.*

So we really only need to consider stacks of spacetime filling branes, this automatically includes all lower dimensional branes as special field configurations. The charges are then classified by

$$\Big\{\text{D–brane charges}\Big\} \overset{1-1}{\updownarrow} \Big\{\text{stacks of D9–, }\overline{\text{D9}}\Big\} \Big/ \text{pair creation \& annihilation} \tag{1.3}$$

## 1.3 D–branes and K–theory

What topological information is stored in a spacetime filling D–brane? Of course it is the gauge bundle. In Type IIB string theory this means that the stack of D9– and $\overline{\text{D9}}$ branes describes really two $U(n_i)$ gauge bundles. Moreover there is a natural way to "add" gauge bundles, which corresponds to adding another stack of D–branes. So pair creation is just the addition of the same gauge bundle to the branes and antibranes and we find (see [64]):

$$\begin{array}{c} \Big\{\text{D–brane charges}\Big\} \\ \Big\updownarrow {\scriptstyle 1-1} \\ \Big\{\text{pairs of gauge bundles } (E,F)\Big\} \Big/ (E,F) \sim (E \oplus H, F \oplus H) \end{array} \tag{1.4}$$

To a $U(n)$ gauge bundle we may associate a vector bundle and vice versa, so instead of gauge bundles we could have talked about vector bundles everywhere. The addition of the gauge bundles is the Whitney sum of vector bundles. For the reader's benefit all those terms will be explained in the following chapters, together with much machinery to actually compute the K–groups for (hopefully) interesting spaces.

# Chapter 2

# Vector Bundles

In this chapter I will introduce the notion of a vector bundle and describe a few basic properties. All this material is well–known but included in an attempt to give a self–contained presentation. I will focus on the real case instead of starting with complex bundles because it allows to visualize simple cases.

## 2.1 Real Vector Bundles

Suppose you are given a (topological) space $X$. Then a vector bundle on $X$ is a vector space over each point $x \in X$ "varying continuously". To make this more precise we require that

- All the vector spaces fit together into the *total space* $E$.
- Locally (in a neighborhood $x \in U \subset X$) the bundle looks like $U \times \mathbb{R}^n$ for some $n$.

So with other words, a vector bundle on the base space $X$ consists of the total space $E$ and a continuous map $\pi : E \to X$ such that the preimage of a point $x \in X$ (the *fiber*) is a vector space. Moreover for each point $x \in X$ there is a neighborhood $x \in U \subset X$ such that $\pi^{-1}(U) \simeq U \times \mathbb{R}^n$ for some $n \in \mathbb{Z}_{\geq}$.

The fiber of $E$ at the point $x \in X$ is also denoted $E_x$, and its dimension (which is constant if $X$ is connected) is called the *rank* of $E$: $\dim(E_x) = \mathrm{rk}(E)$ . Since we will also consider complex vector bundles we will write $\mathrm{rk}_{\mathbb{R}}$ or $\mathrm{rk}_{\mathbb{C}}$ if there is any doubt. A bundle $E$ with $\mathrm{rk}(E) = 1$ is also called a line bundle.

Finally we want to define what a map from one vector bundle $E \to X$ to another $F \to X$. This will give us a notion of "isomorphism", that is when are two vector bundles the same. Of course maps have to preserve the property that the fiber is a vector space. So we define a map $f : E \to F$ (here $E$ and $F$ denote the bundle) as a continuous map of the total spaces (by abuse of notation also denoted $f : E \to F$) that carries fibers into fibers ($f(E_x) \subset F_x$) and is a linear map on the fibers:

$$f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2) \quad \forall\ \lambda_1, \lambda_2 \in \mathbb{R}, v_1, v_2 \in E_x \tag{2.1}$$

The two bundles are isomorphic (denoted $E \simeq F$) if $f$ admits an inverse map.

The simplest example is a vector space considered as a vector bundle over a point. The isomorphism classes are simply labeled by the rank.

Another silly example is the bundle $X \times \mathbb{R}^n$ over any space $X$. It is called the trivial bundle.

Now the easiest nontrivial example is for the base space $X = S^1$. Here we have two different line bundles: The trivial line bundle $S^1 \times \mathbb{R} \to S^1$ and the Möbius strip $M \to S^1$ (that is if you continue the transverse direction of the strip indefinitely). Those two bundles are not isomorphic: Think of the $S^1$ being included in the total space of the bundle as the origin of each fiber. Then $M - S^1$ is connected and $S^1 \times \mathbb{R} - S^1 \times \{0\}$ is not, while an isomorphism would preserve the connectedness.

Now one might think that the "double twisted" line bundle is again a new line bundle since one cannot untwist it. But that is only a speciality of the embedding into $\mathbb{R}^3$. The line bundle itself is trivial as you could either see by embedding it into $\mathbb{R}^4$ or by the following construction: Cut the bundle at one fiber, then rotate one end by $2\pi$ and glue the ends again. This operation does not change the bundle but obviously turns the "double twisted" line bundle into the trivial bundle (provided you rotate in the right direction). The identification that you so get with $S^1 \times \mathbb{R}$ is an isomorphism.

Another inportant example is the tangent bundle of a smooth manifold $X$, denoted $TX$: This is the vector bundle whose fiber is the tangent space at a given point. For example $TS^1 = S^1 \times \mathbb{R}$ (in fact $S^1$, $S^3$ and $S^7$ are the only spheres with trivial tangent bundle).

### The Pullback

There is an important property of vector bundles with respect to continuous maps of the base space. More precisely suppose that you are given a vector bundle $E \to Y$ and a continuous map $f : X \to Y$. Then you can form the pullback bundle $f^*(E)$ over $X$ where the fiber over $x \in X$ is $E_{f(x)}$. Note that $f^*$ is "the other way round": it maps vector bundles on $Y$ to those with base $X$.

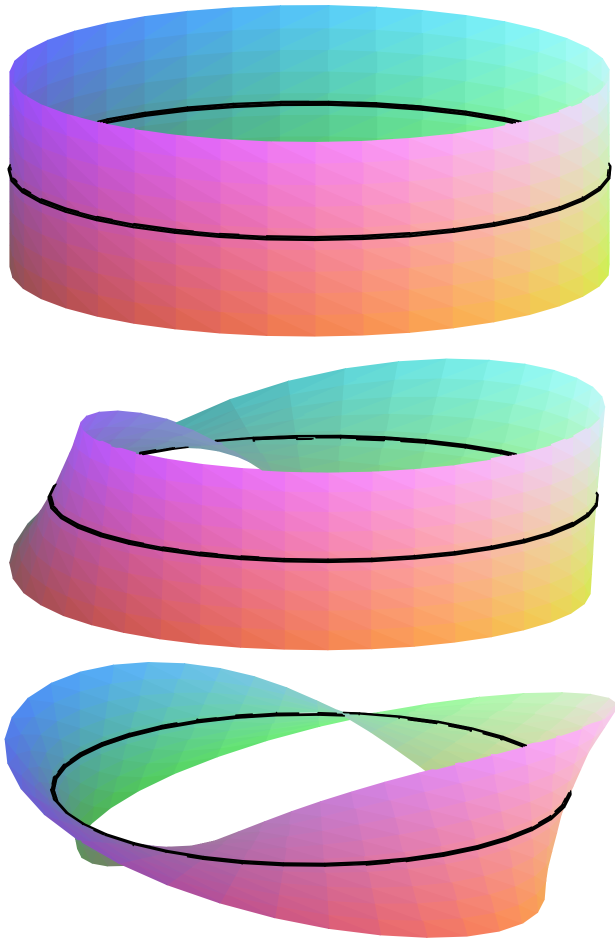


Figure 2.1: Vector bundles on $S^1$

## 2.2 Transition functions

Here is another way to understand a vector bundle $E \to X$ of rank $\mathrm{rk}(E) = n$. Take the base space $X$ and cover it with sufficiently small open sets $U_i$, such that $\pi^{-1}(U_i) \simeq U_i \times \mathbb{R}^n$. Now pick a local trivialization $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$. Over any point $x \in X$ the trivializations differ by a linear map of the fiber, that is on each double overlap $U_{ij} \stackrel{\text{def}}{=} U_i \cap U_j$ there is a matrix–valued function

$$g_{ij} : U_{ij} \to O(n) \quad \text{such that} \qquad \varphi_j \circ \varphi_i^{-1}\big(x, v\big) = \big(x, g_{ij}(x)v\big) \tag{2.2}$$

(In general the $g_{ij}$ are $GL(n, \mathbb{R})$ valued, but you can always choose the trivialisations such that the $g_{ij}$ are orthogonal)

But not any set of such functions gives rise to a vector bundle. Rather they have to "fit together" on multiple overlaps since going from one trivialization to another must not depend on intermediate steps:

$$g_{ij}(x) = g_{ji}(x)^{-1} \quad \forall\, x \in U_{ij} \tag{2.3}$$

$$g_{ik}(x) = g_{ij}(x) g_{jk}(x) \quad \forall\, x \in U_{ijk} \overset{\text{def}}{=} U_i \cap U_j \cap U_k \tag{2.4}$$

In fact also the converse holds: a set of functions satisfying eqns 2.3, 2.4 are the transition functions of some vector bundle.

**Vector bundles on Spheres**

So define

$$\mathrm{Vect}(X) = \{\text{vector bundles on } X\} \Big/ \simeq \tag{2.5}$$

the isomorphism classes of vector bundles, and by $\mathrm{Vect}_n(X)$ the isomorphism classes of rank $n \in \mathbb{Z}_{\geq}$.

As a special case consider $X = S^d = \mathring{D}^d_+ \cup \mathring{D}^d_-$, the union of two open disks overlapping in an annulus $A = S^1 \times \mathbb{R}$ around the equator. Now by a homotopy you can make the annulus arbitrarily thin and therefore you can assume that any transition function is constant in the perpendicular direction of $A$. So really the transition function for a rank $n$ vector bundle is $\varphi_\pm : S^1 \to O(n)$.

Moreover since a vector bundle on the disk alone is always trivial all vector bundles on $S^d$ come from such a $\varphi_\pm$. Of course homotopic $\varphi_\pm$ yields the same vector bundle; Therefore

$$\mathrm{Vect}_n(S^d) = \pi_{d-1}\Big(O(n)\Big) \tag{2.6}$$

For example $\mathrm{Vect}_1(S^1) = \pi_0(O(1)) = \mathbb{Z}_2$ (as sets), so $\mathrm{Vect}_1(S^1)$ has two elements. Those are precisely the trivial line bundle and the "Möbius strip" line bundle from figure 2.1.

## 2.3 Whitney Sum

From what we saw so far we can define the set of isomorphism classes of vector bundles. The purpose of this section is to define an operation on the vector bundles that will give this set a semigroup structure, that is an associative binary operation.

The *Whitney sum* $E \oplus F$ of two vector bundles $E \to X$, $F \to X$ is the vector bundle over $X$ with fiber $(E \oplus F)_x = E_x \oplus F_x\ \forall x \in X$. What does that mean in terms of transition functions? Well let $g_{ij}$ be transition functions describing $E$ and $h_{ij}$ describing $F$. Then

$$(g \oplus h)_{ij} = \begin{pmatrix} g_{ij} & \\ & h_{ij} \end{pmatrix} \in O\big(\operatorname{rk}(E) + \operatorname{rk}(F)\big) \tag{2.7}$$

satisfies again the requirements for a transition function and so defines $E \oplus F$. It is obviously associative. Then $\operatorname{Vect}(X)$ is a semigroup via

$$+ : \operatorname{Vect}(X) \times \operatorname{Vect}(X) \to \operatorname{Vect}(X), \quad [E] + [F] = [E \oplus F] \tag{2.8}$$

Moreover exchanging the two summands in $E \oplus F$ is an isomorphism (the isomorphism is just the permutation matrix acting pointwise on the fiber $(E \oplus F)_x$), so $\operatorname{Vect}(X)$ is an abelian semigroup.

As a trivial example let $E = X \times \mathbb{R}^n$ and $F = X \times \mathbb{R}^m$ then $E \oplus F = X \times \mathbb{R}^{n+m}$. So as a semigroup $\operatorname{Vect}(\{pt\}) = \mathbb{Z}_{\geq}$. Because of this we simply write $n$ for the trivial rank $n$ bundle.

A more interesting example is the following: Remember the "Möbius strip" bundle $M \to S^1$, see figure 2.1. We have $M \oplus M = S^1 \times \mathbb{R}^2$, the rank two trivial bundle. How can one see that? Think of the two Möbius strips overlaid in one picture, the second rotated by $\pi$ (such that the two fiber directions are everywhere perpendicular in $\mathbb{R}^3$). Then the sum $M \oplus M$ is the bundle over $S^1$ with fiber the $\mathbb{R}^2$ perpendicular to the $S^1 \subset \mathbb{R}^3$. But there is a different family of bases for the fibers that does not "wind around" if you follow the $S^1$, for example take the radial and a fixed axial directon of the circle. The map between the two bases is an isomorphism between $M \oplus M$ and $S^1 \times \mathbb{R}^2$.

### Nowhere vanishing Sections

A *section* of a vector bundle $E \to X$ is continuously varying choice of vector from each fiber. So with other words it is a map $s : X \to E$ such that $\pi \circ s = \mathrm{id}_X$. For example a vector field is a section of the tangent bundle.

Now every vector bundle has a section, for example the zero section being the zero vector over each point. However not every bundle has a nowhere vanishing section: For example the tangent bundle of the two sphere $TS^2$ has not, this is known as "you cannot comb the hair of a coconut" or the Poincaré Hopf index theorem.

But if you have a nowhere vanishing section then it generates a line subbundle, and moreover the nowhere vanishing section is really an isomorphism of this bundle with the trivial line bundle. So the original bundle $E \to X$ decomposes as $E = F \oplus 1$.

Especially if you have an rank $n$ vector bundle $E \to X$ and the base space is an $d < n$ dimensional manifold then you can always find a nowhere vanishing section: Just take an arbitrary section and perturb it a little bit to get rid of the zeroes. This proves the following:

**Theorem 1.** *Let $E \to X$ a rank $n$ vector bundle and $\dim(X) = d < n$. Then there exists a vector bundle $F \to X$, $\mathrm{rk}(F) = d$ such that $E = F \oplus (n - d)$.*

So finally we can determine $\mathrm{Vect}(S^1)$ completely: every bundle is the sum of a trivial bundle and one of the two possible line bundles. Let $\theta = S^1 \times \mathbb{R}$ and $M$ the "Möbius strip" then

$$\mathrm{Vect}(S^1) = \left\langle M, \theta \right\rangle_{\mathbb{Z}_\geq} \Big/ (2\theta = 2M) \tag{2.9}$$

the abelian semigroup generated by $M$, $\theta$ modulo the relation that we found in the above example.

## 2.4 Multiplication

Finally there is another operation on vector bundles that will be important in the following. This operation is again induced from some operation on the fibers, just as in the previous section. Given two vector spaces $V_1$, $V_2$ you can form their tensor product $V_1 \otimes V_2$ which is again a vector space of dimension $(\dim V_1)(\dim V_2)$. So tensoring the fibers over each point you get the tensor product $E \otimes F$ of two vector bundles $E \to X$ and $F \to X$.

Now the tensor product of vector spaces is distributive over direct sum of vector spaces, and therefore vector bundles inherit the same property:

$$E \otimes (F_1 \oplus F_2) = E \otimes F_1 \oplus E \otimes F_2 \tag{2.10}$$

In terms of tranisition functions the tensor product is a little bit awkward to formulate: if $g_{ij} : U_{ij} \to O(n)$ and $h_{ij} : U_{ij} \to O(m)$ are two transition functions then

$$(g \otimes h)_{ij} : U_{ij} \to O(nm), \quad x \mapsto \Big( g_{ij}(x)_{uv} h_{ij}(x)_{xy} \Big)_{(u,x)(v,y)} \tag{2.11}$$

thinking of index pairs labelling the coordinates of the tensor product.

A useful special case is the tensor product of a line bundle $L \to X$ with a vector bundle $E \to X$. If $g_{ij} : U_{ij} \to O(1)$ and $h_{ij} : U_{ij} \to O(n)$ are their transition functions then the transition function for the product is simply

$$(g \otimes h)_{ij} : U_{ij} \to O(n), \quad x \mapsto g_{ij}(x) h_{ij}(x) \tag{2.12}$$

Especially the tensor product with the trivial line bundle leaves the vector bundle invariant.

As a more interesting example take the "Möbius strip" line bundle $M \to S^1$. If you cover the $S^1$ as usual by two open intervals intersecting in a small annulus over the equator then you can take the transition funcitons to be locally constant, that is $\pm 1$ over the two connected components of the annulus. The transition functions of the tensor product are then always $+1$ by eq. 2.12. Therefore $M \otimes M = \theta$, the trivial line bundle.

So the semigroup with product structure on $\mathrm{Vect}(S^1)$ can be summarized as follows, compare equation 2.9:

$$\mathrm{Vect}(S^1) = \mathbb{Z}_{\geq}\big[M\big]\Big/(2M = 2, M^2 = 1) \tag{2.13}$$

# Chapter 3

# K–theory

## 3.1 Grothendieck group construction

Given any abelian semigroup one can construct an abelian group by introducing formal differences — just in the same way as you first learn about integers as formal differences of nonnegative numbers. The group thus associated with the semigroup $\mathrm{Vect}(X)$ will be K–theory $KO(X)$ .

Let us look at this construction in more details (based on appendix $G$ of [63]). Suppose you are given a commutative semigroup $S$ with operation $+$, then we would like to define formal differences by

$$a-b=x-y \quad \overset{\text{def}}{\Longleftrightarrow} \quad a+y=x+b \qquad \text{(wrong!)} \tag{3.1}$$

What is the problem? We would not have transitivity! For example:

$$\begin{gathered}\underbrace{\begin{array}{ccc} a-b=u-w & \text{and} & u-w=x-y \\ \Updownarrow & & \Updownarrow \\ a+v=u+b & & u+y=x+w \end{array}} \\ \Rightarrow a+y+(u+v)=x+b+(u+v)\end{gathered} \tag{3.2}$$

and this does not imply that $a+y=x+b \Leftrightarrow a-b=x-y$. So instead define

$$a-b=x-y \quad \overset{\text{def}}{\Longleftrightarrow} \quad \exists s \in S: \quad a+y+s=x+b+s \tag{3.3}$$

This definition of equality is now transitive, reflexive and symmetric. The group that you thus get is called the *Grothendieck group* $\mathfrak{G}(S)$ of $S$, and it satisfies the universal property

**Theorem 2.** *For each group $G$ and homomorphism $\phi : S \to G$ there is a unique $\psi$ such that*

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[dr] & G \\ & \mathfrak{G}(S) \arrow[u, "\psi"'] \end{tikzcd} \tag{3.4}$$

Now K–theory $KO(X) = \mathfrak{G}(\mathrm{Vect}(X))$ is the abelian group of formal differences $[E] - [F]$ of isomorphism classes of vector bundles. In fact we can give a slightly simpler description:

**Theorem 3 (Swann).** *Every vector bundle $V \in \mathrm{Vect}(X)$ is a summand of a trivial bundle.*

**Corollary 1.** *Each element $x \in KO(X)$ can be written as*

$$x = [V] - [n], \quad n \in \mathbb{Z}_{\geq} \tag{3.5}$$

**Examples**

The simplest example is again $X = \{pt\}$ where $KO(\{pt\}) = \mathbb{Z} = \mathfrak{G}(\mathbb{Z}_{\geq})$ .

Now for a more interesting example take $X = S^1$ where we determined all vector bundles in eq. 2.9. By a choice of basis you can take $\theta$ and $M - \theta$ as generators of $KO(S^1)$, subject to the single relation $2(M - \theta) = 0$. Therefore

$$KO(S^1) = \mathbb{Z} \oplus \mathbb{Z}_2 \tag{3.6}$$

Furthermore you can define a multiplication on $KO(X)$ from the multiplication in $\mathrm{Vect}(X)$, so $KO(X)$ is even a ring. The ring structure can be written as ($x = [M]$):

$$\begin{aligned} KO(S^1) &= \mathbb{Z}\left[x\right]\Big/(2x = 2, x^2 = 1) = \\ &= \mathbb{Z}\left[x\right]\Big/\big(2(x-1) = 0, (x-1)^2 = 0\big) \\ &= \mathbb{Z}\left[y\right]\Big/\big(2y = 0, y^2 = 0\big) \end{aligned} \tag{3.7}$$

## 3.2 Compact support

So far I only considered compact spaces, but now take any (maybe non-compact) space $X$. Then we could define $KO(X)$ just as before as the

Grothendieck group of arbitrary vector bundles. However there are two problems with that: Physically we would like some "finite action" condition, and not allow brane configurations that spread throughout the whole space (like a lattice of evenly spaced periodic branes). Mathematically such "infinite" vector bundles would make the K–theory a lot less interesting since they will allow to push nontrivial "twists" of the bundle away to infinity.

The solution is to define K–theory as "K–theory with *compact support*": If $[E]-[F] \in KO(X)$ then we require that there exists a compact $U \subset X$ such that $E|_{X-U} \simeq F|_{X-U}$. It is easy to see that Whitney sum and tensor product of compactly supported vector bundles have again compact support. Moreover if $X$ itself is compact then we get no restriction,

So if $X$ is noncompact then especially $\mathrm{rk}(E)=\mathrm{rk}(F)$. On the other hand side define the *virtual rank*

$$\mathrm{rk}: KO(X) \to \mathbb{Z}, \quad [E]-[F] \to \mathrm{rk}(E)-\mathrm{rk}(F) \tag{3.8}$$

then for any compact space $X$ we have $KO(X)=\ker(\mathrm{rk})\oplus\mathbb{Z}$. Since the $\mathbb{Z}$ summand is not very interesting define

$$\widetilde{KO}(X) = \ker\Big(\mathrm{rk}: KO(X) \to \mathbb{Z}\Big) \tag{3.9}$$

It is called the *reduced K–theory*.

In a concrete string model of course one usually wants to describe D–branes that are localized in space but not in time, so one should think of spacetime as $X \times \mathbb{R}$ and then demand compact support in $X$–direction but not in $\mathbb{R}$–direction. Then of course one can simply contract the time direction and the D–brane charges are just $K(X)$.

Alternatively one might be interested in D–branes extended in various noncompact directions, either as local description of the situation above or as infinitely extended object. Then one has to ask for compact support in the transverse directions, and no restrictions in the parallel directions.

## 3.3 Stabilization

Suppose you have two different vector bundles $E \to X$, $F \to X$. What can be said about their classes $[E],[F] \in KO(X)$? It turns out that they might be equal, even though the vector bundles are not isomorphic. This phenomenon is really at the heart of K–theory: In general it is impossible to determine $\mathrm{Vect}(X)$, but $KO(X)$ carries less information which makes it possible to actually determine it.

To see this consider the tangent bundle $TS^2$ on the sphere $S^2$. We add to it the trivial line bundle $\theta$. Think of $\theta$ as the normal bundle from the usual embedding of $S^2$ in $\mathbb{R}^3$. Then $TS^2 \oplus \theta$ is the vector bundle with fiber $\mathbb{R}^3$ over any point $x \in S^2$. Think of the fiber as the tangent space in the embedding space $\mathbb{R}^3$ at the point $x \in S^2 \subset \mathbb{R}^3$. Then $TS^2 \oplus \theta$ is just $T\mathbb{R}^3$ restricted to the $S^2 \subset \mathbb{R}^3$. But $T\mathbb{R}^3$ is trivial and therefore also $TS^2 \oplus \theta$. So we found:

$$\begin{aligned}[TS^2] = [TS^2] + [\theta] - [\theta] = [TS^2 \oplus \theta] - [\theta] = \\ = [\theta \oplus \theta \oplus \theta] - [\theta] = [\theta \oplus \theta] \quad \in KO(S^2)\end{aligned} \tag{3.10}$$

K–theory does not distinguish between the tangent bundle $TS^2$ and the rank 2 trivial bundle $\theta \oplus \theta$ — while they are clearly not isomorphic, for example $TS^2$ does not have any nowhere vanishing sections.

So $KO(X)$ does only know about the vector bundles "up to addition of other vector bundles", and this is really less information than in $\mathrm{Vect}(X)$. We can reformulate this slightly with the help of theorem 3: It suffices to add trivial bundles.

$$[E] = [F] \in KO(X) \quad \Leftrightarrow \quad \exists n \in \mathbb{Z}_{\geq} : E \oplus n \simeq F \oplus n \tag{3.11}$$

Allowing to add sufficiently large trivial bundles is called stabilization, and K–theory classifies stable isomorphism classes of vector bundles.

# Chapter 4

# From real to complex Bundles

## 4.1 Complex Vector Bundles

Just as one can define vector bundles with fibers $\mathbb{R}^n$ one can also define complex vector bundles. Of course they are not so easily visualized since the real dimension is often too high, but on the other hand side they enjoy nicer properties that will aid in calculations.

From the transition function point of view we get $GL(n,\mathbb{C})$ valued transition functions, and as in the real case one can without limiting the generality choose them to be norm preserving, i.e. unitary:

$$g_{ij} : U_{ij} \to U(n) \quad \text{such that} \qquad \varphi_j \circ \varphi_i^{-1}\big(x,v\big) = \big(x, g_{ij}(x)v\big) \tag{4.1}$$

satisfying

$$g_{ij}(x) = g_{ji}(x)^{-1} \quad \forall\, x \in U_{ij} \tag{4.2}$$

$$g_{ik}(x) = g_{ij}(x)g_{jk}(x) \quad \forall\, x \in U_{ijk} \overset{\text{def}}{=} U_i \cap U_j \cap U_k \tag{4.3}$$

The Whitney sum and tensor product for complex vector bundles can be defined analogously to the real case; Denote the ensuing semigroup $\mathrm{Vect}_{\mathbb{C}}(X)$ . If there is any chance of confusion denote the semigroup of real vector bundles by $\mathrm{Vect}_{\mathbb{R}}(X)$ .

Given any semigroup we can again form its Grothendieck group, so define

$$K(X) \overset{\text{def}}{=} \mathfrak{G}\Big(\mathrm{Vect}_{\mathbb{C}}(X)\Big) \tag{4.4}$$

Swann's theorem 3 has its complex analog and so we can write

$$\begin{aligned} K(X) &= \Big\{[E]-[F]\Big| E,F \in \mathrm{Vect}_{\mathbb{C}}(X)\Big\} \\ &= \Big\{[E]-[n]\Big| E \in \mathrm{Vect}_{\mathbb{C}}(X), n \in \mathbb{Z}_{\geq}\Big\} \end{aligned} \tag{4.5}$$

where $[n]$ now denotes the isomorphism class of the trivial complex vector bundle $\theta$ of rank $\mathrm{rk}_{\mathbb{C}}(\theta) = n$.

## 4.2 Line Bundles and Čech Cohomology

### 4.2.1 Isomorphism and Transition Functions

Isomorphism classes of complex line bundles have an important classification that will lead us to the Chern classes. So suppose you are given an open cover $U_i$ and two different sets of $U(n)$ valued transition functions $g_{ij}$, $h_{ij}$. When do they correspond to isomorphic vector bundles? Precisely if there is a change of trivialization, that is local coordinate transformations

$$\lambda_i : U_i \to U(n) \tag{4.6}$$

such that

$$g_{ij} = \lambda_j^{-1} h_{ij} \lambda_i \tag{4.7}$$

Now in general this is as hard to check as testing isomorphism of the vector bundles, but for line bundles ($n = 1$) we can rewrite this as

$$g_{ij} = \lambda_i \lambda_j^{-1} h_{ij} \tag{4.8}$$

And this has a nice interpretation in terms of Čech cohomology: $\lambda_i \lambda_j^{-1}$ is a Čech coboundary while transition functions $g_{ij}$, $h_{ij}$ are Čech cocycles. So the isomorphism classes of complex line bundles are Čech cocycles modulo coboundaries, i.e. Čech cohomology classes. Let me review that notion:

### 4.2.2 Čech cohomology

Suppose you have a space $X$ together with an open cover $U_i$, and let

$$U_{ij\cdots\ell} = U_i \cap U_j \cap \cdots \cap U_\ell \tag{4.9}$$

So for each $ij \cdots \ell$ ("Čech index") there is an open set. Now consider $G$ valued functions on each set where $G$ is some abelian group. Together with the restriction that turns a a function $f_{i\cdots k} : U_{i\cdots k} \to G$ into a function $f_{i\cdots k\ell} : U_{i\cdots k\ell} \to G$ this is a *sheaf*, that is roughly an object that assigns abelian groups to open sets (see [33] for precise definitions). More examples are

- $\underline{G}$ the (sheaf of) $G$ valued functions.

- $G$ the constant $G$ valued functions.
- $C^0$ the continuous functions (same as $\underline{\mathbb{R}}$)

Especially note the difference between $G$ and $\underline{G}$, one are constant functions while the other consists of all continuous $G$ valued functions.

Now given such a sheaf one can define the Čech cochains, given by a choice of function for each $n+1$–tuple intersection. Write $\underline{G}(U_{i\cdots k})$ for the functions on $U_{i\cdots k}$, then the set of all $n$–cochains is

$$C^n(\underline{G}) = \prod_{i_0,i_1,\dots,i_n} \underline{G}(U_{i_0,i_1,\dots,i_n}) \tag{4.10}$$

(the chains are again abelian groups by applying the $\underline{G}$ group law componentwise) together with the coboundary map:

$$\partial^n : C^n \to C^{n+1}, \quad (\sigma_{i_0,i_1,\dots,i_n}) \mapsto \prod_{j=0}^{n+1} \sigma^{(-1)^j}_{i_0,i_1,\dots,\widehat{i_j},\dots,i_{n+1}}\big|_{U_{i_0 i_1 \cdots i_{n+1}}} \tag{4.11}$$

Note that if you happen to write the group law as addition you would have said

$$\partial^n : C^n \to C^{n+1}, \quad (\sigma_{i_0,i_1,\dots,i_n}) \mapsto \sum_{j=0}^{n+1} (-1)^j \sigma_{i_0,i_1,\dots,\widehat{i_j},\dots,i_{n+1}}\big|_{U_{i_0 i_1 \cdots i_{n+1}}} \tag{4.12}$$

but that is of course only a matter of notation. The important thing is that $\partial^n$ it is a group homomorphism and satisfies $\partial^{n+1} \circ \partial^n = 0$.

Now the Čech coboundaries are the image of $\partial$; the cocycles are the kernel of $\partial$. Equation 4.3 is just the condition for a Čech cocycle in $C^1(\underline{U(n)})$.

The Čech cohomology groups are then

$$\check{H}^n(X;\underline{G}) = \ker\left(\partial^n : C^n \to C^{n+1}\right) \Big/ \operatorname{img}\left(\partial^{n-1} : C^{n-1} \to C^n\right) \tag{4.13}$$

If the open cover $U_i$ is fine enough (and for suitably nice spaces $X$) the Čech and the ordinary cohomology groups (see section 6.1 for the definition) coincide:

$$\check{H}^*(X;\underline{\mathbb{Z}}) = \check{H}^*(X;\mathbb{Z}) = H^*(X;\mathbb{Z}) \tag{4.14}$$

Another result that we will require in the following concerns *fine sheaves*, that is functions that include partitions of unity like $\underline{\mathbb{R}}$ or $\underline{\mathbb{C}}$. For those all the higher cohomology groups vanish:

$$\check{H}^n(X;\underline{\mathbb{R}}) = \check{H}^n(X;\underline{\mathbb{C}}) = 0 \quad \forall\, n \geq 1 \tag{4.15}$$

Note that while $\underline{\mathbb{R}}$ is fine, the sheaf of constant functions $\mathbb{R}$ is not! In general the groups $\check{H}^n(X;\mathbb{R}) = H^n_{\mathrm{DR}}(X)$ do not vanish.

### 4.2.3 Line Bundles

So far we identified line bundles with cohomology groups as

$$\mathrm{Vect}^1_{\mathbb{C}}(X) = \check{H}^1(X; \underline{U(1)}) \tag{4.16}$$

In fact this is an isomorphism of groups, where the group law in $\mathrm{Vect}^1_{\mathbb{C}}(X)$ is the multiplication (of course this is very different from the semigroup law in $\mathrm{Vect}_{\mathbb{C}}(X)$ coming from the Whitney sum). Now there is a way to rewrite this in terms of more accessible cohomology groups by the long exact coefficient sequence in cohomology. This is a general property of cohomology and works as follows:

Suppose you have a short exact sequence of the coefficient groups, for example

$$0 \to \mathbb{Z} \hookrightarrow \mathbb{R} \xrightarrow{\exp} U(1) \to 0 \tag{4.17}$$

where the third map is $\exp : \mathbb{R} \to U(1), t \mapsto e^{2\pi i t}$. Then this induces a short exact sequence of the corresponding sheaves (see [33] for details)

$$0 \to \underline{\mathbb{Z}} \hookrightarrow \underline{\mathbb{R}} \xrightarrow{\exp} \underline{U(1)} \to 0 \tag{4.18}$$

Such a short exact sequence of the coefficient groups induces the following long exact sequence for Čech cohomology groups:

$$\begin{aligned} \cdots \to H^n\big(X; \underline{\mathbb{Z}}\big) \to H^n\big(X; \underline{\mathbb{R}}\big) \to H^n\big(X; \underline{U(1)}\big) \to \\ \to H^{n+1}\big(X; \underline{\mathbb{Z}}\big) \to H^{n+1}\big(X; \underline{\mathbb{R}}\big) \to H^{n+1}\big(X; \underline{U(1)}\big) \to \cdots \end{aligned} \tag{4.19}$$

Now $\underline{\mathbb{R}}$ is fine and therefore we get the following piece of the long exact sequence:

$$0 \to H^1\big(X; \underline{U(1)}\big) \xrightarrow{\simeq} H^2\big(X; \underline{\mathbb{Z}}\big) \to 0 \tag{4.20}$$

So there is a group isomorphism that identifies those groups. Therefore

$$\mathrm{Vect}^1_{\mathbb{C}}(X) = \check{H}^2(X; \underline{\mathbb{Z}}) = \check{H}^2(X; \mathbb{Z}) = H^2(X; \mathbb{Z}) \tag{4.21}$$

The group homomorphism $c_1 : \mathrm{Vect}^1_{\mathbb{C}}(X) \to H^2(X; \mathbb{Z})$ is called the *first Chern class*, and its generalization will occupy the next section.

## 4.3 Chern character

The purpose of this section is to define the generalization of eq. 4.21 for arbitrary vector bundles. It is a map

$$\begin{aligned} c : \mathrm{Vect}(X) &\to H^{\mathrm{ev}}(X;\mathbb{Z}), \\ E &\mapsto 1 + c_1(E) + c_2(E) + \cdots , \quad c_i(E) \in H^{2i}(X) \end{aligned} \tag{4.22}$$

called the *total Chern class*. Here we defined

$$\begin{aligned} H^{\mathrm{ev}}(X;\mathbb{Z}) &= \bigoplus_{i=0} H^{2i}(X;\mathbb{Z}) \\ H^{\mathrm{odd}}(X;\mathbb{Z}) &= \bigoplus_{i=0} H^{2i+1}(X;\mathbb{Z}) \end{aligned} \tag{4.23}$$

the cohomology (as Čech cohomology or CW cohomology, see section 6.1) in even resp. odd degrees. The cohomology groups come with a multiplication (the cup product), but we will not need its precise definition.

Now we do not want just any map, but it should come with certain good properties:

1. $c_i(E) = 0 \quad \forall\, i > \mathrm{rk}_{\mathbb{C}}(E)$.
2. If $E$ and $F$ are isomorphic vector bundles then $c(E) = c(F)$ and furthermore behaves well with respect to pullbacks: For each map $f : Y \to X$ of base spaces $f^*(c(E)) = c(f^*(E))$.
3. It behaves well with respect to the Whitney sum: $c(E \oplus F) = c(E)c(F)$ (This requires the cup product).
4. For the tautological line bundle (see section 4.4) $L \to S^2$ we have $c_1(L)$ the generator of $H^2(S^2)$ (Normalization).

**Fact 1.** *Those properties define the total Chern class uniquely, and the so–defined $c_1$ is the one from eq. 4.21.*

I will not try to prove this fact; instead we will use the properties to derive some simple properties.

First consider a trivial line bundle $\theta = X \times \mathbb{C}$. There we can take all transition functions to be $+1$, that is the neutral element in $\check{H}(X; \underline{U(1)}) = H^2(X;\mathbb{Z})$. Therefore $c_1(\theta) = 0$ and we get

$$c(\theta) = 1 \quad \Rightarrow \quad c(n\theta) = c(\theta \oplus \cdots \oplus \theta) = \Big(c(\theta)\Big)^n = 1^n = 1 \tag{4.24}$$

so the total Chern class of any trivial bundle is just 1. This implies that the Chern class only depends on the stable isomorphism class of the vector bundle: Assume that $E \not\simeq F$ but $E \oplus n \simeq F \oplus n$

$$\Rightarrow \quad c(E \oplus n) = c(F \oplus n) \quad \Rightarrow \quad c(E) \cdot 1 = c(F) \cdot 1 \tag{4.25}$$

Moreover the following map is well–defined:

$$c : K(X) \to H^{\mathrm{ev}}(X; \mathbb{Z}), \quad [E] - [n] \mapsto c(E) \tag{4.26}$$

**The Chern Character**

There is a close relative of the total Chern class, the *Chern character*. This is also a map

$$\begin{aligned} &ch : \mathrm{Vect}(X) \to H^{\mathrm{ev}}(X; \mathbb{Q}), \\ &\quad E \mapsto ch_0(E) + ch_1(E) + ch_2(E) + \cdots, \quad ch_i(E) \in H^{2i}(X; \mathbb{Q}) \end{aligned} \tag{4.27}$$

but rather satisfies the nicer property

$$ch(E \oplus F) = ch(E) + ch(F) \quad \in H^{\mathrm{ev}}(X; \mathbb{Q}) \tag{4.28}$$

$$ch(E \otimes F) = ch(E)\, ch(F) \tag{4.29}$$

For a line bundle $L \to X$ this is just (note the fractions, because of them we need $H^{\mathrm{ev}}(X; \mathbb{Q})$ instead of $H^{\mathrm{ev}}(X; \mathbb{Z})$):

$$ch(L) = \exp c(L) = \exp\Big(c_1(L)\Big) = 1 + c_1(L) + \frac{1}{2} c_1(L)^2 + \cdots \tag{4.30}$$

where $c_1(L)$ really denotes its image in $H^2(X; \mathbb{Q})$. As a consistency check remember that we identified the (additive) group law in $H^2(X)$ with the tensor product in $\mathrm{Vect}^1_{\mathbb{C}}(X)$. The exponential above makes eq. 4.29 work:

$$\begin{aligned} ch(L_1) ch(L_2) &= \exp\Big(c_1(L_1)\Big) \exp\Big(c_1(L_2)\Big) = \\ &= \exp\Big(c_1(L_1) + c_1(L_2)\Big) = \exp\Big(c_1(L_1 \otimes L_2)\Big) = ch(L_1 \otimes L_2) \end{aligned} \tag{4.31}$$

Now for general vector bundles the Chern character is determined by the Chern classes, but not as easily as in eq. 4.30. However in general characteristic classes are determined by how they act on line bundles and naturality, that is $f^*(ch(E)) = ch(f^*(E))$. So although far from obvious there is no ambiguity, and one can show that:

$$\begin{aligned} ch_0(E) &= \mathrm{rk}_{\mathbb{C}}(E) \\ ch_1(E) &= c_1(E) \\ ch_2(E) &= \frac{1}{2}\Big(c_1(E)^2 - 2c_2(E)\Big) \\ ch_3(E) &= \frac{1}{6}\Big(c_1(E)^3 - 3c_1(E)c_2(E) + c_3(E)\Big) \\ &\vdots \end{aligned} \tag{4.32}$$

Of course the Chern character induces a map

$$ch : K(X) \to H^{\mathrm{ev}}(X;\mathbb{Q}), \quad [E]-[F] \mapsto ch(E) - ch(F) \tag{4.33}$$

which is a ring homomorphism thanks to eqns. 4.28, 4.29. Now the single most useful result concerning the Chern character is the following (see [7])

**Theorem 4.** *The induced map*

$$ch : K(X) \otimes \mathbb{Q} \to H^{\mathrm{ev}}(X;\mathbb{Q}) \tag{4.34}$$

*is bijective. With other words the free parts of $K(X)$ and $H^{\mathrm{ev}}(X;\mathbb{Z})$ are the same. Moreover*

$$H^{\mathrm{ev}}(X;\mathbb{Z})_{\mathrm{Tor}} = 0 \quad \Rightarrow \quad K(X)_{\mathrm{Tor}} = 0 \tag{4.35}$$

This is of course how the K–theoretic description of D–brane charges contains ordinary de Rahm cohomology (cf. [50]). If you ignore torsion and the correct charge quantization you can reduce everything to computations with differential forms.

## 4.4 Computation: Spheres

Armed with theorem 4 it is trivial to determine the K–groups for all spheres $S^n$. Their cohomology groups are

$$H^i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{else} \end{cases} \quad \Rightarrow H^{\mathrm{ev}}(S^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases} \tag{4.36}$$

So especially $H^{\mathrm{ev}}(S^n;\mathbb{Z})$ is torsion free and thus

$$K(S^n) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases} \tag{4.37}$$

Let us focus on the $S^2$ case; As usual one of the $\mathbb{Z}$ summands is just the virtual dimension:

$$K(S^2) = \mathbb{Z} \oplus \mathbb{Z} \quad \Rightarrow \quad \widetilde{K}(S^2) = \mathbb{Z} \tag{4.38}$$

So there must be a nontrivial bundle that generates the other $\mathbb{Z}$. This generator is the *tautological line bundle* $\lambda \to S^2$ on $S^2 = \mathbb{C}\mathrm{P}^1$: Think of $\mathbb{C}\mathrm{P}^1$ as the set of complex planes $\mathbb{C} \subset \mathbb{C}^2$, then the fiber of the tautological bundle over a point $\mathbb{C} \subset \mathbb{C}^2$ is just this $\mathbb{C}$. Alternatively describe the bundle by the transition function along the equator $S^1$; then the identity map $g_{NS} : S^1 \to U(1), z \mapsto z$ describes the tautological line bundle.

Remember that $\mathrm{Vect}^1_{\mathbb{C}}(S^2)$ with tensor product is a group; Its generator is $\lambda$. What is the relation with the tangent bundle? The tangent bundle can be described by the transition function $g_{NS} : S^1 \to U(1), z \mapsto z^{-2}$, so $TS^2 = \lambda^{-2}$.

Any complex vector bundle $E \to X$ can be thought of as a real vector bundle of real rank $2\,\mathrm{rk}_{\mathbb{C}}(E)$ by forgetting the complex structure of the fiber. Denote the corresponding real vector bundle $E_{\mathbb{R}} \to X$, then this also defines a map

$$K(X) \to KO(X), \quad [E] - [F] \mapsto [E_{\mathbb{R}}] - [F_{\mathbb{R}}] \tag{4.39}$$

Of course this map is not surjective, as its image can only have even virtual rank. It is also not injective, for example take

$$r : K(S^2) = \mathbb{Z} \oplus \mathbb{Z} \to KO(S^2) = \mathbb{Z} \oplus \mathbb{Z}_2 \tag{4.40}$$

The complex tangent bundle $TS^2$ is $(1, -2) \in K(S^2)$ and as we have seen the real tangent bundle $TS^2_{\mathbb{R}}$ is $(2, 0) \in KO(S^2)$. From this we can describe the *realification* explicitly:

$$r : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2, \quad (a, b) \mapsto (2a, b \bmod 2) \tag{4.41}$$

One can also turn a real vector bundle into a complex one by thinking of the $O(n)$ valued transition functions as $U(n)$ valued. This is called *complexification* and can be written as $E \otimes_{\mathbb{R}} \mathbb{C}$. This operation obviously doubles the real rank; its connection with the above can be described as

$$E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = E \oplus \overline{E} \tag{4.42}$$

## 4.5 Coherent Sheaves

Suppose you want to compactify Type II string theory preserving some supersymmetry. Then from the supersymmetry variations one knows that there must be a constant spinor; So the task is to find a compact smooth manifold (real 6 dimensional) with a single constant spinor.

Simply trying to solve $\nabla\psi = 0$ is hopeless. However there is an equivalent characterization: you need a Calabi–Yau manifold, that is a complex manifold (with analytic transition functions) of complex dimension 3 that is in addition Kähler and has $c_1 = 0$.

It gets even simpler than that, one can restrict[1] oneself to polynomial transition functions instead of arbitrary power series. So really it suffices to consider a much smaller class of functions, and one that is easy to compute with. Schematically

$$\text{smooth} \supset \text{analytic} \supset \text{algebraic} \tag{4.43}$$

But those simplifications all came as implications of the equations we were trying to solve; Physical intuition tells us to start within smooth or maybe continuous functions.

So back to our topic, suppose you have an analytic or algebraic manifold $X$ and you want to compute $K(X)$. This is the Grothendieck group of vector bundles with continuous transition functions. But you could have just as well considered vector bundles with analytic or algebraic transition functions, they also yield well–defined groups $K_{\text{an}}(X)$, $K_{\text{alg}}(X)$ . They may be useful to compute $K(X)$ in a simpler way, for example by using the obvious "forgetting map"

$$K(X) \leftarrow K_{\text{an}}(X) \leftarrow K_{\text{alg}}(X) \tag{4.44}$$

Algebraic geometry knows another generalization of vector bundle, called *coherent sheaf*. So in addition we also have the Grothendieck group of coherent sheaves $K_{\text{coh}}(X)$ .

What is the relation between all these groups? A partial answer is the following (see [34])

**Theorem 5.** *If $X$ is algebraic, then $K_{\text{an}}(X) = K_{\text{alg}}(X) = K_{\text{coh}}(X)$*

So really there is only one question: What is the forgetting map $K_{\text{an}}(X) \to K(X)$? Unfortunately there is no good answer, in general there it is neither surjective nor injective.

As an example consider the following (this was also mentioned in [57]). Let $\Sigma$ be a nonsingular Riemann surface. Then from theorem 4 we know that $K(\Sigma) = \mathbb{Z} \oplus \mathbb{Z}$. On the other hand (see [37]):

$$K_{\text{an}}(\Sigma) = \mathbb{Z} \oplus \text{Pic}(\Sigma) \tag{4.45}$$

[1] if the Kähler class is integral

Now the $\mathbb{Z}$ factor is again only the virtual rank; The interesting part of the forgetting map is the $d : \mathrm{Pic}(\Sigma) \to \mathbb{Z}$. This is the degree $d(D)$ for an divisor $D \in \mathrm{Pic}(\Sigma)$, and therefore $d$ is surjective. But it is not injective, for example (see [37]):

$$d^{-1}(0) = \mathrm{Pic}^0(\Sigma) = \mathbb{C}^g \Big/ \mathbb{Z}^{2g} \tag{4.46}$$

So what happens is the following: $K_{\mathrm{an}}(\Sigma)$ still knows about the moduli of the analytic vector bundles. Sometimes bundles that can be deformed into each other by continuously changing moduli correspond to different classes in $K_{\mathrm{an}}(\Sigma)$, but of course are the same in $K(\Sigma)$.

It is comparatively easy to see that the forgetting map is not surjective. For this you have to know that on a Kähler manifold the cohomology groups decompose further as

$$H^i(X;\mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X) \tag{4.47}$$

and for any analytic vector bundle $E \to X$

$$ch(E) \in \bigoplus_{p} H^{p,p}(X) \tag{4.48}$$

So if some $H^{p,q}(X) \neq 0$ with $p \neq q$ and $p+q \in 2\mathbb{Z}$ (e.g. $H^{2,0}(T^4) = \mathbb{C}$) there is via theorem. 4 a class $\xi \in K(X)$ such that the $H^{p,q}(X)$ component of $ch(\xi)$ is nonzero. This class can then not be represented by analytic vector bundles, so does not come from any class in $K_{\mathrm{an}}(X)$.

Now for a complex 3 dimensional Calabi–Yau manifold there are no non-vanishing $H^{p,q}(X)$ with $p+q \in 2\mathbb{Z}$ and $p \neq q$. It is a tempting conjecture that the forgetting map is at least surjective in this case. Unfortunately I do not know how to prove it[2].

[2]This is related to the Hodge conjecture, but hopefully easier.

# Chapter 5

# Algebraic Topology

## 5.1 CW complexes

We want to work with a class of topological spaces that is more general than smooth manifolds (so that e.g. we can pinch a subspace and still have a "well-formed" space, but on the other hand side we want to exclude pathological sets (There are really too many things that can go wrong). One nice class of spaces to study topology on are CW complexes, which I am going to define in this section.

A *CW complex* is a collection of disks (of arbitrary dimensions), glued together at the boundaries. Restricting to the disks of dimension $d$ or less yields a filtration of the space $\cdots \subset \Sigma_{d-1} \subset \Sigma_d \subset \cdots$. The subspace $\Sigma_d$ is called the $d$–skeleton. This is best defined by the recursive description

1. The 0–skeleton is a set of points (i.e. 0–disks)

$$\Sigma_0 = \left\{ e_1^{(0)}, \dots, e_{n_1}^{(0)} \right\} \tag{5.1}$$

2. The $d+1$–skeleton $\Sigma_{d+1}$ consists of $\Sigma_d$ together with $d+1$–disks $e_1^{(d+1)}, \dots, e_{n_{d+1}}^{(0)}$ glued to $\Sigma_d$ via attaching maps

$$f_i^{(d+1)} : \partial e_i^{(d+1)} \to \Sigma_d, \qquad i = 1, \dots, n_{d+1} \tag{5.2}$$

That is the boundaries of the disks $e_i^{(d+1)}$ are glued to the $d$–skeleton.

This is best visualized by an example:

**Example 1.** *The real projective plane* $\mathbb{RP}^2$ *is the surface you get from gluing one disk to the boundary of the Möbius strip (The boundary of the Möbius strip is a single circle). You can get a simpler CW structure by contracting*

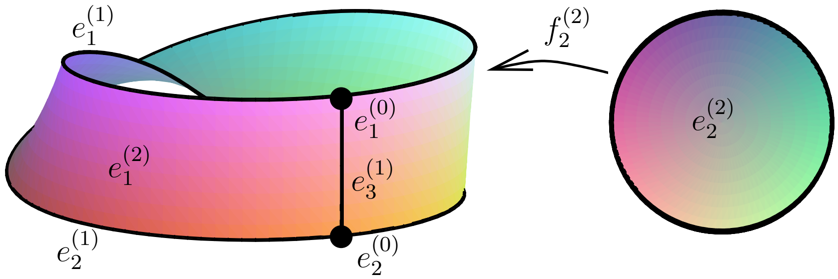


Figure 5.1: A CW complex for $\mathbb{R}\mathrm{P}^2$

*the Möbius strip to a circle (this is a homotopy): Then you only need one cell in each dimension. The boundary of the 2–cell is then winding twice over the one–skeleton* $\Sigma_1 = S^1$.

## 5.2 Suspension

Let us start with defining the cone of a space $CX$. This is the cylinder $X \times I$ ($I$ is the interval $[0, 1]$) with one end shrunk to a point. The tip of the cone is the new basepoint:

$$CX = \big(X \times I\big)\Big/\big(X \times \{0\}\big) \tag{5.3}$$

Obviously $CX$ is contractible to a point, and by itself not a very interesting space. Now the *suspension* $SX$ is roughly the cone with the other end also contracted: But remember that we agreed to work within "spaces with basepoint", so which basepoint do we pick? The canonical solution is to contract the line $\{*\} \times I$ which joins the two endpoints and take this as the new basepoint. Note that $\{*\} \times I$ was contractible in the first place, so shrinking it to a point is just a homotopy.

$$SX = \big(X \times I\big)\Big/\big(X \times \partial I \cup \{*\} \times I\big) \tag{5.4}$$

Another way to think of the suspension $SX$ is two cones $CX$ glued together at their base $X$. This should be clear from figure 5.2. Moreover it should be obvious that $SS^1 = S^2$ (as depicted in fig. 5.2). This is the special case of the following identity:

**Proposition 1.** $S(S^n) = S^{n+1}$.

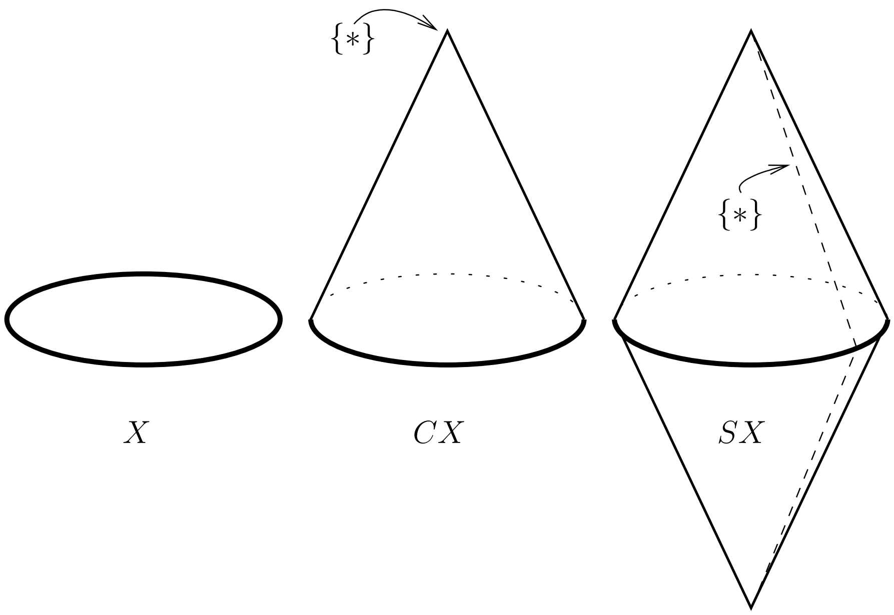


Figure 5.2: Cone and suspension

### 5.2.1 Wedge and smash

Let me describe yet another, more formal way to denote the suspension. This relies on the following two basic operations to combine two spaces into one: The first operation is to join just the two basepoints into the new basepoint, called the "wedge":

$$X \vee Y = \big(X \cup Y\big)\Big/\big(\{x_0\} \cup \{y_0\}\big) \tag{5.5}$$

Think of $X \vee Y$ as a subset of $X \times Y$ and let

$$= \big(X \cup Y\big)\Big/\big(X \vee Y\big) \tag{5.6}$$

This is called "smash"; The basepoint of $X \wedge Y$ is the contracted subset. By unraveling the definitions one sees that

**Lemma 1.** *Smash is associative:* $\big(X \wedge Y\big) \wedge Z = X \wedge \big(Y \wedge Z\big)$.

and (back to our topic):

**Lemma 2.** *Suspension is smash with a circle:* $SX = X \wedge S^1$.

Especially we do not have to worry about the order in multiple suspensions. From proposition 1, lemma 1 and 2 we find purely algebraically that

**Corollary 2.** $(S)^n X = \overbrace{SS\cdots S}^{n} X = X \wedge S^n$

### 5.2.2 Suspension and transition functions

So why should we be interested in suspensions? Here is a partial answer. Remember that we could define vector bundles as a set of transition functions, defined on the overlap of coordinate charts. E.g. on the two–sphere the real vector bundles were maps from the equator to $O(n)$. Think of the northern and southern hemisphere as cones over the equator, and you are immediately led to the following generalization:

**Theorem 6.** *(Isomorphism classes of) vector bundles on $SX$ are in one–to–one correspondence with (homotopy classes of) maps from $X$ to $O(n)$:*

$$\mathrm{Vect}^n(SX) = [X; O(n)] \tag{5.7}$$

Of course this does not depend on the group $O(n)$ and we can generalize it to arbitrary $G$–bundles:

**Theorem 7.** *Equivalence classes of $G$–bundles on $SX$ are in one–to–one correspondence with $[X; G]$.*

But we want to describe bundles on $X$ and not on its suspension $SX$! If we could undo suspensions then it would be just $[S^{-1}X, G]$. Unfortunately this cannot be so easy[1]:

**Proposition 2.** *There is no space $X$ such that $SX \simeq S^0$.*

*Proof.* From the definition follows that $SX$ has one connected component $\forall X$. But $S^0$ has two components. □

Note that you cannot argue with the "dimension": Although the explicit construction of the suspension increases what you would call "dimension" it is not a homotopy invariant, think $I \simeq \{pt\}$.

There is a close relative of suspension, that is forming the loop space. To any space $X$ we associate the space of loops $\Omega X$ (starting and ending at the basepoint), with the new basepoint the constant loop. It is not hard to see that

**Theorem 8.** *Loop is adjoint to suspend:*

$$[SX; Y] \simeq [X; \Omega Y] \tag{5.8}$$

[1]However we could enlarge the notion of "space" to include formal "de-suspensions". This would lead us to the definition of spectra, but I will stay on a more elementary level.

So if we cannot "unsuspend", can we "unloop"? At least formally we would then have that $G$–bundles are described by $[X; \Omega^{-1}G]$. Of course naively the loop space will always be "infinite–dimensional" so it seems doubtful that, say, a finite dimensional Lie group $G$ is the loop space of anything. But it would be sufficient if it were up to homotopy! And indeed this is possible and will be the topic of the remainder of this chapter.

### 5.2.3 Classifying spaces

So let $G$ be any group (discrete or continuous) and suppose we are given a $G$–bundle $EG$ that is contractible. Such a thing exists, although I will not try to give an explicit construction. I call the base $BG$:

**Fact 2.** *Given a group $G$ there exists a bundle*

$$\begin{tikzcd} G \arrow[r] & EG \arrow[d] \\ & BG \end{tikzcd} \tag{5.9}$$

*with the total space $EG$ contractible.*

This bundle gives rise to the following long exact sequence for homotopy groups:

$$\cdots \to \underbrace{\pi_{n+1}(EG)}_{=0} \to \pi_{n+1}(BG) \to \pi_n(G) \to \underbrace{\pi_n(EG)}_{=0} \to \cdots \tag{5.10}$$

So we learn that

$$[S^n; \Omega BG] = [SS^n; BG] = \pi_{n+1}(BG) = \pi_n(G) = [S^n; G] \tag{5.11}$$

Especially $\Omega BG$ has the same homotopy groups as $G$. Of course this does not prove that $\Omega BG \simeq G$ — it is true nevertheless:

**Theorem 9.** *$\Omega BG \simeq G$ and $G$–bundles are in one–to–one correspondence with homotopy classes of maps $[X; BG]$. (Hence the name classifying space, it classifies $G$–bundles).*

and in case you wonder about any choices involved:

**Theorem 10.** *$BG$ is unique up to homotopy*

The proofs are long and would lead us far astray; I will not reproduce them but instead discuss a few examples.

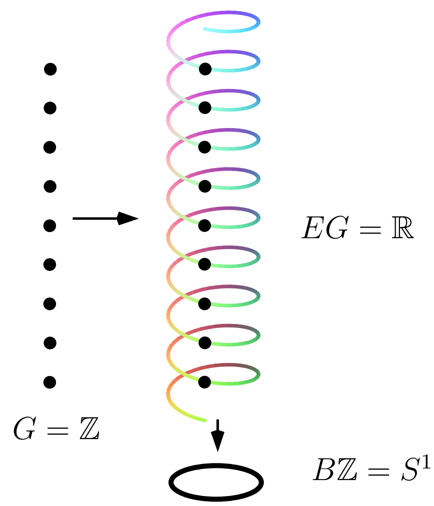


Figure 5.3: Classifying space for $\mathbb{Z}$

**Example 2.** *$E\mathbb{Z} = \mathbb{R}$ and $B\mathbb{Z} = S^1$. This can be seen from figure 5.3. Note that $\Omega B\mathbb{Z} = \Omega S^1 \simeq \mathbb{Z}$ since loops on the circle are determined (up to homotopy) by their winding number.*

This example was chosen for its simplicity, unfortunately $EG$ is in general not a finite dimensional manifold:

**Example 3.** *Take $G = \mathbb{Z}_2$, then $E\mathbb{Z}_2 = S^\infty$ with the $\mathbb{Z}_2$–action the antipodal map. The classifying space is*

$$B\mathbb{Z}_2 = S^\infty/\mathbb{Z}_2 = \mathbb{R}\mathrm{P}^\infty \tag{5.12}$$

*Note that $S^\infty$ (the unit sphere in a separable Hilbert space) is contractible, this is known as Kuypers theorem.*

$B\mathbb{Z}_2$ is the classifying space for real line bundles; the classifying space for complex line bundles is

**Example 4.** *Take $G = U(1)$, then $EU(1) = S^\infty$ and $BU(1) = \mathbb{C}\mathrm{P}^\infty$. The $U(1)$–action on*

$$S^\infty = \left\{ (z_1, z_2, \dots) \middle| z_i \in \mathbb{C}, \sum_{i=1}^{\infty} |z_i|^2 = 1 \right\} \tag{5.13}$$

*is just multiplication with a phase, $z_i \mapsto e^{i\varphi} z_i$.*

*BU* **and** *BO*

So how does this classifying space story help us? Well of course there are the classifying spaces for $O(n)$ and $U(n)$:

$$\mathrm{Vect}^n_{\mathbb{R}}(X) = [X; BO(n)] \qquad \mathrm{Vect}^n_{\mathbb{C}}(X) = [X; BU(n)] \tag{5.14}$$

I want to define classifying spaces for K–theory:

$$\widetilde{KO}(X) = [X; BO] \qquad \widetilde{K}(X) = [X; BU] \tag{5.15}$$

and since the real and complex case are very much alike I will focus on real bundles. So what are these classifying spaces? First, remember theorem 3: For every vector bundle $V \in \mathrm{Vect}^n_{\mathbb{R}}(X)$ we can find another vector bundle $W$ such that $V \oplus W = X \times \mathbb{R}^d$, the trivial bundle. So we can think of $V$ as picking a $n$–plane in every fiber $\mathbb{R}^n$ of the trivial bundle.

**Definition 1.** *The (real) Grassmannian $G_{d,n}$ is the space of $n$–planes in $\mathbb{R}^d$.*

The rotation group $O(d)$ in $\mathbb{R}^d$ acts transitively on the Grassmannian with stabilizer $O(n-d)\times O(n)$, the rotations perpendicular to the plane and rotations within the plane. Therefore

$$G_{d,n} = \frac{O(d)}{O(n-d) \times O(n)} \tag{5.16}$$

So $G_{d,n}$ is a smooth manifold of dimension $n(d-n)$.

We have seen that $G_{d,n}$ is the classifying space for rank $n$ subbundles of $X \times \mathbb{R}^d$. Furthermore we have the inclusion $G_{d,n} \subset G_{d+1,n}$ as "the $n$–planes in $\mathbb{R}^{d+1}$ that are orthogonal to the $(d+1)^{\text{th}}$ direction". So really the Grassmannians form a directed system and we can define the limit

$$BO(n) = \lim_{d\to\infty} G_{d,n} \tag{5.17}$$

In more down to earth terms we can form the union of all those Grassmannianns, and it will be the classifying space for the rank $n$ subbundles of arbitrarily large trivial bundles, i.e. the classifying space of rank $n$ vector bundles.

Of course you have to form the union of higher and higher dimensional Grassmannians, so $BO(n)$ is certainly not finite dimensional. But there is a way to think of the infinite union in a way that you only have to look at finite dimensional pieces, by "fattening" the Grassmannians into $G_{n,d} \times I$ and then gluing them head–to–tail (figure 5.2.3). For obvious reasons this

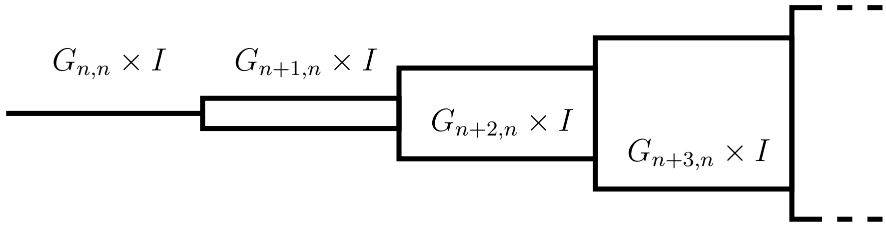


Figure 5.4: $BO(n)$ as a union of Grassmannians

is called the "telescope" and is (up to homotopy) the same as $BO(n)$. The basepoint is the leftmost point.

If you map a compact space $X$ into the telescope you can get only finitely far to the right — remember that the basepoint has to map to the basepoint. But the map into $BO(n)$ is a vector bundle, and if it eventually fits into one segment of the telescope, that is one Grassmannian. This is how the telescope knows about theorem 3, every vector bundle is a subbundle of a finite–dimensional trivial bundle.

Given a rank $n$ vector bundle we can form a rank $n+1$ vector bundle by adding the trivial linebundle. In other words, we have an inclusion $BO(n) \subset BO(n+1)$, which we can use to form the limit

$$BO = \lim_{n\to\infty} BO(n) = \lim_{n\to\infty} \lim_{d\to\infty} G_{d,n} \tag{5.18}$$

So $[X; BO]$ classifies stable vector bundles, that is vector bundles up to addition of trivial bundles. But this is nothing else than $\widetilde{KO}(X)$, since each element of $\widetilde{KO}(X)$ can be written as "vector bundle minus trivial bundle of the same rank", cf. corollary 1.

Unreduced K–theory is basically the same, you just have to keep track of the virtual rank of the bundle separately:

$$KO(X) = [X; \mathbb{Z} \times BO] \qquad K(X) = [X; \mathbb{Z} \times BU] \tag{5.19}$$

# Chapter 6

# Cohomology

So far we have only used K–theory to associate groups to spaces, invariant under continuous deformations. But there is much more, K–theory is really a cohomology theory. This is of utmost importance if you want to actually calculate the K–groups since it lets you employ various techniques which apply to any cohomology theory. It is also physically interesting since it nicely generalizes the properties of de Rahm cohomology. For example all cohomology theories satisfy "excision", which is a topological version of locality.

I will not assume that the reader is familiar with cohomology theories beyond usual de Rahm theory. But neither will it be possible to give a thorough presentation that encompasses all aspects. Instead I will highlight the constructions with a view towards computing the actual cohomology groups.

## 6.1 Ordinary (CW) Homology and Cohomology

So let $X$ be a finite CW complex made from the cells $e_i^{(d)}$ where $i = 1 \dots n_d$ indexes the cells in dimension $d \in \mathbb{Z}$ ($n_d = 0$ for $d < 0$ and for sufficiently large $d$). To it we can associate the free abelian group generated by the cells — this is just one integer for each cell, with addition of the $\sum n_d$–tuple as group action. This group is obviously graded by the dimension of the cell:

$$C_d \stackrel{\text{def}}{=} \bigoplus_{i=1}^{n_d} \mathbb{Z} e_i(d) \qquad (6.1)$$

Now remember that each cell $e_i^{(d)}$ comes with a map $f_i^{(d)} : S^{d-1} \to \Sigma^{(d-1)}$ that specifies how it is attached to the $d-1$ skeleton. We can use this to define "how often" the boundary of $e_i^{(d)}$ is mapped onto each $d-1$ cell, counting orientation reversal by a sign:

**Definition 2.** *For each cell $e_i^{(d)}$ let the boundary $\partial e_i^{(d)}$ be the sum of its boundary components in $C_{d-1}$.*

This obviously induces a map $\partial : C_d \to C_{d-1}$. Boundaries themselves do not have boundaries, so $\partial^2 : C_d \to C_{d-2}$ is the 0 map. Let me illustrate this with an example:

**Example 5.** *Take the 6 cell CW complex for $\mathbb{RP}^2$, as in figure 5.1. The boundaries are (with some choice for the orientations):*

$$\begin{aligned}
\partial e_i^{(0)} &= 0 \\
\partial e_1^{(1)} &= e_2^{(0)} - e_1^{(0)} \\
\partial e_2^{(1)} &= e_1^{(0)} - e_2^{(0)} \\
\partial e_3^{(1)} &= e_1^{(0)} - e_2^{(0)} \\
\partial e_1^{(2)} &= e_1^{(1)} + e_3^{(1)} - e_2^{(1)} + e_3^{(1)} = e_1^{(1)} - e_2^{(1)} + 2e_3^{(1)} \\
\partial e_2^{(2)} &= e_1^{(1)} + e_2^{(1)}
\end{aligned} \tag{6.2}$$

The boundary map $\partial$ is a linear map from the lattice $C_d \simeq \mathbb{Z}^{n_d}$ to $C_{d-1} \simeq \mathbb{Z}^{n_{d-1}}$. Therefore we can write the *chain complex* $C = (C_d, \partial)$ with matrices:

$$\cdots \longleftarrow 0 \xleftarrow{(0\ 0)} \mathbb{Z}^2 \xleftarrow{\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}} \mathbb{Z}^3 \xleftarrow{\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 0 \end{pmatrix}} \mathbb{Z}^2 \longleftarrow 0 \longleftarrow \cdots \tag{6.3}$$

(A string of maps is called *complex* if going twice is 0). Now the *homology* of the complex is defined as the successive quotients

$$H_i(C) = \ker(\partial : C_d \to C_{d-1}) \Big/ \operatorname{img}(\partial : C_{d+1} \to C_d) \tag{6.4}$$

By the base change $\tilde{e}_1^{(0)} = e_1^{(0)} + e_2^{(0)}$, $\tilde{e}_2^{(0)} = e_2^{(0)}$ we can simplify eq. 6.3 to

$$\cdots \longleftarrow 0 \xleftarrow{(0\ 0)} \mathbb{Z}^2 \xleftarrow{\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \end{pmatrix}} \mathbb{Z}^3 \xleftarrow{\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 0 \end{pmatrix}} \mathbb{Z}^2 \longleftarrow 0 \longleftarrow \cdots \tag{6.5}$$

Obviously

$$H_0(C) = \mathbb{Z} \qquad H_2(C) = 0 \tag{6.6}$$

What about $H_1$? We have

$$\begin{aligned}
\ker(\partial : C_1 \to C_0) &= \operatorname{span}\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle \\
\operatorname{img}(\partial : C_2 \to C_1) &= \operatorname{span}\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\rangle \\
&= \operatorname{span}\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} \right\rangle \qquad (6.7)
\end{aligned}$$

$\Rightarrow H_1(C) = \mathbb{Z}_2$. Although far from obvious **the resulting homology groups do not depend on any of the choices made** — they are topological invariants of the space. We write

$$H_i(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) = H_i(\mathbb{R}\mathrm{P}^2) = H_i(C) \qquad (6.8)$$

Note how much easier it would have been to use the CW complex for $\mathbb{R}\mathrm{P}^2$ which has only one cell in each dimension, see example 1:

$$\cdots \longleftarrow 0 \xleftarrow{0} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \longleftarrow 0 \longleftarrow \cdots \qquad (6.9)$$

Now instead of using the cells as generators for our chain complex we could have used the linear functionals on the cells, that is $\mathbb{Z}$–linear maps $\{e_i^{(d)}\} \to \mathbb{Z}$. They form the dual lattice $\tilde{C}_d$ to the $C_d$. But now given a map $\varphi : C_d \to \mathbb{Z}$, what is its "boundary"? The only way to form a new linear functional out of $\varphi$ and $\partial$ is $\varphi \circ \partial : C_{d+1} \to \mathbb{Z}$. Thus we have a *coboundary* $\tilde{\partial} : \tilde{C}_d \to \tilde{C}_{d+1}$ and the corresponding *cochain complex* $\tilde{C} : (\tilde{C}_d, \tilde{\partial})$. The homology of the cochain complex is called cohomology and called $H^i$.

**Exercise 1.** *Check that*

$$H^i(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & i = 2 \\ 0 & i = 1,\ i > 2 \text{ and } i < 0 \\ \mathbb{Z} & i = 0 \end{cases} \qquad (6.10)$$

For simplicity's sake I never mentioned basepoints so far; What we saw so far is *unreduced* (co)homology. The *reduced* (co)homology is "trivial over the basepoint" but otherwise the same. So the basepoint (say, $e_1^0$) is set to zero in the chain complex $C$ and its dual $\tilde{e}_1^0$ is set to zero in the cochain complex $\tilde{C}$. The resulting (co)homology groups are denoted by $\tilde{H}_i$ resp. $\tilde{H}^i$. The difference is not much: one can show that for any compact space $X$

$$\begin{aligned}
H_0(X; \mathbb{Z}) &= \tilde{H}_0(X; \mathbb{Z}) \oplus \mathbb{Z} \qquad & H_i(X; \mathbb{Z}) &= \tilde{H}_i(X; \mathbb{Z})\ \forall i \neq 0 \\
H^0(X; \mathbb{Z}) &= \tilde{H}^0(X; \mathbb{Z}) \oplus \mathbb{Z} \qquad & H^i(X; \mathbb{Z}) &= \tilde{H}^i(X; \mathbb{Z})\ \forall i \neq 0 \qquad (6.11)
\end{aligned}$$

### 6.1.1 Cohomology of a Suspension

Given a space $X$, what is the (co)homology of its suspension $SX$? Let $e_i^{(d)}$ be a CW complex for $X$, and we want to describe a CW complex for $X \times I / X \times \partial I$. In dimensions $d \geq 1$ we can use the cells $\tilde{e}_i^{(d)} \simeq e_i^{(d-1)} \times I$, with gluing maps induced from $X$. In dimension $d = 0$ we then need two points, the endpoints of the interval. We can take one of them to be the new basepoint, up to homotopy this is the suspension of $X$ as in eq. 5.4.

The associated chains and cochains of $SX$ is the same as for $X$ except in dimension $d = 0$, only shifted up in dimension. Moreover the boundary maps are also the same. Therefore the homology and cohomology is also the same, only shifted by one. To compute the (co)homology for $d = 0$ use the CW complex with the interval $* \times I \subset X \times I$ contracted to a point, as in our definition of suspension. Then there is only one point, the basepoint. Therefore $\tilde{H}_0(SX) = 0 = \tilde{H}^0(SX)$. So we can also think of the dimension 0 (co)homology as being shifted up from dimension $-1$:

$$\tilde{H}^i(SX) = \tilde{H}^{i-1}(X), \qquad \tilde{H}_i(SX) = \tilde{H}_{i-1}(X) \quad \forall i \in \mathbb{Z} \tag{6.12}$$

If we would have computed unreduced (co)homology we would have gotten the slightly less symmetric result

$$H^i(SX) = \begin{cases} H^{i-1}(X) & i \neq 0 \\ \mathbb{Z} & i = 0 \end{cases} \qquad H_i(SX) = \begin{cases} H_{i-1}(X) & i \neq 0 \\ \mathbb{Z} & i = 0 \end{cases} \tag{6.13}$$

### 6.1.2 Useful Identities

Often either homology or cohomology is accessible while we want to know the other. For this there are two basic tricks to convert one into the other. But first we have to split the (co)homology groups into its torsion and its free part:

**Definition 3.** *Let $G$ be a finitely generated abelian group. Then the torsion subgroup $G_{\mathrm{Tor}}$ is the subgroup of elements of finite order:*

$$G_{\mathrm{Tor}} = \left\{ g \in G \,\middle|\, \exists n \in \mathbb{Z} : ng = 1 \right\} \tag{6.14}$$

*Thus $G/G_{\mathrm{Tor}}$ is a free abelian group (i.e. $G/G_{\mathrm{Tor}} \simeq \mathbb{Z}^k$ for some $k$). So pick a set of representatives $g_1, \dots, g_k \in G$ for the generators of $G_{\mathrm{Tor}}$ and let*

$$G_{\mathrm{Free}} = \left\langle g_1, \dots, g_k \right\rangle_{\mathbb{Z}} \subset G \tag{6.15}$$

Obviously $G = G_{\text{Free}} \oplus G_{\text{Tor}}$, but $G_{\text{Free}}$ is not uniquely determined:

**Example 6.** *Let $G$ be the finitely generated abelian group with the generators $a$ of infinite order and $b$ of order 2. Then $G \simeq \mathbb{Z} \oplus \mathbb{Z}_2$, and $G_{\text{Tor}} = \{1, b\}$. But there are two different choices*

$$G_{\text{Free},1} = \big\langle a \big\rangle_{\mathbb{Z}} \qquad G_{\text{Free},2} = \big\langle a + b \big\rangle_{\mathbb{Z}} \tag{6.16}$$

With these definitions we can state the

**Theorem 11 (Universal Coefficient Theorem).**

$$H^i(X; \mathbb{Z}) = H_i(X; \mathbb{Z})_{\text{Free}} \oplus H_{i-1}(X; \mathbb{Z})_{\text{Tor}} \tag{6.17}$$

Actually this is only a corollary of the real Universal Coefficient Theorem, as applied to $\mathbb{Z}$ coefficients (compare with [14]). The UCT is simply a consequence of how the coboundaries are induced from the boundary maps and so does not depend on any further properties of the space.

The other formula does use special properties of the base space: it is only valid for orientable manifolds:

**Theorem 12 (Poincaré Duality).** *Let $X$ be a compact oriented manifold of dimension* $\dim X = d$*, then*

$$H^i(X; \mathbb{Z}) \simeq H_{d-i}(X; \mathbb{Z}) \tag{6.18}$$

### 6.1.3 Compact Support

So far we only considered *compact* spaces. But we will need K–theory for noncompact spaces. In this case I will only be interested in cohomology with compact support. This means

- For ordinary cohomology: only cochains over a finite subcomplex (a compact subspace $Y \subset X$) are non–zero.
- For K–theory: if $[E] - [F] \in KO(X)$ then there is a compact subset $Y \subset X$ (the "support" of $[E] - [F]$) such that $E|_{X-Y} \simeq F|_{X-Y}$, that is the formal difference vanishes outside of some compact subspace.

This only works if we can compactify the space $X$ into a nice space $\bar{X}$ (i.e. a finite CW complex) by adding "points at infinity", as for example $\mathbb{R}^n$ (you can compactify it to a sphere by adding a point at infinity, or to the disk $D^n$ by adding a $S^{n-1}$). This excludes pathological spaces like the Riemann surface with infinitely many holes.

Cohomology of such a noncompact space is then the *relative cohomology* $H^*(X) = H^*(\bar{X}, \partial\bar{X})$ or $KO^*(X) = KO^*(\bar{X}, \partial\bar{X})$ where cochains or vector bundles are trivial over $\partial\bar{X}$. Obviously you can shrink $\partial\bar{X}$ to a single point without changing the compactly supported cohomology. If you take this point $\infty$ as the basepoint you recover the reduced cohomology:

$$X \text{ non–compact} \quad \Rightarrow \quad KO^*(X) = \widetilde{KO}^*(X \cup \{\infty\}) \tag{6.19}$$

## 6.2 Generalized Cohomology Theories

Since we are ultimately interested in K–theory (which is a generalized cohomology theory), I will restrict myself to cohomology rather than discuss homology and cohomology simultaneously. This is just to simplify notations, practically every formal property of cohomology has its counterpart in homology. For more details the reader is invited to consult [59].

I have described a very explicit realisation of cohomology based on cell decomposition of the space. However the resulting groups are homotopy invariants, and especially do not depend on the chosen cell decomposition. Why is that so? The real reason is that the cohomology groups are really determined by a few simple properties (the Eilenberg–Steenrod axioms). All the explicit cell decompositions only provide a way to compute groups that satisfy these axioms, and hence are the cohomology groups. Here are the axioms:

A reduced cohomology theory is a collection of maps (really cofunctors) $k^n$ that map topological spaces with basepoint to some abelian groups such that

- There is a natural equivalence $k^{n-1} \circ S \simeq k^n$
- For each $Y \subset X$ (with basepoint $x_0 \in Y \subset X$) the following sequence is exact:

$$k^n(X/Y) \to k^n(X) \to k^n(Y) \tag{6.20}$$

This suggests to try the following

**Definition 4.** *For any space $X$ let*

$$\begin{aligned} \widetilde{KO}^{-n}(X) &= \widetilde{KO}(S^n X) \qquad \forall n \in \mathbb{Z}_{\geq} \\ \widetilde{K}^{-n}(X) &= \widetilde{K}(S^n X) \end{aligned} \tag{6.21}$$

One can show that they satisfy eq. 6.20, so one gets almost a (reduced) cohomology theory. The problem of course is that the $\widetilde{K}^n$ are only defined for $n < 0$ while a cohomology theory would be "doubly infinite", that is we need $\widetilde{K}^n\ \forall\, n \in \mathbb{Z}$.

Now so far the $\widetilde{K}^n$ are basically defined in a way to fit the axioms, nothing really depends on the properties of vector bundles. But now we will make use of the most characteristic property of K–theory, Bott periodicity (theorem 24):

$$\begin{aligned}&\widetilde{KO}^{-8}(X) = \widetilde{KO}(X)\\&\widetilde{K}^{-2}(X) = \widetilde{K}(X)\end{aligned} \tag{6.22}$$

Armed with this result we simply define

**Definition 5.** *Let $n \in \mathbb{Z}$ arbitrary, and choose $k \in \mathbb{Z}_{\geq}:\ 8k+n>0$ (resp. $2k+n>0$). Then define*

$$\begin{aligned}&\widetilde{KO}^{-n}(X) = \widetilde{KO}(S^{8k+n}X)\\&\widetilde{K}^{-n}(X) = \widetilde{K}(S^{2k+n}X)\end{aligned} \tag{6.23}$$

and get a generalized (reduced) cohomology theory. The unreduced case is analogous. But once we know that we are dealing with a cohomology theory we can utilize all the machinery that is known to deal with cohomology theories. One of the most useful power tools will be the topic of the following sections.

## 6.3 Spectral Sequences

A *spectral sequence* is something like a long exact sequence, only (much) more complicated. For the sake of completeness I will describe how the technology works in this chapter, and we will apply it to K–theory in the next. However I will not try to describe the innards (which are rather overwhelming for the first time, see [48]) but merely how to apply the spectral sequence in a simple example.

Now spectral sequences[1] usually appear as concrete instances of the following

**Theorem 13 (Generic Theorem).** *There is a spectral sequence with*

$$E_2^{p,q} = [\text{something computable}] \tag{6.24}$$

[1] I will restrict myself here to spectral sequences for cohomology, as usual homology is the same with arrows reversed

*converging to* [something interesting]$^i$

For example let us compute $H^i\big(SU(3);\mathbb{Z}\big)$, one possibility is the following: As topological spaces

$$SU(2) = S^3 \tag{6.25}$$

and

$$SU(3)/SU(2) = S^5 \tag{6.26}$$

that is $SU(3)$ is a $S^3$–bundle over $S^5$. So $SU(3)$ satisfies the conditions in the following:

**Theorem 14 (Leray Serre Spectral Sequence).** *Let $X$ be a $F$–bundle over $B$, with simply connected base $B$. Then there is a spectral sequence with*

$$E_2^{p,q} = H^p\Big(B, H^q(F;\mathbb{Z})\Big)$$

*converging to $H^i(X;\mathbb{Z})$*

So plugging in the cohomology for $S^3$ and $S^5$ we find

$$E_2^{p,q} = \begin{array}{c|cccccc} q{=}3 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\ q{=}2 & 0 & 0 & 0 & 0 & 0 & 0 \\ q{=}1 & 0 & 0 & 0 & 0 & 0 & 0 \\ q{=}0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\ \hline & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 \end{array} \tag{6.27}$$

with all other $(p,q) \in \mathbb{Z}^2$ entries zero. Now the first step in evaluating the spectral sequence is to calculate further tableaus $E_r^{p,q}$ with $3 \le r < \infty$ from $E_2^{p,q}$. For that each tableau comes with maps (group homomorphisms) $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ with $d_r^2 = 0$. A few example $d_2$'s are shown in eq. 6.27. Now the next tableau is the $d_r$ cohomology of the previous:

$$E_{r+1}^{p,q} = \ker\Big(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}\Big)\Big/\operatorname{img}\Big(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}\Big) \tag{6.28}$$

In general this of course depends on the $d_r$'s, but in our example there are enough zeroes to fix all differentials unambiguously, e.g.

$$E_3^{0,3} = \ker\Big(d_2 : \mathbb{Z} \to 0\Big)\Big/\operatorname{img}\Big(d_2 : 0 \to \mathbb{Z}\Big) = \mathbb{Z} \tag{6.29}$$

So we can continue infinitely, and since the nonvanishing $(p,q)$ entries are only for a finite range of $p$'s the tableaus have to stay the same at some point (the spectral sequence *collapses*):

$$E_3^{p,q} = E_4^{p,q} = E_\infty^{p,q} = \begin{tikzcd}[row sep=small, column sep=small] q{=}3 & \mathbb{Z} \arrow[rrrdd, "d_3"] \arrow[rrrrddd, "d_4"'] & 0 & 0 & 0 & 0 & \mathbb{Z} \\ q{=}2 & 0 & 0 & 0 & 0 & 0 & 0 \\ q{=}1 & 0 & 0 & 0 & 0 & 0 & 0 \\ q{=}0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\ & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 \end{tikzcd} \tag{6.30}$$

So what does this tell us about the cohomology of $SU(3)$? Well $E_\infty^{p,q}$ is the *associated graded complex* to $H^*(SU(3);\mathbb{Z})$. This means that there is a *filtration* of $H^*(SU(3);\mathbb{Z})$, i.e. a sequence of subgroups

$$H^i(SU(3);\mathbb{Z}) = F_0^i \supset F_1^i \supset \cdots \supset F_n^i \supset 0 \tag{6.31}$$

such that the associated graded complex, i.e. the successive quotients are $E_\infty^{p,q}$:

$$F_p^{p+q} \Big/ F_{p+1}^{p+q} = E_\infty^{p,q} \tag{6.32}$$

So pictorially you have to "sum up" the diagonals in the tableau $E_\infty^{p,q}$. For example read off $H^3(SU(3);\mathbb{Z})$:

$$E_\infty^{p,q} = \begin{array}{c|cccccc} q{=}3 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\ q{=}2 & 0 & 0 & 0 & 0 & 0 & 0 \\ q{=}1 & 0 & 0 & 0 & 0 & 0 & 0 \\ q{=}0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\ \hline & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 \end{array}$$

$$\begin{aligned} F_0^3/F_1^3 = \mathbb{Z} \quad &\Rightarrow \quad F_0^3 = H^3(SU(3);\mathbb{Z}) = \mathbb{Z} \\ F_1^3/F_2^3 = 0 \quad &\Rightarrow \quad F_1^3 = 0 \\ F_2^3/F_3^3 = 0 \quad &\Rightarrow \quad F_2^3 = 0 \\ F_3^3/0 = 0 \quad &\Rightarrow \quad F_3^3 = 0 \end{aligned} \tag{6.33}$$

Thus we find

$$H^i\Big(SU(3);\mathbb{Z}\Big) = \begin{cases} \mathbb{Z} & i = 0,3,5,8 \\ 0 & \text{else} \end{cases} \tag{6.34}$$

## 6.4 The Atiyah–Hirzebruch Spectral Sequence

So how does this spectral sequence technology help us to compute interesting K–groups? There is a spectral sequence to compute any generalized cohomology from ordinary cohomology, the Atiyah–Hirzebruch–Whitehead spectral sequence. Specializing to K–theory this sequence is known as Atiyah–Hirzebruch spectral sequence (AHSS), derived in [7].

**Theorem 15 (Atiyah Hirzebruch S.S.).** *Let $X$ be a finite CW complex (or compact manifold). Then there is a spectral sequence with*

$$E_2^{p,q} = \begin{cases} H^p(X;\mathbb{Z}) & q \text{ even} \\ 0 & q \text{ odd} \end{cases} \tag{6.35}$$

*converging to $K^i(X)$.*

Consider $X = \mathbb{RP}^5$ as an example. First note that each $E_r^{p,q}$ tableau is 2–periodic in $q$. Then each second row is zero, so only the $d_r$ with $r$ even can possibly change the tableau. Thus

$$E_2^{p,q} = E_3^{p,q} = \begin{array}{c|cccccc} & \vdots & & \vdots & & \vdots & \vdots \\ q{=}2 & \mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z} \\ q{=}1 & 0 & 0 & 0 & 0 & 0 & 0 \\ q{=}0 & \mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z} \\ \hline & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 \end{array} \tag{6.36}$$

There is one $d_3$ where not either domain or range vanish automatically. However there is no group homomorphism $\mathbb{Z}_2 \to \mathbb{Z}$ except the zero map, and thus

$$E_5^{p,q} = \begin{array}{c|cccccc} q{=}4 & \mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z} \\ q{=}3 & 0 & 0 & 0 & 0 & 0 & 0 \\ q{=}2 & \mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z} \\ q{=}1 & 0 & 0 & 0 & 0 & 0 & 0 \\ q{=}0 & \mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z} \\ \hline & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 \end{array} \tag{6.37}$$

Now assume $d_5 : \mathbb{Z} \to \mathbb{Z}, x \mapsto nx$ is not the zero map ($n \neq 0$). Then we would have $E_6^{0,4} = E_\infty^{0,4} = 0$ and $E_6^{5,0} = E_\infty^{5,0} = \mathbb{Z}_n$. Compare this with theorem 4: The free part of $H^p(X;\mathbb{Z})$ has to appear in $K^i(X)$. This shows the following

**Theorem 16.** *In the AHSS the higher differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ do not change the free parts $E_{r+1,\mathrm{Free}}^{p,q}$. With other words $\mathrm{img}(d_3) \in E_{r,\mathrm{Tor}}^{p+r,q-r+1}$ or $d_3 \otimes \mathbb{Q} = 0$.*

So we find

$$E_\infty^{p,q} = \begin{array}{c|cccccc} q{=}4 & \mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z} \\ q{=}3 & 0 & 0 & 0 & 0 & 0 & 0 \\ q{=}2 & \mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z} \\ q{=}1 & 0 & 0 & 0 & 0 & 0 & 0 \\ q{=}0 & \mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z} \\ \hline & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 \end{array} \tag{6.38}$$

So there is a filtration (ignoring the odd rows) $K(\mathbb{R}\mathrm{P}^5) = F_0^0 \subset F_1^0 \subset F_2^0 \subset 0$ with

$$\begin{array}{rcl} F_0^0/F_1^0 = \mathbb{Z} & \Rightarrow & F_0^0 = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \text{ or } \mathbb{Z} \oplus \mathbb{Z}_4 \\ F_1^0/F_2^0 = \mathbb{Z}_2 & \Rightarrow & F_1^0 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \text{ or } \mathbb{Z}_4 \\ F_2^0/0 = \mathbb{Z}_2 & \Rightarrow & F_2^0 = \mathbb{Z}_2 \end{array} \tag{6.39}$$

So we cannot resolve the extension ambiguity! In fact one can fix this ambiguity by using properties of the tensor product (see e.g. [30] for details). The answer is

$$K^i(\mathbb{R}\mathrm{P}^5) = \begin{cases} \mathbb{Z} & i = 1 \\ \mathbb{Z} \oplus \mathbb{Z}_4 & i = 0 \end{cases} \tag{6.40}$$

## 6.5 The order of the torsion subgroup

So far the only difference between $K(X)$ and $H^{\mathrm{ev}}(X;\mathbb{Z})$ we saw in examples was in eq. 6.40. There in both cases the torsion subgroup was of order 4 but came with the two different group structures (either $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$).

Can one give a simple example where the order of the torsion subgroup is actually different? This is indeed possible but of course not by analyzing the AHSS, where the higher differentials are very hard to determine. Instead we use a relatively special property of complex K–theory, the *Künneth theorem* [3]:

$$0 \longrightarrow \bigoplus_{i+j=m} K^i(X) \otimes K^j(Y) \longrightarrow K^m(X \times Y) \longrightarrow \\ \longrightarrow \bigoplus_{i+j=m+1} \mathrm{Tor}\left(K^i(X), K^j(Y)\right) \longrightarrow 0 \tag{6.41}$$

where all indices are modulo 2. This is the same as the Künneth theorem in ordinary cohomology, just with $H$ instead of $K$.

Now let us apply this to $X \times Y = \mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5$. The K–groups of $\mathbb{R}\mathrm{P}^3$ are just

$$K^i(\mathbb{R}\mathrm{P}^3) = \begin{cases} \mathbb{Z} & i=1 \\ \mathbb{Z}\oplus\mathbb{Z}_2 & i=0 \end{cases} \tag{6.42}$$

this is an easy application of the AHSS. The K–groups of $\mathbb{R}\mathrm{P}^5$ were determined in eq. 6.40, so the Künneth theorem yields

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underset{\displaystyle \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2}{\underset{\shortparallel}{\Big[\mathbb{Z}\otimes\big(\mathbb{Z}\oplus\mathbb{Z}_4\big)\Big]\oplus\Big[\big(\mathbb{Z}\oplus\mathbb{Z}_2\big)\otimes\mathbb{Z}\Big]}} & \longrightarrow & K^1(\mathbb{R}\mathrm{P}^3\times\mathbb{R}\mathrm{P}^5) & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 \\ 0 & \longrightarrow & \underset{\displaystyle \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2}{\underset{\shortparallel}{\Big[\mathbb{Z}\otimes\mathbb{Z}\Big]\oplus\Big[\big(\mathbb{Z}\oplus\mathbb{Z}_2\big)\otimes\big(\mathbb{Z}\oplus\mathbb{Z}_4\big)\Big]}} & \longrightarrow & K^0(\mathbb{R}\mathrm{P}^3\times\mathbb{R}\mathrm{P}^5) & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \tag{6.43}$$

There is an ambiguity for $K^1$. We can fix it by the following result (see [66, 60])

**Theorem 17.** *Let $X$ be an even–dimensional orientable manifold. Then the is a duality between the torsion parts of $K^0$ and $K^1$:*

$$K^0(X)_{\mathrm{Tor}} = K^1(X)_{\mathrm{Tor}} \tag{6.44}$$

This fixes the above ambiguity and we arrive at the following result:

$$\begin{aligned} K^1(\mathbb{R}\mathrm{P}^3\times\mathbb{R}\mathrm{P}^5) &= \mathbb{Z}^2\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2^2 \\ K^0(\mathbb{R}\mathrm{P}^3\times\mathbb{R}\mathrm{P}^5) &= \mathbb{Z}^2\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2^2 \end{aligned} \tag{6.45}$$

Compare this with the ordinary cohomology of $\mathbb{R}\mathrm{P}^3\times\mathbb{R}\mathrm{P}^5$ which one can determine analogously by Künneth theorem and Poincaré duality:

$$H^i(\mathbb{R}\mathrm{P}^3\times\mathbb{R}\mathrm{P}^5) = \begin{cases} \mathbb{Z} & i=8 \\ \mathbb{Z}_2\oplus\mathbb{Z}_2 & i=7 \\ \mathbb{Z}_2 & i=6 \\ \mathbb{Z}\oplus\mathbb{Z}_2^2 & i=5 \\ \mathbb{Z}_2\oplus\mathbb{Z}_2 & i=4 \\ \mathbb{Z}\oplus\mathbb{Z}_2 & i=3 \\ \mathbb{Z}_2\oplus\mathbb{Z}_2 & i=2 \\ 0 & i=1 \\ \mathbb{Z} & i=0 \end{cases} \Rightarrow \begin{cases} H^{\mathrm{ev}}(\mathbb{R}\mathrm{P}^3\times\mathbb{R}\mathrm{P}^5) = \mathbb{Z}^2\oplus\mathbb{Z}_2^5 \\ H^{\mathrm{odd}}(\mathbb{R}\mathrm{P}^3\times\mathbb{R}\mathrm{P}^5) = \mathbb{Z}^2\oplus\mathbb{Z}_2^5 \end{cases}$$

(6.46)

So especially the order $|K(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5)_{\mathrm{Tor}}| = 16$ while $|H^{\mathrm{ev}}(\mathbb{R}\mathrm{P}^3 \times \mathbb{R}\mathrm{P}^5)_{\mathrm{Tor}}| = 2^5$

## 6.6 Computation: The Quintic

### 6.6.1 Lefschetz Hyperplane Theorem

Of course real projective spaces are not possible string theory backgrounds, even if one were to give up supersymmetry. In fact $\mathbb{R}\mathrm{P}^5$ is not spin, and the $\mathbb{R}\mathrm{P}^{2n}$ are not even orientable.

Really we would like to compactify on a Calabi–Yau manifold, which would preserve the minimum supersymmetry in 4 dimensions. Now almost all known Calabi–Yau manifolds are hypersurfaces or complete intersections in toric varieties. Unfortunately those will not have torsion in $H^{\mathrm{ev}}$, and therefore by theorem 4 torsion free K–groups. Here the torsion in $H^{\mathrm{ev}}$ can be determined by the

**Theorem 18 (Lefschetz Hyperplane Theorem).** *Let $X$ be a $\dim_{\mathbb{C}}(X) = n$ dimensional compact, complex manifold and let $V \subset X$ a hypersurface given as the zero–set of a positive line bundle (i.e. by a polynomial equation). Then*

$$H_i(V;\mathbb{Z}) = H_i(X;\mathbb{Z}) \quad \forall\ i \in 0, \ldots, n-2 \tag{6.47}$$

So especially for $V$ a complex 3 fold ($n = 4$) we have

$$\begin{aligned} H_1(X;\mathbb{Z})_{\mathrm{Tor}} = H_1(V;\mathbb{Z})_{\mathrm{Tor}} \quad &\Rightarrow \quad H^2(X;\mathbb{Z})_{\mathrm{Tor}} = H^2(V;\mathbb{Z})_{\mathrm{Tor}} \\ H_2(X;\mathbb{Z})_{\mathrm{Tor}} = H_2(V;\mathbb{Z})_{\mathrm{Tor}} \quad &\Rightarrow \quad H^3(X;\mathbb{Z})_{\mathrm{Tor}} = H^3(V;\mathbb{Z})_{\mathrm{Tor}} \end{aligned} \tag{6.48}$$

using the Universal Coefficient Theorem. But Poincaré duality then determines all the torsion in $H^*(V;\mathbb{Z})$.

### 6.6.2 The Quintic

So we need a Calabi–Yau manifold with torsion in the ordinary cohomology groups (this discussion is based on [16]). An example for this is the quotient of some Calabi–Yau (with torsion free cohomology) by a freely acting holomorphic group action. For example take the Fermat quintic

$$Q \stackrel{\text{def}}{=} \Big\{[z_0 : z_1 : \cdots : z_4] \Big| \sum_{i=0}^{4} z_i^5 = 0\Big\} \subset \mathbb{C}\mathrm{P}^4 \tag{6.49}$$

with the $G = \mathbb{Z}_5$ group action

$$g\Big([z_0 : z_1 : \cdots : z_4]\Big) = [z_0 : \alpha z_1 : \alpha^2 z_2 : \alpha^3 z_3 : \alpha^4 z_4], \qquad \alpha = e^{\frac{2\pi i}{5}} \tag{6.50}$$

The group $G$ acts freely on $Q$: The only fixed point $[1 : 0 : 0 : 0 : 0] \in \mathbb{C}\mathrm{P}^4$ of the ambient space is missed by the hypersurface eq. 6.49. Therefore the quotient $X \stackrel{\text{def}}{=} Q/G$ is again a smooth manifold.

Now we need the cohomology groups of the quotient $X$ to apply the AHSS. Since the quintic $Q$ was simply connected (as is every complete intersection), we can determine the quotient's fundamental group from the long exact homotopy sequence (for $Q$ as a bundle over $Q$ with fiber $G$):

$$\cdots \to \underbrace{\pi_1(G)}_{=0} \to \underbrace{\pi_1(Q)}_{=0} \to \pi_1(X) \to \underbrace{\pi_0(G)}_{=G} \to \underbrace{\pi_0(Q)}_{=0} \to \underbrace{\pi_0(X)}_{=0} \tag{6.51}$$

Since $Q$ is a complete intersection, $h^{1,1}(Q) = h^{1,1}(\mathbb{C}\mathrm{P}^4) = 1$. The quotient $X$ is still Kähler (the Kähler class $\omega = \partial\bar{\partial} \log ||Z||^2$ is $G$-invariant), so that $h^{1,1}(X) = h^{1,1}(Y) = 1$.

The complex structure deformations $h^{2,1}(Q)$ correspond to the monomials modulo PGL(4) (the automorphisms of the ambient space) and rescaling of the equation. Here there are $\binom{5+5-1}{5} = 126$ monomials, and $|\mathrm{PGL}(4)| = 24$. Therefore $h^{2,1}(Q) = 126 - 24 - 1 = 101$. The complex structure deformations of the quotient are the $G$-invariant monomials, straightforward counting gives 26. But now by treating every coordinate separately in the $G$-action the full PGL(4) is broken to the diagonal subgroup (4 parameters). Therefore $h^{2,1}(X) = 26 - 4 - 1 = 21$. An independent way (which does not rely on the rather naive counting of complex structure deformations) is to calculate the Euler number

$$\begin{aligned} &\chi(Q) = 2\Big(h^{2,1}(Q) - h^{1,1}(Q)\Big) = 200 \\ &\Rightarrow\ \chi(X) = \chi(Q/G) = \frac{1}{|G|}\chi(Q) = 40 = 2\Big(h^{2,1}(X) - h^{1,1}(X)\Big) \\ &\Rightarrow\ h^{2,1}(X) = 21 \end{aligned} \tag{6.52}$$

This can be summarized in the Hodge diamond

$$h^{p,q}(X) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 21 & & 21 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{6.53}$$

Now we have to find the torsion part of the cohomology groups. For every manifold $H^1(X;\mathbb{Z})$ is torsion free, since the torsion part is dual to the torsion part in $H_0(X;\mathbb{Z}) = \mathbb{Z}$. Furthermore $H_1(X;\mathbb{Z})$ is the abelianization of $\pi_1(X) = \mathbb{Z}_5$ which was already abelian. Therefore $H_1(X;\mathbb{Z}) = \mathbb{Z}_5$. By the universal coefficient theorem $H^2(X;\mathbb{Z})_{\text{tors}} \simeq H_1(X;\mathbb{Z})_{\text{tors}} = \mathbb{Z}_5$.

The hard part is the torsion in $H^3$ (Poincaré duality then determines the rest). We are going to use the following sequence [26]:

$$0 \;\to\; \Sigma_2 \;\to\; H_2(X;\mathbb{Z}) \;\to\; H_2(\mathbb{Z}_5) \;\to\; 0 \tag{6.54}$$

where $\Sigma_2$ is the image of $\pi_2(X)$ in $H_2(X;\mathbb{Z})$. With other words $\Sigma_2$ are the homology classes that can be represented by 2–spheres.

So we need to determine $\pi_2(X)$ first. We know that on the covering space $\pi_2(Q) = H_2(Q) = \mathbb{Z}$ (The Hurewicz isomorphism theorem) since $Q$ is simply connected. But every map $f : S^2 \to X$ can be lifted to $\tilde{f} : S^2 \to Q$ since the $S^2$ is simply connected. That is the $S^2$ cannot wrap the nontrivial $S^1 \subset X$. More formally we can use the homotopy long exact sequence:

$$\cdots \;\to\; \underbrace{\pi_2(G)}_{=0} \;\to\; \pi_2(Q) \;\to\; \pi_2(X) \;\to\; \underbrace{\pi_1(G)}_{=0} \;\to\; \cdots \tag{6.55}$$

to show that $\pi_2(X) = \pi_2(Q) = \mathbb{Z}$.

The group homology $H_2(\mathbb{Z}_5) = 0$, therefore eq. (6.54) determines an isomorphism $\Sigma_2 \simeq H_2(X;\mathbb{Z})$. We know already that the free part $H_2(X,\mathbb{Z})_{\text{free}} = \mathbb{Z}$ from the Hodge diamond. But then the map $\pi_2(X) \to \Sigma_2$ must have been injective since the domain is $\mathbb{Z}$ and the image at least $\mathbb{Z}$. Therefore $\Sigma_2 = \mathbb{Z}$ and the torsion part $H^3(X;\mathbb{Z})_{\text{tors}} \simeq H_2(X,\mathbb{Z})_{\text{tors}} = 0$.

We have seen that

$$H^i(X,\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 6 \\ \mathbb{Z}_5 & i = 5 \\ \mathbb{Z} & i = 4 \\ \mathbb{Z}^{44} & i = 3 \\ \mathbb{Z} \oplus \mathbb{Z}_5 & i = 2 \\ 0 & i = 1 \\ \mathbb{Z} & i = 0 \end{cases} \tag{6.56}$$

A more systematic way to compute the (co)homology of the quotient is the Cartan–Leray spectral sequence (theorem 30) as was pointed out in [18, 19].

### 6.6.3 K–theory of the quotient

From the AHSS it is obvious that either the $\mathbb{Z}_5$ torsion part survives to K–theory or vanishes (there is no subgroup except the trivial group). Therefore $K(X)_{\text{tors}} = \mathbb{Z}_5$ or 0. We can fix this ambiguity by the following

**Theorem 19.** $H^2(X;\mathbb{Z})_{\text{Tor}} \neq 0 \quad \Rightarrow \quad K(X)_{\text{Tor}} \neq 0$

*Proof.* Let $H^2(X;\mathbb{Z})_{\text{Tor}} \neq 0$. Since $H^2(X;\mathbb{Z})$ classifies line bundles (eq. 4.21) there exists a

$$E \in \text{Vect}^1_{\mathbb{C}}(X): \quad 0 \neq c_1(E) \in H^2(X;\mathbb{Z})_{\text{Tor}} \tag{6.57}$$

We will show that $0 \neq [E] - [1] \in K(X)$ is torsion. This naturally consists of two steps:

1. By assumption $c([E] - [1]) = 1 + c_1(E) \in H^{\text{ev}}(X;\mathbb{Z})$ does not vanish. But the total Chern class (eq. 4.26) is a group homomorphism, and therefore $[E] - [1] \neq 0$.

2. The image $c_1(E)$ in $H^2(X;\mathbb{Q})$ vanishes. Therefore

$$ch([E]-[1]) = ch(E) - ch(1) = e^{c_1(E)} - 1 = 0 \in H^{\text{ev}}(X;\mathbb{Q}) \tag{6.58}$$

   But then $[E] - [1] \in K(X)_{\text{Tor}}$: Otherwise you could complete it to a basis for $K(X)_{\text{Free}}$, and the Chern isomorphism (theorem 4) maps a basis of $K(X)_{\text{Free}}$ to a basis of $H^{\text{ev}}(X;\mathbb{Q})$.

□

Using Chern isomorphism and duality this determines the K–groups of $X = Q/G$ completely:

$$K^i(X) = \begin{cases} \mathbb{Z}^{44} \oplus \mathbb{Z}_5 & i = 1 \\ \mathbb{Z}^4 \oplus \mathbb{Z}_5 & i = 0 \end{cases} \tag{6.59}$$

# Chapter 7

# Bott Periodicity

In this chapter I will explain the missing link: The periodicity that allows us to complete the "half" long exact sequence in K–theory into the complete long exact sequence. For this I will use a technical connection with Clifford algebras. Indeed we will see that the periodicity with order 8 for real and 2 for complex K–theory is precisely the periodicity known for real and complex Clifford algebras. This connection with Clifford algebras will be useful to actually compute various K–groups.

## 7.1 Clifford Algebras

The Clifford algebras are the well–known $\gamma$–matrix algebra for arbitrary dimension and signature:

**Definition 6.** *The Clifford algebra $\mathbf{C}_{\mathbb{R}}^{p,q}$ is the real algebra generated by $\gamma_1, \dots, \gamma_{p+q}$ subject to the relations*

$$\begin{aligned} \gamma_i\gamma_j &= -\gamma_j\gamma_i & \forall\ i \neq j \\ \gamma_i^2 &= -1 & \forall\ i \in \{1, \dots, p\} \\ \gamma_i^2 &= +1 & \forall\ i \in \{p+1, \dots, p+q\} \end{aligned} \tag{7.1}$$

*Furthermore let its complexification*

$$\mathbf{C}_{\mathbb{C}}^{p} \stackrel{\text{def}}{=} \mathbf{C}_{\mathbb{R}}^{p,0} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbf{C}_{\mathbb{R}}^{0,p} \otimes \mathbb{C} \tag{7.2}$$

The complexification does not depend on the signature: If $\gamma^2 = -1$ then $(i\gamma)^2 = 1$. Let us compute a few examples. Obviously $\mathbf{C}_{\mathbb{R}}^{0,0} = \mathbb{R}$. The first interesting cases are

$$\begin{aligned} \mathbf{C}_{\mathbb{R}}^{1,0} &= \mathbb{R}[\gamma]\big/(\gamma^2+1=0) \;=\; \mathbb{C} \\ \mathbf{C}_{\mathbb{R}}^{0,1} &= \mathbb{R}[\gamma]\big/(\gamma^2-1=0) \;=\; \mathbb{R}[\gamma]\big/\big((\gamma+1)(\gamma-1)=0\big) \\ &= (\gamma-1)\mathbb{R} \oplus (\gamma+1)\mathbb{R} \;=\; \mathbb{R} \oplus \mathbb{R} \end{aligned} \tag{7.3}$$

The notation here keeps track of how the multiplication works: Both $\mathbb{C}$ and $\mathbb{R}\oplus\mathbb{R}$ can be thought of as pairs of real numbers and componentwise addition. But the multiplication is either "as complex numbers" or componentwise, and those possibilities are not related by a basis transformation.

Determining all Clifford algebras is essentially a finite task thanks to the following identities (see [6, 44]):

**Proposition 3.**

$$\mathbf{C}_{\mathbb{R}}^{p+n,q+n} \simeq \mathrm{Mat}_{2^n}\left(\mathbf{C}_{\mathbb{R}}^{p,q}\right) \tag{7.4}$$

*and*

$$\mathbf{C}_{\mathbb{R}}^{p+8,q} \simeq \mathbf{C}_{\mathbb{R}}^{p,q+8} \simeq \mathrm{Mat}_{16}\left(\mathbf{C}_{\mathbb{R}}^{p,q}\right) \tag{7.5}$$

So all Clifford algebras are determined by the following table:

| $n$ | $\mathbf{C}_{\mathbb{R}}^{n,0}$ | $\mathbf{C}_{\mathbb{R}}^{0,n}$ |
|---|---|---|
| 0 | $\mathbb{R}$ | $\mathbb{R}$ |
| 1 | $\mathbb{C}$ | $\mathbb{R}\oplus\mathbb{R}$ |
| 2 | $\mathbb{H}$ | $\mathrm{Mat}_2(\mathbb{R})$ |
| 3 | $\mathbb{H}\oplus\mathbb{H}$ | $\mathrm{Mat}_2(\mathbb{C})$ |
| 4 | $\mathrm{Mat}_2(\mathbb{H})$ | $\mathrm{Mat}_2(\mathbb{H})$ |
| 5 | $\mathrm{Mat}_4(\mathbb{C})$ | $\mathrm{Mat}_2(\mathbb{H})\oplus\mathrm{Mat}_2(\mathbb{H})$ |
| 6 | $\mathrm{Mat}_8(\mathbb{R})$ | $\mathrm{Mat}_4(\mathbb{H})$ |
| 7 | $\mathrm{Mat}_8(\mathbb{R})\oplus\mathrm{Mat}_8(\mathbb{R})$ | $\mathrm{Mat}_8(\mathbb{C})$ |

Table 7.1: List of Clifford algebras

## 7.2 Clifford Modules and K–theory

In physics the Clifford algebra usually appears as a matrix algebra acting on spinor fields. But physics never really depends on the explicit form of the chosen matrices, they only have to satisfy the (anti)commutator relations. So we should think of the $\gamma$–matrices as one special representation of the abstract Clifford algebra. Generalizing this we can consider Clifford algebras acting on arbitrary vector bundles, not only the spin bundle:

**Definition 7.** *A $\mathbf{C}_{\mathbb{R}}^{p,q}$ vector bundle on a space $X$ is a pair $(E,\rho)$ where $E \in \mathrm{Vect}_{\mathbb{R}}(X)$ and $\rho : \mathbf{C}_{\mathbb{R}}^{p,q} \to \mathrm{End}(E)$ is an algebra homomorphism (representation of the Clifford algebra); Let $\mathrm{Vect}_{\mathbb{R}}^{p,q}(X)$ be the set of such bundles.*

Note that the Clifford algebra acts fiberwise: $\rho(\gamma_i)(E_x) \subset E_x$. Now one can define the Grothendieck group of such bundles, however we need something more elaborate. For this we require more structure on the bundles:

**Definition 8.** *A gradation on a $\mathbf{C}^{p,q}_{\mathbb{R}}$ vector bundle $(E,\rho) \in \mathrm{Vect}^{p,q}_{\mathbb{R}}(X)$ is a map $\eta : E \to E$ (i.e. $\eta \in \mathrm{End}(E)$) such that*

- $\eta^2 = 1$
- $\eta\rho(\gamma_i) = -\rho(\gamma_i)\eta \qquad \forall\ i \in \{1, \dots, p+q\}$

Now $KO^{p,q}(X)$ is roughly the "Grothendieck group of gradations":

**Definition 9.** *$KO^{p,q}(X)$ is the group generated by triples $(E, \eta_1, \eta_2)$ where $E \in \mathrm{Vect}^{p,q}_{\mathbb{R}}(X)$, and $\eta_{1,2}$ are gradations on $E$ subject to the relations*

- $(E, \eta_1, \eta_2) + (F, \xi_1, \xi_2) = (E \oplus F, \eta_1 \oplus \xi_1, \eta_2 \oplus \xi_2)$
- $(E, \eta_1, \eta_2) = 0$ *if $\eta_1$ is homotopic to $\eta_2$ within the gradations of $E$.*

Although the definition does not talk of "isomorphism classes of vector bundles" it does really only depend on the triples up to isomorphism, for more details consult [44]:

**Lemma 3.** *We have the following identities in $KO^{p,q}(X)$:*

$$(E, \eta_1, \eta_2) + (E, \eta_2, \eta_1) = 0 \tag{7.6}$$

$$E \simeq E', \eta_1 \simeq \eta_1', \eta_2 \simeq \eta_2' \ \Rightarrow\ (E, \eta_1, \eta_2) = (E', \eta_1', \eta_2') \in KO^{p,q}(X) \tag{7.7}$$

$$(E, \eta_1, \eta_2) + (E, \eta_2, \eta_3) = (E, \eta_1, \eta_3) \tag{7.8}$$

As a consequence we get

**Proposition 4.** *Any element of $KO^{p,q}(X)$ can be represented by a triple $(E, \eta_1, \eta_2)$.*

*Proof.* This is a easy consequence of the previous lemma:

$$\begin{aligned} KO^{p,q}(X) \ni (E, \eta_1, \eta_2) - (F, \xi_1, \xi_2) &= \\ = (E, \eta_1, \eta_2) + (F, \xi_2, \xi_1) &= (E \oplus F, \eta_1 \oplus \xi_2, \eta_2 \oplus \xi_1) \end{aligned} \tag{7.9}$$

□

Here is another way to think about $KO^{p,q}(X)$. A $\mathbf{C}_{\mathbb{R}}^{p,q}$ vector bundle $(E,\rho)$ with graduation $\eta$ can be thought of as a $\mathbf{C}_{\mathbb{R}}^{p,q+1}$ vector bundle $(\hat{E},\hat{\rho})$ by taking $\hat{\rho}(\gamma_{p+q+1})=\eta$. So the possible gradations on $(E,\rho)\in\mathrm{Vect}_{\mathbb{R}}^{p,q}(X)$ are in one–to–one correspondence with the possible $\mathbf{C}_{\mathbb{R}}^{p,q+1}$–module structures. Therefore (see [43, 44]):

**Theorem 20.** *$KO^{p,q}(X)$ can also be described by triples $(E,F,\alpha)$ where $E,F\in\mathrm{Vect}_{\mathbb{R}}^{p,q+1}(X)$ and the vector bundle map $\alpha$ with $\alpha(E)=F$ is an isomorphism of the underlying $\mathbf{C}_{\mathbb{R}}^{p,q}$ vector bundles.*

What is the relation now with the obvious Grothendieck group of $\mathbf{C}_{\mathbb{R}}^{p,q}$ vector bundles and why did we choose such a complicated definition? Let $KO^{(p,q)}(X)$ be the obvious Grothendieck group of $\mathrm{Vect}_{\mathbb{R}}^{p,q}(X)$, then $KO^{p,q}(X)$ is defined to fit into the exact sequence

$$\begin{aligned}KO^{(p,q+1)}(X\times\mathbb{R})\to KO^{(p,q)}(X\times\mathbb{R})\to&\\ \to KO^{p,q}(X)\to KO^{(p,q+1)}(X)\to KO^{(p,q)}(X)&\end{aligned}\tag{7.10}$$

### Grothendieck group of $\mathbf{C}_{\mathbb{R}}^{p,q}$ vector bundles

In section 7.3 we will see what the groups $KO^{p,q}(X)$ are in terms of ordinary K–groups. And indeed they are different from the $KO^{(p,q)}(X)$. As an example let us determine the $KO^{(0,q)}(X)$ since this will be important later on.

Let us start with $KO^{(0,0)}(X)$. This is the Grothendieck group of vector bundles with an action of $\mathbf{C}_{\mathbb{R}}^{0,0}=\mathbb{R}$, that is multiplication by a real scalar. So obviously $KO^{(0,0)}(X)=KO(X)$, the Grothendieck group of ordinary vector bundles.

Now more interesting is $KO^{(0,1)}(X)$, which is generated by vector bundles with an action of $\mathbf{C}_{\mathbb{R}}^{0,1}=\mathbb{R}\oplus\mathbb{R}$. Especially there are the two orthogonal projectors $(1,0)$ and $(0,1)\in\mathbf{C}_{\mathbb{R}}^{0,1}$. So every $\mathbf{C}_{\mathbb{R}}^{0,1}$ vector bundle $(E,\rho)$ decomposes into

$$E=E_1\oplus E_2\stackrel{\mathrm{def}}{=}\Big(\rho(1,0)E\Big)\oplus\Big(\rho(0,1)E\Big)\tag{7.11}$$

The subbundles $E_1$, $E_2$ are otherwise independent and we conclude that $KO^{(0,1)}(X)=KO(X)\oplus KO(X)$.

Finally consider $KO^{(0,2)}(X)$, i.e. vector bundles with $\mathbf{C}_{\mathbb{R}}^{0,2}=\mathrm{Mat}_2(\mathbb{C})$ action. In the following section we will see that those are the same as vector bundles with $\mathbb{C}$ action (lemma 4). But a real vector bundle with an action of $\mathbb{C}$ is nothing but a complex vector bundle, thus $KO^{(0,2)}(X)=K(X)$.

The other $KO^{(0,q)}(X)$ are determined analogously from the Clifford algebra in table 7.1 and inherit the same periodicity, they are listed in table 7.2. We can turn the $KO^{(p,q)}(X)$ into a cohomology theory by the usual definition

$$\begin{array}{ll} KO^{(0,0)}(X) = KO(X) & KO^{(0,1)}(X) = KO(X) \oplus KO(X) \\ KO^{(0,2)}(X) = KO(X) & KO^{(0,3)}(X) = K(X) \\ KO^{(0,4)}(X) = KH(X) & KO^{(0,5)}(X) = KH(X) \oplus KH(X) \\ KO^{(0,6)}(X) = KH(X) & KO^{(0,7)}(X) = K(X) \end{array}$$

Table 7.2: List of the $\mathbf{C}_{\mathbb{R}}^{0,q}$ K–groups

$$KO^{(p,q),-i}(X) = KO^{(p,q)}(X \times \mathbb{R}^i) \tag{7.12}$$

but table 7.2 and its analog for $p \neq 0$ makes it clear that we will not gain anything new.

**The Basepoint**

Remember that we want to describe spaces with basepoint, and for them the natural cohomology theory is reduced cohomology. The difference is of course minor: The restriction of the bundles (with whatever structure they also carry) has to be trivial over the basepoint $* \in X$. Thus the reduced K–theory $\widetilde{KO}^{p,q}(X)$ is either described by

1. the triples $(E, \eta_1, \eta_2)$ such that $\eta_1|_{E_*} = \eta_2|_{E_*}$.
2. the triples $(E, F, \alpha)$ such that $E_* = F_*$ as $\mathbf{C}_{\mathbb{R}}^{p,q+1}$ modules.

## 7.3 The Periodicity

I will not prove the periodicity with mathematical rigor, rather I will try to present some of the results that lead to it since they will be important in the following. The first is

**Theorem 21.** *$KO^{p,q}(X)$ depends only on $p - q \mod 8$.*

This is the origin of the 8 in the periodicity, and it is really coming from the Clifford algebras. But this is not so trivial since the Clifford algebra are not really periodic mod 8, instead they satisfy eq. 7.5. The theorem follows from the following lemma:

**Lemma 4.** *The following semigroups are isomorphic:*

1. $S_1 \overset{\text{def}}{=}$ *equivalence classes of* $\mathbf{C}_{\mathbb{R}}^{p,q}$ *vector bundles.*

2. $S_1 \overset{\text{def}}{=}$ *equivalence classes of* $\mathbf{C}_{\mathbb{R}}^{p,q}$ *vector bundles with action of* $\mathrm{Mat}_n(\mathbb{R})$.

*The same is true without the* $\mathbf{C}_{\mathbb{R}}^{p,q}$ *action.*

*Proof.* Let $E$ be a vector bundle with an action $\rho : \mathrm{Mat}_n(\mathbb{R}) \to \mathrm{End}(E)$. Let $e_{ij} \in \mathrm{Mat}_n(\mathbb{R})$ be the matrix with entries 1 at position $(i, j)$ and 0 otherwise. Then

$$E = \rho(1)E = \rho\left(\sum_{i=1}^{n} e_{ii}\right) E = \bigoplus_{i=1}^{n} \rho(e_{ii})E \overset{\text{def}}{=} \bigoplus_{i=1}^{n} E_i \tag{7.13}$$

Furthermore the permutation matrices $\pi_{ij} = 1 - e_{ii} - e_{jj} + e_{ij} + e_{ji}$ (exchanging entry $i$ and $j$) are invertible and thus induce isomorphisms

$$E \simeq \rho(\pi_{ij})E \quad \Rightarrow \quad E_1 \simeq E_2 \simeq \cdots \simeq E_n \tag{7.14}$$

With this in mind the following maps are the desired isomorphisms:

$$S_1 \to S_2, E \mapsto \left(\bigoplus_{i=1}^{n} E,\ \rho\right) \tag{7.15}$$

$$S_2 \to S_1, \left(\bigoplus_{i=1}^{n} E_i,\ \rho\right) \mapsto E_1 \tag{7.16}$$

The case with $\mathbf{C}_{\mathbb{R}}^{p,q}$ action is analogous. $\square$

The next theorem is really the fundamental one, and its generalisation (theorem 25) will be important in the following chapters.

**Theorem 22.**

$$KO^{p,q+1}(X) = KO^{p,q}(X \times \mathbb{R}) \tag{7.17}$$

The proof is very technical and can be found in [44].

Now we only need a way to make contact with the ordinary K–theory to get the periodicity there. The key is to understand $KO^{0,0}(X) \ni (E, \eta_1, \eta_2)$. The gradations $\eta_i$ only have to satisfy $\eta_i^2 = 1$ since there are no Clifford algebra generators (gamma matrices). So we can decompose each fiber of $E$ into irreps of $\mathbb{Z}_2$. Of course we cannot simultaneusly diagonalize $\eta_1$ and $\eta_2$ — but we can do so after stabilisation! Let $\mathbf{1} = \mathbf{1}_{\mathrm{rk}(E)}$ denote the trivial gradation on $E$, then

$$\begin{aligned}(E, \eta_1, \eta_2) &= (E \oplus E, \eta_1 \oplus \mathbf{1}, \eta_2 \oplus \mathbf{1}) = (E \oplus E, \eta_1 \oplus \mathbf{1}, \mathbf{1} \oplus \eta_2) = \\ &= (E, \eta_1, \mathbf{1}) + (E, \mathbf{1}, \eta_2) = (E, \eta_1, \mathbf{1}) - (E, \eta_2, \mathbf{1})\end{aligned} \tag{7.18}$$

Let (up to a choice of basis)

$$\eta_i = \mathrm{diag}(\overbrace{-1,\ldots,-1}^{n_i},1,\ldots,1) \tag{7.19}$$

Then

$$\begin{aligned}(E,\eta_i,\mathbf{1}) = &\big(\ker(1+\eta_i), -\mathbf{1}_{n_i}, \mathbf{1}_{n_i}\big) + \\ &+ \big(\ker(1-\eta_i), \mathbf{1}_{\mathrm{rk}(E)-n_i}, \mathbf{1}_{\mathrm{rk}(E)-n_i}\big) = \\ =&\big(\ker(1+\eta_i), -\mathbf{1}_{n_i}, \mathbf{1}_{n_i}\big)\end{aligned} \tag{7.20}$$

So every class can be decomposed into a difference of triples $(E,-\mathbf{1},\mathbf{1})$. With this representation we can define maps

$$\begin{aligned}&KO^{0,0}(X) \to KO(X),\ (E,-\mathbf{1},\mathbf{1}) - (F,-\mathbf{1},\mathbf{1}) \mapsto [E]-[F] \\ &KO(X) \to KO^{0,0}(X),\ [E]-[F] \mapsto (E,-\mathbf{1},\mathbf{1}) - (F,-\mathbf{1},\mathbf{1})\end{aligned} \tag{7.21}$$

Those maps are obviously inverse of each other; We have shown

**Theorem 23.**

$$KO^{0,0}(X) = KO(X) \tag{7.22}$$

Armed with these results Bott periodicity follows:

**Theorem 24 (Bott Periodicity).**

$$KO^{-n-8}(X) = KO^{-n}(X) \qquad \forall\ n \in \mathbb{Z}_{\geq} \tag{7.23}$$

*Proof.*

$$\begin{aligned}KO^{-n-8}(X) = KO(X\times\mathbb{R}^{n+8}) = KO^{0,0}(X\times\mathbb{R}^{n+8}) = \\ = KO^{0,n+8}(X) = KO^{0,n}(X) = KO^{-n}(X)\end{aligned} \tag{7.24}$$

∎

# Chapter 8

# Equivariant K–theory

## 8.1 Equivariant Vector Bundles

So far we only considered K–theory on ordinary spaces. But in string theory most solvable compactifications are not bona fide manifolds, but orbifolds. For us an orbifold is a quotient $X/G$ of a manifold $X$ by a discrete group such that the fields are equivariant. Equivariant means that the fields are not invariant, but rather transform with some representation of $G$: $\varphi(gx) = r(g)\varphi(x)$. The fields that do not transform with the trivial representation (that is those that are not single-valued on the quotient $X/G$) are called *twisted sector* fields.

So rather than being sections of ordinary bundles on the topological quotient $X/G$, physics is described by bundles on the original space $X$ that transform with $G$:

**Definition 10.** *Let $X$ be a $G$–space, i.e. a space with $G$–action. An equivariant vector bundle $E$ on $X$ is a ordinary vector bundle together with a group action*

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[d, "g"'] & X \arrow[d, "g"] \\ E \arrow[r, "\pi"] & X \end{tikzcd} \tag{8.1}$$

So there is a matrix–valued function $r(g, x)$ acting on the fibers as

$$E_{gx} = r(g, x)E_x \tag{8.2}$$

You can add equivariant vector bundles as usual; By the same Grothendieck group construction as before we can turn this semigroup into a group. This group is called equivariant K–theory and denoted $K_G(X)$ for complex bundles or $KO_G(X)$ for real bundles.

There is the obvious map $KO_G(X) \to KO(X)$ forgetting the group action. But it is in general neither surjective nor injective: A given (ordinary) vector bundle might not allow a group action that is compatible with the group action on the base, or might allow two inequivalent group actions. However there are two important special cases when we may say more:

1. If the $G$–action on the base $X$ is trivial then the forgetting map is onto. Moreover you can pick $x_0 \in X$ and decompose $r(\cdot, x_0)$ into irreducible representations of $G$. Up to a choice of basis the irreps determine the action, therefore
$$K_G(X) = K(X) \otimes R(G) \qquad KO_G(X) = KO(X) \otimes RO(G) \quad (8.3)$$
where $R(G)$ resp. $RO(g)$ is the real representation ring (resp. real representation ring) of $G$.

2. If the $G$–action is free then you can choose a basis for $E_x$ and $E_{gx}$ such that $r(x, g) = 1$. Thus
$$K_G(X) = K(X/G) \qquad KO_G(X) = KO(X/G) \quad (8.4)$$
or more generally $K_{H\times G}(X) = K_H(X/G)$.

Unfortunately already quite simple equivariant groups can be surprisingly hard to compute if the group action is neither trivial nor free on the base. For example a physically interesting question would be to compute $KO_{\mathbb{Z}_2}(\mathbb{R}^{p,q})$ where $\mathbb{R}^{p,q}$ is $\mathbb{R}^{p+q}$ with involution
$$\begin{aligned}&\mathbb{Z}_2 \times \mathbb{R}^{p+q} \to \mathbb{R}^{p+q},\\ &\quad (x_1, \ldots, x_p, x_{p+1}, \ldots, x_{p+q}) \mapsto (-x_1, \ldots, -x_p, x_{p+1}, \ldots, x_{p+q}) \quad (8.5)\end{aligned}$$
With our present tools (that is basically the definition and general properties of cohomology theories) I do not know how to determine them completely. However we can add more structure in addition to the group action, and this will finally allow us to determine the $KO_{\mathbb{Z}_2}(\mathbb{R}^{p,q})$. This will be the topic of the following sections.

## 8.2 real vs. Real K–theory

A nice unified treatment of real and complex K–theory was suggested in [4]. The idea is roughly to consider complex bundles together with an antiholomorphic involution. Depending on the details of the involution this contains the usual real and complex vector bundles as special cases. The most general way to introduce this is simply replace "space" everywhere by "space with involution", together with suitable anti–linearity so that it generalizes complex conjugation (Following [9] called "*Real* space"):

**Definition 11.** *A Real space $X$ is a (topological) space together with a continuous map $\tau : X \to X$ such that $\tau \circ \tau = \mathrm{id}_X$. If $X$, $Y$ are Real spaces then a Real map $f : X \to Y$ is a map of the underlying ordinary spaces that commutes with $\tau$.*

*A Real vector bundle is a complex vector bundle $E \to X$ (i.e. with total space $E$ and base $X$) such that $E$, $X$ are Real spaces, $E \to X$ a Real map and the involution $\tau|_{E_x} : E_x \to E_{\tau(x)}$ is anti–linear on the fibers:*

$$\tau\big(\lambda v\big) = \bar{\lambda}\tau(v) \qquad \forall\ \lambda \in \mathbb{C},\ v \in E_x \tag{8.6}$$

You can add Real vector bundles, so by the usual Grothendieck construction we get Real K–theory $KR(X)$.

The two special cases I mentioned earlier are whether the involution acts trivially or freely on the base space:

1. If $\tau : X \to X$ is trivial ($\tau(x) = x$) then Real vector bundles on $X$ are the same as real vector bundles. One can see this as follows: Given a Real vector bundle $E$ the involution $\tau$ maps each fiber into itself, so you can define the $\tau$–invariant subbundle $E^{\mathbf{R}}$. This is an ordinary real vector bundle with $\mathrm{rk}_{\mathbb{R}}(E^{\mathbf{R}}) = \frac{1}{2}\,\mathrm{rk}_{\mathbb{R}}(E)$.

   Conversely given an ordinary real vector bundle $F$ you can form its complexification $F \otimes_{\mathbb{R}} \mathbb{C}$, that is the complex vector bundle you can build out of the $GL(n, \mathbb{R})$ transition functions. The complex conjugation on $F \otimes_{\mathbb{R}} \mathbb{C}$ is the usual complex conjugation on the fibers. This is well–defined precisely because the a priori $GL(n, \mathbb{C})$ transition functions are in $GL(n, \mathbb{R}) \subset GL(n, \mathbb{C})$.

   Complexification and taking the real subbundle are mutually inverse, therefore

   $$KR^*(X) = KO^*(X) \tag{8.7}$$

2. If $X = Y \sqcup Y$ (disjoint union) with $\tau(y_1, y_2) = (y_2, y_1)$ exchanging the components, then Real vector bundles on $X$ are the same as complex vector bundles on $Y$ and therefore $KR^*(X) = K^*(Y)$. The correspondence is obviously by restricting a bundle on $X$ to one copy of $Y$. Conversely if you are given a bundle $E$ on $Y$ you can form the bundle $E \sqcup \bar{E}$ on $X$, this is obviously the inverse.

Note that it is not sufficient for $\tau$ to act freely. For example take $X = S^1 \times S^1$ with the involution the antipodal map on the first $S^1$ and the identity on the second $S^1$. Then $X/\tau = S^1 \times S^1 = T^2$ and

$$KR(X) = \mathbb{Z} \oplus \mathbb{Z}_2 \neq \mathbb{Z} \oplus \mathbb{Z} = K(T^2) \tag{8.8}$$

By now we have seen enough computational tools to confirm the above result for $K(T^2)$ easily, but how to compute $KR(X)$? For this one has to realize that $KR(X) = KSC(S^1)$ (this is the definition of K–theory for self–conjugate bundles $KSC$, compare [4]) and the $KSC$–groups for spheres are known (see [4, 32, 10]).

In string theory a background spacetime that comes only with a $\mathbb{Z}_2$ group action would be an orbifold, as in the previous section. But there is the possibility to combine the geometric group action with orientation reversal of the string worldsheet (also called parity). If one investigates the induced action on the Chan–Paton factors (see [31]) one realizes that this just amounts to complex conjugation in the above sense. Such a model is called an *orientifold* and one can check that the K–groups agree with the possible D–brane charges, see [35].

## 8.3 Equivariant Real Bundles

Now that we have defined Real K–theory and equivariant K–theory the obvious thing to do is to combine them. There is a more subtle point that is not so obvious: instead of just making Real K–theory equivariant, we will do equivariant K–theory for Real spaces. Especially we allow for Real groups, that is groups together with an involution:

**Definition 12.** *A Real group $G$ is a group with involution $\tau : G \to G$ such that*

$$\tau(g_1)\tau(g_2) = \tau(g_1 g_2) \qquad \forall\ g_1, g_2 \in G \tag{8.9}$$

An equivariant Real vector bundle $E$ is then a Real vector bundle on which the Real group $G$ acts by a Real map. For $v \in E_x$ this means ($gv \in E_{gx}$, $\tau(v) \in E_{\tau(x)}$):

$$\tau\Big(g\big(\lambda v\big)\Big) = \tau\Big(\lambda g\big(v\big)\Big) = \bar{\lambda}\tau\Big(gv\Big) = \bar{\lambda}\tau(g)\tau(v) \quad \forall\ \lambda \in \mathbb{C}, g \in G \tag{8.10}$$

Another point of view is the following: Take as group $\mathbb{Z}_2 \times G$ and let the subgroup $\{0\} \times G$ act by complex linear maps on the underlying complex vector bundle, and the subgroup $\{1\} \times G$ by complex antilinear maps. This suggests the following generalization:

**Definition 13.** *Let $H$ be a group and $\theta : H \to \mathbb{Z}_2$ a group homomorphism (the* augmentation *map). Then a $H$–equivariant Real vector bundle $(E, \rho)$ is a complex vector bundle $E$ together with a group action $\rho : H \to \mathrm{End}(E^{\mathbf{R}})$ on the underlying real vector bundle such that $\forall\ h \in H$:*

- *$\rho(h)$ is complex linear if $\theta(h) = 0$.*
- *$\rho(h)$ is complex antilinear if $\theta(h) = 1$, i.e.*

$$\rho(h)\big(\lambda v\big) = \bar{\lambda}\rho(h)\big(v\big) \qquad \forall\ \lambda \in \mathbb{C}, v \in E_{x_0} \tag{8.11}$$

The Grothendieck group of such bundles is then equivariant Real K–theory $KR_H(X)$. The notation is such that the group includes the $\mathbb{Z}_2$–involution. So in section 8.2 we should have called the Real K–theory $KR_{\mathbb{Z}_2}(X)$. Note that there are two different notations in the literature, ours is the of [45, 54] while [4] does not include the involution in the group subscript.

If you define $\widetilde{\mathrm{End}}(E)$ as the group of endomorphisms with augmentation $\theta$ such that $f \in \widetilde{\mathrm{End}}(E)$ acts complex linear if $\theta(f) = 0$ and complex antilinear if $\theta(f) = 1$ then the definition of Real vector bundles can be written as follows: It is a complex vector bundle with an action $\rho : H \to \widetilde{\mathrm{End}}(E)$. Here $\rho$ is a homomorphism of augmented groups, that is a group homomorphism that is compatible with the augmentation.

The advantage of using these augmentations is that now also complex K–theory is naturally included in Real K–theory, simply take $\theta : \{1\} \to \mathbb{Z}_2$ the trivial homomorphism $\theta(1) = 0$. Or more generally

$$\theta(g) = 0\ \forall\, g \in G \quad \Rightarrow \quad KR^*_G(X) = K^*_G(X) \tag{8.12}$$

## 8.4 Equivariant Real Bundles and Clifford Algebras

Finally let us include Clifford algebras. So let $G$ be a group together with a map $\theta : G \to \mathbb{Z}_2$ and an action on the Clifford algebra $\mathbf{C}_{\mathbb{R}}^{p,q}$, that is every $g \in G$ acts by an real–linear augmentation map on the vector space spanned by the $\gamma_i$ that preserves the Clifford algebra:

$$\begin{aligned} g(\gamma_i)g(\gamma_j) &= -g(\gamma_j)g(\gamma_i) && \forall\ i \neq j \\ g(\gamma_i)^2 &= -1 && \forall\ i \in \{1, \ldots, p\} \\ g(\gamma_i)^2 &= +1 && \forall\ i \in \{p+1, \ldots, p+q\} \end{aligned} \tag{8.13}$$

Then a Real equivariant $\mathbf{C}_{\mathbb{R}}^{p,q}$ vector bundle is a Real equivariant vector bundle $E$ and a representation $\rho : \mathbf{C}_{\mathbb{R}}^{p,q} \to \mathrm{End}(E)$ (complex–linear maps $E \to E$) compatible with the $G$ actions:

$$r(g) \circ \rho(\gamma_i) = \rho(g\gamma_i) \circ r(g) \qquad \forall\, g \in G \tag{8.14}$$

To make this more precise rewrite the above equation with indices: Given $g \in G$ there is a matrix representing the action on the Clifford algebra $r^{\gamma}_{ij} \in \mathrm{Mat}(p+q,\mathbb{R})$ such that $g(\gamma_i) = r^{\gamma}_{ij}\gamma_j$. Furthermore for each point $x \in X$ of the base space there is another matrix $r_{nm} : E_x \to E_{gx}$ representing the group action on the fiber. In indices we have $r_{nm} \in \mathrm{Mat}(\mathrm{rk}_{\mathbb{R}}(E),\mathbb{R})$ acting on the underlying real vector space of the complex vector space $E_x$. Equation 8.14 then reads:

$$r_{nm}\rho(\gamma_i)_{ml}v_l = \rho(r^{\gamma}_{ij}\gamma_j)_{nm}r_{ml}v_l \qquad \forall\, v \in E_x \tag{8.15}$$

By the ordinary Grothendieck group construction we then get the equivariant Real $\mathbf{C}_{\mathbb{R}}^{p,q}$ K–theory . But this notation hides the $G$–action on the Clifford algebra, so instead let $V$ is the $G$ vector space spanned by the generators $\gamma_i \in \mathbf{C}_{\mathbb{R}}^{p,q}$ and we will talk of equivariant Real $\mathbf{C}(V)$ bundles.

Now as in section 7.2 we are not really interested in the ordinary Grothendieck group but rather in the “Grothendieck group of gradations”. What is a gradation in the equivariant context? We require that an equivariant Real $\mathbf{C}_{\mathbb{R}}^{p,q}$ vector bundle with gradation is the same as a equivariant Real $\mathbf{C}_{\mathbb{R}}^{p,q+1}$ vector bundle. Here both Clifford algebras come with $G$ actions, so we really want that an equivariant Real $\mathbf{C}(V)$ vector bundle with gradation is the same as an equivariant Real $\mathbf{C}(V \oplus 1)$ vector bundle. Here the 1 denotes the one–dimensional vector space with the trivial group action. Because of that we require that the gradation commutes with the group action.

**Definition 14.** *Let $V$ be the $G$–vector space spanned by the $\gamma_i \in \mathbf{C}_{\mathbb{R}}^{p,q}$. Then let $KR^V_G(X)$ be the free group generated by triples $(E,\eta_1,\eta_2)$ (where $E$ is an equivariant Real $\mathbf{C}_{\mathbb{R}}^{p,q}$ vector bundle and $\eta_1,\eta_2$ two gradations) subject to the relations as in definition 9.*

If you replace Real by real everywhere you arrive at the analogous definition:

**Definition 15.** *Let $V$ be the $G$–vector space spanned by the $\gamma_i \in \mathbf{C}_{\mathbb{R}}^{p,q}$. Then let $KO^V_G(X)$ be the free group generated by triples $(E,\eta_1,\eta_2)$ (where $E$ is an equivariant real $\mathbf{C}_{\mathbb{R}}^{p,q}$ vector bundle and $\eta_1,\eta_2$ two gradations) subject to the relations above.*

**Examples**

Let us consider a few examples to understand the definition. Let $G = \mathbb{Z}_2 = \{1, \tau\}$ with $\theta = \mathrm{id}_{\mathbb{Z}_2}$ and take the trivial Clifford algebra $\mathbf{C}_{\mathbb{R}}^{0,0} \;\Rightarrow\; V = \mathbb{R}^0$. Then obviously

$$KR^V_{\mathbb{Z}_2}(X) = KR(X) \tag{8.16}$$

Now for a more interesting example. Take $G$ as above, but now acting trivially on the base space (so without the Clifford algebra this would reduce to KO–theory). Let $V = \mathbb{R}^{p,q}$ be the vector space spanned by the generators of $\mathbf{C}_{\mathbb{R}}^{0,p+q}$, the $\mathbb{Z}_2$–action being the usual on $\mathbb{R}^{p,q}$. Then let $(E, \rho)$ be a Real vector bundle with $\mathbf{C}(V)$ action.

Now we would like to restrict the bundle $E$ to the $\tau$–invariant subbundle $E^{\mathbf{R}} = \ker(1-\tau) \subset E$ and thereby construct an isomorphism $KR^V_{\mathbb{Z}_2}(X) \to KO^{0,p+q}(X)$ as in eq. 8.7. But this only works if $\tau$ and $\rho(\gamma_i)$ commute, since otherwise the $\mathbf{C}(V)$ action does not restrict to an action on $E^{\mathbf{R}}$. By assumption and eq. 8.14

$$\begin{array}{ll} \tau \circ \rho(\gamma_i) = \rho(-\gamma_i) \circ \tau = -\rho(\gamma_i) \circ \tau & \forall\, i \in \{1, \ldots, p\} \\ \tau \circ \rho(\gamma_i) = \rho(\gamma_i) \circ \tau & \forall\, i \in \{p+1, \ldots, p+q\} \end{array} \tag{8.17}$$

So only the $\mathbf{C}(\mathbb{R}^{0,q}) \subset \mathbf{C}(V)$ acts nicely on $E^{\mathbf{R}}$. The trick to make it work is to define a new

$$\tilde{\rho}(\gamma_i) = \left\{ \begin{array}{ll} i\rho(\gamma_i) & \forall\, i \in \{1, \ldots, p\} \\ \rho(\gamma_i) & \forall\, i \in \{p+1, \ldots, p+q\} \end{array} \right. \tag{8.18}$$

since this takes care of the extra minus sign (remember that $\tau$ is antilinear:

$$\tau \circ \tilde{\rho}(\gamma_i) = -i\tau \circ \rho(\gamma_i) = \tilde{\rho}(\gamma_i) \circ \tau \qquad \forall\, i \in \{1, \ldots, p\} \tag{8.19}$$

But $\tilde{\rho}$ is no longer a representation of $\mathbf{C}_{\mathbb{R}}^{0,p+q}$ because

$$\Big(\tilde{\rho}(\gamma_i)\Big)^2 = -1 \qquad \forall\, i \in \{1, \ldots, p\} \tag{8.20}$$

However we can take $\tilde{\rho}$ to be a representation of $\mathbf{C}_{\mathbb{R}}^{p,q}$. Obviously the assignment $\rho \leftrightarrow \tilde{\rho}$ is invertible and to this end the old result eq. 8.7 generalizes to

$$KR^V_{\mathbb{Z}_2}(X) = KO^{p,q}(X) \tag{8.21}$$

We can generalize this immediately if we let $G = \mathbb{Z}_2 \times H$ with $H$ some arbitrary other group, and $\theta : G \to \mathbb{Z}_2$ being the projection on the second factor. Because $H$ commutes with the $\mathbb{Z}_2$ everything goes though in the same way and we get:

$$KR^V_{\mathbb{Z}_2\times H}(X) = KO^{\tilde{V}}_H(X) \tag{8.22}$$

where now $V$ generates $\mathbf{C}^{0,p+q}_{\mathbb{R}}$ and comes with a $\mathbb{Z}_2 \times H$–action such that the first factor $\mathbb{Z}_2$ acts as on $\mathbb{R}^{p,q}$, and $\tilde{V}$ generates $\mathbf{C}^{p,q}_{\mathbb{R}}$ and comes with the induced $H$–action.

### The fundamental Theorem

The generalization of theorem 22 is the following:

**Theorem 25.** *Let $V$, $W$ be the $G$ vector spaces spanned by the generators $\gamma_i$, $\tilde{\gamma}_j$ of two Clifford algebras, and let $\gamma_i^2 = 1$ (i.e. $V$ generates $\mathbf{C}^{0,q}_{\mathbb{R}}$). Then*

$$KR^{W\oplus V}_G(X) = KR^W_G(X \times V) \tag{8.23}$$

The proof can be found in [45, 43] and will not be repeated here. But let me describe the map that leads to the isomorphism:

An element of $KR^{V\oplus W}_G(X)$ is a tuple $(E, g; v, w; \eta_1, \eta_2)$ where

1. $E$ is the underlying complex vector bundle.
2. $g$ stands for the group action on $E$.
3. $v$, $w$ denote the Clifford algebra actions of the Clifford algebra generated by $V$ and $W$ on $E$.
4. $\eta_1$, $\eta_2$ are two gradations.

Then let $S(V)$ be the unit sphere in $V$, $B(V)$ the unit disk thought of as the upper half sphere $S(V \oplus 1)$, see figure 8.1. So we can denote the points of $B(V)$ by $(v, \varphi)$ where $||v||^2 = v^2 = 1$ and $0 \leq \varphi \leq \frac{\pi}{2}$. With this define

$$\begin{aligned} t : KR^{V\oplus W}_G(X) &\to KR^W_G\big(X \times B(V), X \times S(V)\big), \\ \Big(E, g; v, w; \eta_1, \eta_2\Big) &\mapsto \\ &\Big(\pi^*(E), g; w; v\cos(\varphi) + \eta_1 \sin(\varphi), v\cos(\varphi) + \eta_2 \sin(\varphi)\Big) \end{aligned} \tag{8.24}$$

Note that $\xi_i \stackrel{\text{def}}{=} v\cos(\varphi) + \eta_i \sin(\varphi)$ is again a gradation since

$$\begin{aligned} \xi_i^2 = \Big(v\cos(\varphi) + \eta_i \sin(\varphi)\Big)^2 &= v^2\cos^2(\varphi) + \eta_i^2 \sin^2(\varphi) = \\ &= \cos^2(\varphi) + \sin^2(\varphi) = 1 \end{aligned} \tag{8.25}$$

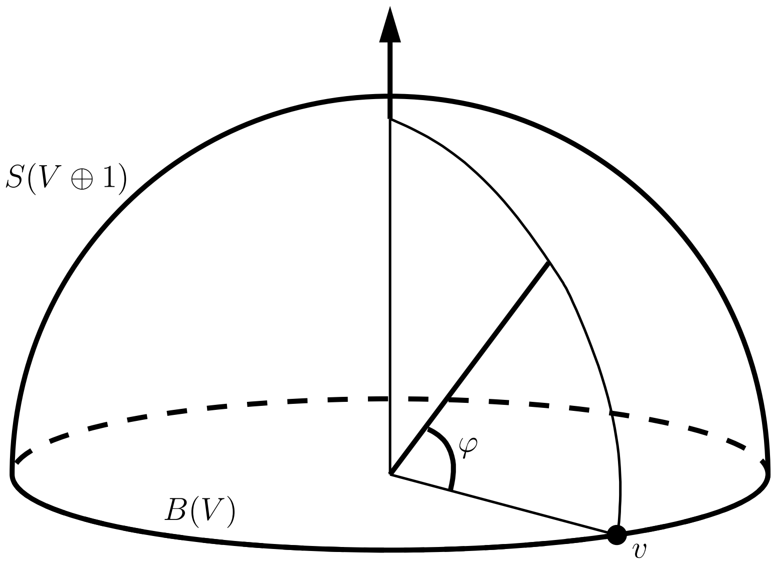


Figure 8.1: $B(V)$ as half $S(V \oplus 1)$

and it commutes with the $G$ action: Let $(x, v) \in X \times B(V)$ and let $w_{(x,v)} \in \pi^*(E)|_{(x,v)}$ a vector over $(x, v)$. Then

$$\begin{aligned} g \circ \xi_i(w_{(x,v)}) &= g \circ \Big(v\cos(\varphi) + \eta_i \sin(\varphi)\Big)(w_{(x,v)}) = \\ &= \Big(g(v)\cos(\varphi) + \eta_i \sin(\varphi)\Big) \circ g(w_{(x,v)}) = \\ &= \Big(g(v)\cos(\varphi) + \eta_i \sin(\varphi)\Big)(g(w)_{(g(x),g(v))}) = \\ &= \xi_i\big(g(w)_{(g(x),g(v))}\big) = \xi_i \circ g(w_{(x,v)}) \end{aligned} \tag{8.26}$$

Note that in eq. 8.25 we need $v^2 = 1$, i.e. $V$ generates $\mathbf{C}_{\mathbb{R}}^{0,q}$ as is one of the prerequisites of the theorem.

## 8.5 $KO_{\mathbb{Z}_2}(\mathbb{R}^{p,q})$

Now let us return to the computation of $KO_{\mathbb{Z}_2}(\mathbb{R}^{p,q})$ that was promised at the end of section 8.1. This will be based on the following

**Theorem 26.** *Let $V = \mathbb{R}^{p+q,0}$ the span of the generators of $\mathbf{C}_{\mathbb{R}}^{p,q}$, and let $X$ be a space with trivial $\mathbb{Z}_2$ action. Then*

$$KO_{\mathbb{Z}_2}^V(X) = KO^{(p,q+1)}(X) \tag{8.27}$$

*Proof.* Let $g$ be the generator of $\mathbb{Z}_2$. It satisfies by eq. 8.14:

$$r(g)\rho(\gamma_i) = \rho(g\gamma_i)r(g) = -\rho(\gamma_i)r(g) \tag{8.28}$$

So a $\mathbb{Z}_2$ equivariant $\mathbf{C}(V)$ vector bundle $E$ is the same as an ordinary $\mathbf{C}(V \oplus 1)$ vector bundle $\tilde{E}$ by setting $\rho(\gamma_{p+q+1}) = r(g)$.

But remember that $KO^V_{\mathbb{Z}_2}(X)$ is generated by triples $(E, \eta_1, \eta_2)$. By the above remark we can turn $E$ into a $\mathbf{C}^{p,q+1}_{\mathbb{R}}$ vector bundle, but then the $\eta_i$ are no longer gradations since they commute with $\rho(\gamma_{p+q+1})$ instead of anticommute. However let

$$\tilde{\eta}_i = r(g) \circ \eta_i \quad \in \mathrm{End}(E) \tag{8.29}$$

Note that for this to be in $\mathrm{End}(E)$, i.e. mapping fibers $E_x$ to itself we needed that the $G$ action fixes the base $X$.

The $\tilde{\eta}_i$ now commute with the whole $\mathbf{C}(V \oplus 1)$ action, so we can think of them as gradations of a $\mathbf{C}^{0,0}_{\mathbb{R}}$ action on the $\mathbf{C}^{p,q+1}_{\mathbb{R}}$ vector bundle. By the same argument that lead to theorem 23 (i.e. $KO^{0,0}(X) = KO(X)$) we get the analogous theorem with “real bundles” replaced by “real bundles with $\mathbf{C}^{p,q+1}_{\mathbb{R}}$ action”. □

This result enables us to dispose of one $\mathbb{Z}_2$ action: Given a space $X$ with trivial $\mathbb{Z}_2$ action we find:

$$\begin{aligned}
KO_{\mathbb{Z}_2}(X \times \mathbb{R}^{p,q}) &= KR_{\mathbb{Z}_2\times\mathbb{Z}_2}(X \times \mathbb{R}^{p,q}) && \text{by eq. 8.22}\\
&= KR_{\mathbb{Z}_2\times\mathbb{Z}_2}(X \times \mathbb{R}^{0,q} \times \mathbb{R}^{p,0}) && \text{first } \mathbb{Z}_2 \text{ acts trivial}\\
&= KR^V_{\mathbb{Z}_2\times\mathbb{Z}_2}(X \times \mathbb{R}^{q}) && \text{by theorem. 25}\\
&= KO^{\tilde{V}}_{\mathbb{Z}_2}(X \times \mathbb{R}^{q}) && \text{by eq. 8.22}\\
&= KO^{(\tilde{V}\oplus 1)}(X \times \mathbb{R}^{q}) && \text{by theorem 26}\\
&= KO^{(0,p+1)}(X \times \mathbb{R}^{q})
\end{aligned} \tag{8.30}$$

where $V$ is the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ vector space where the first factor acts trivial, i.e. as $\mathbb{R}^{0,p}$ and the second factor as $\mathbb{R}^{p,0}$, and $\mathbf{C}(V) = \mathbf{C}^{0,p}_{\mathbb{R}}$. $\tilde{V}$ is $V$ with the $\mathbb{Z}_2$ action coming from the second factor and generates the same Clifford algebra.

Specializing to $X$ being a single point we can now read off all K–groups from table 7.2:

$$
KO_{\mathbb{Z}_2}(\mathbb{R}^{p,q}) = \begin{array}{c|cccccccc}
q{=}7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q{=}6 & 0 & 0 & \mathbb{Z} & \mathbb{Z}_2 & \mathbb{Z}_2^2 & \mathbb{Z}_2 & \mathbb{Z} & 0 \\
q{=}5 & 0 & 0 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2^2 & \mathbb{Z}_2 & 0 & 0 \\
q{=}4 & \mathbb{Z}^2 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
q{=}3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q{=}2 & \mathbb{Z}_2^2 & \mathbb{Z}_2 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & \mathbb{Z}_2 \\
q{=}1 & \mathbb{Z}_2^2 & \mathbb{Z}_2 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z}_2 \\
q{=}0 & \mathbb{Z}^2 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\hline
 & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 & p{=}6 & p{=}7
\end{array} \tag{8.31}
$$

# Chapter 9

# Twisted K–theory

## 9.1 D-branes, H-flux and Gerbes

So far we only considered D–branes in ordinary spacetime, that is a Riemannian manifold. So of all NS–NS sector fields in the theory (see table 1.1) only the metric had a non–trivial vacuum value, although there are more massless bosonic modes. Specifically in the Type II (A and B) superstring NS–NS sector there is in addition to the metric also an antisymmetric 2–tensor, the $B_{\mu\nu}$.

What is the geometric interpretation of a 2-form field? Consider the bosonic part of the nonlinear sigma model action (compare with eq. 1.1), and ignoring factors of $2\pi\alpha'$:

$$S[f] = \int_\Sigma \mathrm{d}^2 Vol(G) + \int_{\partial\Sigma} A + \int_\Sigma B \tag{9.1}$$

The background fields enjoy the usual gauge symmetry $A \mapsto A + \mathrm{d}\Lambda$ and the "fancy gauge symmetry"

$$\begin{aligned} B &\mapsto B + \mathrm{d}\lambda \\ A &\mapsto A + \lambda \end{aligned} \tag{9.2}$$

So we know how to interpret the usual gauge transformations: $A$ is not really a 1-form (i.e. not globally defined), but instead only defined locally (on each coordinate patch). On overlapping coordinate patches the representing 1-form may differ by a gauge transformation since gauge equivalent 1-forms are physically equivalent.

Pictorially we represent this as (see [29]):

$$\begin{array}{c|ccccc} \Omega^1 & A_i & \to & \boxed{0} & & \\ & & & \uparrow & & \\ \Omega^0 & & & \Lambda_{ij} & \to & \boxed{0} \\ \hline & U_i & & U_{ij} & & U_{ijk} \end{array} \tag{9.3}$$

where the indices $i$, $j$, ... are Čech indices for the open cover $U_i$, and $\Omega^p$ denotes the sheaf of differential $p$-forms. Moreover the vertical arrows are de Rahm differentials (i.e. the usual d), and horizontal arrows Čech differentials $\partial$ (see 4.11). Finally the $\boxed{0}$ denote equation imposed at that position. So really eq. 9.3 contains the two equations

$$\big(\partial A\big)_{ij} = \mathrm{d}\Lambda_{ij} \quad \Leftrightarrow \quad A_i - A_j = \mathrm{d}\Lambda_{ij} \tag{9.4}$$

$$\big(\partial \Lambda\big)_{ijk} = 0 \quad \Leftrightarrow \quad \Lambda_{ij} + \Lambda_{jk} - \Lambda_{ik} = 0 \tag{9.5}$$

This is just the gauge transformation rule and the requirement for the transformations to be well defined.

Actually eq. 9.3 is not quite right: we neglected charge quantization. By the usual Dirac string argument we can actually measure

$$\oint A \mod 2\pi \tag{9.6}$$

in quantum mechanical experiments. So we have to ensure that the result is ambiguous only up to multiples of $2\pi$ if we use different trivializations to compute the same loop:

$$\oint A_i \equiv \oint A_j \mod 2\pi \quad \Rightarrow \quad \oint \mathrm{d}\Lambda_{ij} \equiv 0 \mod 2\pi \tag{9.7}$$

Forcing $\Lambda_{ij}$ to be a single valued function would be too strong, we need to allow all multiples of $2\pi$ as values of the circle integral. So really there is a function

$$h_{ij} : U_{ij} \to U(1), \quad \Lambda_{ij} = \mathrm{d}\log h_{ij} \tag{9.8}$$

So the improved version of the table in eq. 9.3 is the following:

$$\begin{array}{c|ccccc} \Omega^1 & A_i & \to & \boxed{0} & & \\ & & & \uparrow & & \\ \underline{\underline{U(1)}} & & & h_{ij} & \to & \boxed{0} \\ \hline & U_i & & U_{ij} & & U_{ijk} \end{array} \tag{9.9}$$

As a consequence of the equations in eq. 9.9 there is a globally well defined 2-form $F = \mathrm{d}A_i = \mathrm{d}A_j$. Furthermore $F$ is obviously closed and therefore defines a class in de Rahm cohomology $H^2_{\mathrm{DR}}(X;\mathbb{R})$.

Now the B–field has an analogous Čech description: The $B_i$ are locally defined on each coordinate patch, and fit together on double overlaps up to "fancy gauge transformations" eq. 9.2.

$$\begin{array}{c|ccccccc}
\Omega^2 & B_i & \rightarrow & \boxed{0} & & & & \\
 & & & \uparrow & & & & \\
\Omega^1 & A_i & \rightarrow & \boxed{\alpha_{ij}} & \rightarrow & \boxed{0} & & \\
 & & & \uparrow & & \uparrow & & \\
\underline{U(1)} & & & h_{ij} & \rightarrow & \boxed{g_{ijk}} & \rightarrow & \boxed{0} \\
\hline
 & U_i & & U_{ij} & & U_{ijk} & & U_{ijk\ell}
\end{array} \tag{9.10}$$

Again we suppress necessary pull backs to the brane worldvolume, those will not be important in the following.

The $\boxed{0}$'s represent the equations

$$B_i - B_j = \mathrm{d}\alpha_{ij} \tag{9.11}$$

$$\alpha_{ij} + \alpha_{jk} - \alpha_{ik} = \mathrm{d}\log g_{ijk} \tag{9.12}$$

$$g_{jk\ell}\, g^{-1}_{ik\ell}\, g_{ij\ell}\, g^{-1}_{ijk} = 1 \tag{9.13}$$

those are the obvious analog of eq. 9.9. But there are more equations because the $A$ field also has to transform under the "fancy gauge transformations". Those equations are at the remaining two $\Box$'s:

$$A_i - A_j = \alpha_{ij} + \mathrm{d}\log h_{ij} \tag{9.14}$$

$$h_{jk} h^{-1}_{ik} h_{ij} = g_{ijk} \tag{9.15}$$

As above there is a globally defined closed 3-form field $H = \mathrm{d}B_i$.

Of course we know the geometric interpretation of eq. 9.9: $A$ is the connection of a $U(1)$ principal bundle with transition functions $h_{ij}$, and $F$ is the curvature of the connection. By analogy $B$ has to be the connection on some other object with curvature $H$. This underlying object is called *gerbe* (see [38] for more details).

Now for the $U(1)$ gauge bundle the transition functions define a characteristic class

$$[h_{ij}] \in \check{H}^1\Big(X; \underline{U(1)}\Big) \simeq \check{H}^2\big(X;\mathbb{Z}\big) \tag{9.16}$$

This is of course nothing else than the first Chern class of the associated complex line bundle. Its image in de Rahm cohomology is

$$[F] \in H^2_{\mathrm{DR}}\big(X\big) \tag{9.17}$$

But $[F]$ alone carries less information than the first Chern class in $\check{H}^2(X;\mathbb{Z})$, there may be nonisomorphic bundles with the same curvature $F$.

Similarly the gerbe comes with a characteristic class

$$[g_{ijk}] \in \check{H}^2\Big(X; \underline{U(1)}\Big) \simeq \check{H}^3\big(X;\mathbb{Z}\big) \tag{9.18}$$

and its image in de Rahm cohomology is $[H] \in H^3_{\mathrm{DR}}(X)$. Since the map to de Rahm cohomology loses information again there may be distinct gerbes with the same curvature form $H$.

## 9.2 Twisted K–theory

Our goal is of course to describe the different D–brane charges in a given background. Again we assume the validity of conjecture 1, that is we really only have to consider spacetime filling D–branes. So fix a Riemannian manifold and a $B$–field (a gerbe) on it. Then a $U(1)$ gauge bundle on this is almost a usual $U(1)$ bundle, but instead of the cocycle condition the transition functions $h_{ij} : U_{ij} \to U(1)$ satisfy

$$h_{jk} h_{ik}^{-1} h_{ij} = g_{ijk} \tag{9.19}$$

Call such a gauge bundle a twisted gauge bundle . As before we can use the $U(1)$ transition functions for a $U(1)$ principal bundle or a complex line bundle, so we may just as well talk about the associated (twisted) vector bundle:

**Definition 16.** *Let $U_i$ be an open cover for $X$ and fix a twist cocycle*

$$g_{ijk} : U_{ijk} \to U(1) \quad \in \check{H}\Big(X; \underline{U(1)}\Big) \tag{9.20}$$

*A twisted (complex) vector bundle is then given by a set of $U(n)$ transition functions $h_{ij} : U_{ij} \to U(n)$ subject to*

$$h_{jk} h_{ik}^{-1} h_{ij} = g_{ijk} \tag{9.21}$$

From the transition function description of the Whitney sum eq. 2.7 it is clear that this carries over to twisted bundles. Moreover the sum of two $g_{ijk}$ twisted vector bundles is again a $g_{ijk}$ twisted vector bundle.

Now the gerbe has curvature $H \in H^3_{\mathrm{DR}}(X)$, and by abuse of notation let $H$ also denote its characteristic class in $\check{H}^3(X;\mathbb{Z})$. Then we write $\mathrm{Vect}^H(X)$

for the semigroup of $H$ twisted vector bundles. By the usual Grothendieck group construction we arrive at twisted K–theory:

$$K^{[H]}(X) \stackrel{\text{def}}{=} \mathfrak{G}\Big(\operatorname{Vect}^H(X)\Big) \tag{9.22}$$

One can show that twisted K–theory depends only on the cohomology class of the twist cocycle.

Although the formal properties of twisted K–theory $K^{[H]}(X)$ usually are similar to ordinary $K(X)$, there is one important difference. The usual $K(X)$ is a ring via the tensor product of vector bundles. However from the transition function approach one can see that the tensor product of two twisted vector bundles with twist class $g^{(1)}_{ijk}$, $g^{(2)}_{ijk}$ is itself a twisted vector bundle with twist class $g^{(1)}_{ijk}g^{(2)}_{ijk}$. With other words the characteristic classes add:

$$\otimes : \operatorname{Vect}^{H^{(1)}}(X) \times \operatorname{Vect}^{H^{(2)}}(X) \to \operatorname{Vect}^{H^{(1)}+H^{(2)}}(X) \tag{9.23}$$

Since twisted K–theory contains the usual K–theory as special case $H = 0$ (i.e. $g_{ijk} = 1$) we can also say that $K^{[H]}(X)$ is a $K(X)$ module.

## 9.3 Obstruction to finite dimensionality

Although $K^{[H]}(X)$ is again a nice cohomology theory it is not really "formal differences of twisted vector bundles". Only for $H \in H^3(X;\mathbb{Z})_{\text{Tor}}$ can we find finite dimensional twisted vector bundles (see [42, 65]), in general this only works if we allow infinite dimensional fibers.

The reason is roughly the following: Associated to the short exact sequence of coefficient groups

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0 \tag{9.24}$$

where the second arrow is multiplication by $n \in \mathbb{Z}$ we get the long exact sequence for cohomology groups

$$\cdots \to H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}_n) \xrightarrow{\beta} \underbrace{H^3(X;\mathbb{Z})}_{\ni H} \xrightarrow{n} H^3(X;\mathbb{Z}) \to \cdots \tag{9.25}$$

Now given a rank $n$ twisted bundle (i.e. the fiber is $\mathbb{C}^n$) one can construct a class $y \in H^2(X;\mathbb{Z}_n)$ such that $\beta(y) = H$. But going twice is zero in a long exact sequence and therefore

$$n\beta(y) = nH = 0 \quad \Rightarrow \; H \in H^3(X;\mathbb{Z})_{\text{Tor}} \tag{9.26}$$

Here is how to construct the class $y$. Assume you have a twisted vector bundle $h_{ij}: U_{ij} \to U(n)$ with

$$\partial h_{ijk} = h_{ij} h_{jk} h_{ki} = g_{ijk} \tag{9.27}$$

Then we can form $SU(n)$ valued transition functions

$$\tilde{h}_{ij} = \frac{h_{ij}}{\big(\det h_{ij}\big)^{\frac{1}{n}}} \stackrel{\text{def}}{=} h_{ij} q_{ij} \tag{9.28}$$

They of course also are not real transition functions but rather

$$\tilde{h}_{ij} \tilde{h}_{jk} \tilde{h}_{ki} = g_{ijk} q_{ij} q_{jk} q_{ki} \stackrel{\text{def}}{=} y_{ijk} \tag{9.29}$$

The $y_{ijk}$ are really $\mathbb{Z}_n$ valued (take the determinant of both sides) and they are the Čech representative for the desired class $y$. Let us check this:

$$\begin{aligned} &g_{ijk} q_{ij} q_{jk} q_{ki} = y_{ijk} \\ &\quad \Rightarrow \quad \log g_{ijk} + \log q_{ij} + \log q_{jk} + \log q_{ki} \equiv \log y_{ijk} \mod 2\pi i\mathbb{Z} \end{aligned} \tag{9.30}$$

Now apply the Čech coboundary operator $\partial$:

$$\begin{aligned} \Rightarrow \quad & \log g_{jk\ell} - \log g_{ik\ell} + \log g_{ij\ell} - \log g_{ijk} = \\ & \qquad = \log y_{jk\ell} - \log y_{ik\ell} + \log y_{ij\ell} - \log y_{ijk} + \partial\Big(\cdots\Big)_{ijk\ell} \end{aligned} \tag{9.31}$$

The right hand side is $\beta(y)$ + cocycle, the left hand side is $H$. Therefore as desired

$$\beta(y) = H \quad \in H^3(X; \mathbb{Z}) \tag{9.32}$$

## 9.4 Branes on Group Manifolds

### 9.4.1 For Physicists

There is a very nice interplay between CFT and twisted K–theory if one studies D–branes on group manifolds, which I will try to describe here (see [28, 27]). Those can be described by the Wess–Zumino–Witten models and are tractable theories because of the group action, even though they live on non-trivial spaces with $H$–flux.

So let $G$ be some Lie group. Then the basic result about models on $G$ is that D–branes have to wrap on submanifolds that are conjugacy classes (or twisted conjugacy classes, but those will not be important in the following)

$$C_u \stackrel{\text{def}}{=} \left\{ huh^{-1} \middle| h \in G \right\} \tag{9.33}$$

of the underlying group. So for simplicity choose $G = SU(2)$, the simplest interesting case (for more complicated examples see [28]). Topologically $SU(2) = S^3$, and there are two distinct possibilities for $C_u$:

1. $u = \pm 1$, i.e. $u$ is in the center. Then $C_{\pm 1} = \{\pm 1\}$ is 0 dimensional. (D0–branes)

2. $u \neq \pm 1$, then the conjugacy classes $C_u$ are 2 spheres in $S^3$. (D2–branes)

Moreover there is a quantization condition for the radii of the $S^2$ because otherwise there would be a phase ambiguity (see [2]).

Now the CFT analysis tells you that the D0 and the D2 branes all come with the same type of charge, and all their charges are multiples of the $C_1 = \{1\}$ D–brane. Especially the $S^2$ with $\ell$ quanta of radius carry charge $\ell + 1$. But the $G = SU(2)$ has only finite size given by the level $k$ of the WZW model.

This suggests the following picture: Start with $k+1$ D0–branes at $1 \in G$. You must be able to continuously transform this into D2–branes of larger and larger radius. But the D2 at radius $k$ is the degenerate conjugacy class $C_{-1} = \{-1\} \subset G$. Because of orientation this degenerate case carries D0 charge $-1$, and we therefore have to identify

$$(k + 1) \times \text{D0–charge} = -1 \times \text{D0–charge} \tag{9.34}$$

So the D–brane charge group is actually $\mathbb{Z}_{k+2}$.

### 9.4.2 For Mathematicians

One can interpret this level $k$ WZW model as having $k + 2$ units of $B$–field flux.[1] That is $H = \mathrm{d}B = k + 2 \in H^3(S^3, \mathbb{Z})$. Of course this $H$ is not at all torsion, and we really first have to make sure that we can work with infinite rank bundles. Fortunately this is actually possible, I refer to [53, 15] for details.

[1] This really relies on a semiclassical argument, so one cannot really differentiate between $k$ and $k + 2$. But the large $k$ behavior is fixed.

Now we would like to compute the twisted K–groups from that and compare it with the above result. But so far we do not know any useful result to compute the groups. The idea is to employ the Atiyah–Hirzebruch–Whitehead spectral sequence together with the following knowledge of $d_3$, see [52, 5]:

**Theorem 27 (Rosenberg).** *Let $X$ be a finite CW complex (or compact manifold). Then there is a spectral sequence with*

$$E_2^{p,q} = \begin{cases} H^p(X,\mathbb{Z}) & q \text{ even} \\ 0 & q \text{ odd} \end{cases} \tag{9.35}$$

*converging to $K^{[H],i}(X)$. Moreover $d_3^{0,q} : H^0(X,\mathbb{Z}) \to H^3(X,\mathbb{Z})$ is cup product with $[H] \in H^3(X,\mathbb{Z})$.*

So we start with the tableau, periodic in $q$:

$$E_2^{p,q} = E_3^{p,q} = \begin{tikzcd} q{=}2 & \mathbb{Z} \arrow[ddrrr, "d_3"] & 0 & 0 & \mathbb{Z} \\ q{=}1 & 0 & 0 & 0 & 0 \\ q{=}0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\ & p{=}0 & p{=}1 & p{=}2 & p{=}3 \end{tikzcd} \tag{9.36}$$

By the above theorem the $d_3$ is just multiplication by $k+2$. Therefore

$$E_4^{p,q} = E_\infty^{p,q} = \begin{array}{c|cccc} q{=}2 & 0 & 0 & 0 & \mathbb{Z}_{k+2} \\ q{=}1 & 0 & 0 & 0 & 0 \\ q{=}0 & 0 & 0 & 0 & \mathbb{Z}_{k+2} \\ \hline & p{=}0 & p{=}1 & p{=}2 & p{=}3 \end{array} \tag{9.37}$$

and we find

$$K^{[H],i}\Big(SU(2)\Big) = \begin{cases} \mathbb{Z}_{k+2} & i \text{ odd} \\ 0 & i \text{ even} \end{cases} \tag{9.38}$$

This matches nicely the physical prediction.

In fact it is possible to compute the D–brane charges for many more Lie groups, but apart from the $SU(2)$ case Rosenberg's theorem is not good enough to fix them uniquely unless one could determine the higher differentials. Only recently the $K^{[H],*}(SU(3))$ was determined by M. Hopkins as described by [47] in agreement with the physical prediction.

Finally note that eq. 9.38 illustrates the discussion in the previous section: If one could represent the twisted bundles by finite rank objects then $K^{[H]}(SU(2))$ would be at least $\mathbb{Z}$ to account for the virtual rank. But since we just determined $K^{[H]}(SU(2)) = 0$ there must be twisted bundles which cannot have finite rank.

## 9.5 Twisted equivariant K–theory

In general string compactifications with tractable CFT description are either boring (like on flat space) or orbifolds, the WZW models in the previous section are somewhat of an exception. So we would like to extend the definition of twisted K–theory to orbifolds. By the same arguments as in section 8.1 we need equivariant twisted K–theory. I will describe such K–theories in the following sections, the physical application is in section 9.7.

But what is precisely equivariant twisted K–theory? Suppose you are given a $G$ space $X$, then we know the description of $K_G(X)$ as Grothendieck group of equivariant vector bundles. Alternatively you could try the following: Remember from section 5.2.3 that there exists a contractible space $EG$ with free $G$ action. Then you would expect that $X$ and $X \times EG$ have the same K–groups and therefore

$$K_G^*(X) = K_G^*(X \times EG) = K^*\left(\frac{X \times EG}{G}\right) \quad \text{wrong!} \tag{9.39}$$

Unfortunately something goes wrong because of the infinite dimensionality of $EG$ and eq. 9.39 holds only up to completion of the K–rings (for details on this subtle issue see [9]). Nevertheless I will use it for motivation.

Now given this connection with ordinary K–theory we can employ the usual twisting: Choose a twist class[2]

$$[H] \in H^3\left(\frac{X \times EG}{G}; \mathbb{Z}\right) = H_G^3(X; \mathbb{Z}) \simeq H_G^2\big(X; U(1)\big) \tag{9.41}$$

and think of equivariant twisted K–theory as usual twisted K–theory on $(X \times EG)/G$. Let us for now specialize to $X = \{pt\}$, then

$$H_G^2\big(\{pt\}; U(1)\big) = H^2\left(\frac{\{pt\} \times EG}{G}; U(1)\right) = H^2\big(BG; U(1)\big) \tag{9.42}$$

The essential observation here is that we can rewrite this cohomology group as a purely group theoretic object, see [17, 61, 58]:

$$H^*\big(BG; U(1)\big) = H^*\big(G, U(1)\big) \tag{9.43}$$

[2]Equivariant ordinary cohomology is defined by the Borel construction

$$H_G^*(X) = H^*\left(\frac{X \times EG}{G}\right) \tag{9.40}$$

for arbitrary coefficients.

For the group cohomology $H^*\big(G, U(1)\big)$ we have to specify how the group in the first slot acts on the group in the second; In our case eq. 9.43 $G$ acts on $U(1)$ trivially. Indeed nontrivial group actions would correspond to cohomology of $BG$ with local coefficients.

How does this help us? So far we have identified the twist class as an element of $H^2\big(G, U(1)\big)$. Now use the following theorem (see [17]):

**Theorem 28.** *Let $A$ be an abelian group and $K$ an arbitrary group acting on $A$. Then the (equivalence classes of) extensions*

$$0 \to A \to E \to K \to 1 \tag{9.44}$$

*are in one to one correspondence with $H^2(K, A)$.*

that is for $K$ acting trivially on $A$:

**Corollary 3.** *The central extensions ($K$ acts trivially on $A$) are in one to one correspondence with $H^2(K, A)$ (where $K$ acts trivially on $A$).*

So especially $H^2\big(G, U(1)\big)$ corresponds bijectively to central extensions

$$1 \to U(1) \to \tilde{G} \to G \to 0 \tag{9.45}$$

that is to each $[H] \in H^2_G\big(\{pt\}; U(1)\big)$ we can associate such an extension $\tilde{G}_{[H]}$. Now we want to identify the $U(1)$ with a complex phase for a group action on a complex vector space (see [1]):

**Definition 17.** *An $[H]$ twisted representation (or projective representation) $\tilde{r}_{[H]} : \tilde{G}_{[H]} \to \mathbb{C}^n$ is a representation of $\tilde{G}_{[H]}$ that restricts to usual multiplication on the central $U(1)$.*

Now twisted equivariant K–theory of a point should be the Grothendieck group of $[H]$ twisted $G$ vector bundles $\mathrm{Vect}_G^{[H]}$. Of course the case $X = \{pt\}$ is not very interesting, but the same argument would have worked if the twist class $[H] \in H^2_G\big(X; U(1)\big)$ is the pull back of some class on a point via the projection $X \to \{pt\}$. So we define

**Definition 18.** *Let $[H] \in H^2_G\big(X; U(1)\big)$ be induced from $H^2_G\big(\{pt\}; U(1)\big)$. Then the twisted equivariant K–theory is*

$$K_G^{[H]}(X) \overset{\text{def}}{=} \mathfrak{G}\Big(\mathrm{Vect}_G^H(X)\Big) \tag{9.46}$$

## 9.6 Twisted Real equivariant K–theory

So far we only discussed previously known K–theories. However the extension of twisted equivariant K–theory to Real spaces has not been considered so far in the literature.

To define such a K–theory we need the following data:

- A group $G$ with augmentation $\theta : G \to \mathbb{Z}_2$.
- A $G$–space $X$.
- A twist class $[H] \in H^2\big(G, U(1)\big)$ where now $g \in G$ acts on $U(1)$ by complex conjugation if $\theta(g) = 1$.

The twist class determines a (non–central) group extension

$$1 \to U(1) \to \tilde{G} \to G \to 0 \tag{9.47}$$

The twisted Real representations of $G$ are ordinary Real representations of $\tilde{G}$ such that the $U(1)$ is multiplication by a phase, compare definition 17. The action of $G$ on the $U(1)$ is necessary because some $g \in G$ act antilinear. The Grothendieck group of complex vector bundles with twisted Real action of $G$ is then twisted Real equivariant K–theory

$$KR_G^{[H]}(X) \tag{9.48}$$

The natural question to ask is whether this is again a generalized cohomology theory and to compute this in interesting cases.

Especially we will be interested in the case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ with augmentation $\theta(a, b) = b$. Let the generators of $G$ be $g, \tau$ with $\theta(g) = 0, \theta(\tau) = 1$. Finally let the twist be such that $\rho : G \to \widetilde{\mathrm{End}}(E)$ satisfies

$$\rho(g)\rho(\tau) = -\rho(\tau)\rho(g) \tag{9.49}$$

Now the key idea to analyze this is the following (similar versions were independently suggested by M. Atiyah, G. Segal and B. Totaro): Let $D_8$ be the group

$$D_8 = \Big\{ g, \tau, s \Big| g\tau = s\tau g,\ g^2 = \tau^2 = s^2 = 1,\ sg = gs,\ s\tau = \tau s \Big\} \tag{9.50}$$

with the obvious augmentation counting the number of $\tau$'s. Then the Real equivariant K–theory $KR_{D_8}(X)$ decomposes into bundles with $s$ acting by $\rho(s) = \pm 1$. If $\rho(s) = +1$ then this is just an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$, and if $\rho(s) = -1$ this is just the desired twisted action of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore

$$KR^*_{D_8}(X) = KR^*_{\mathbb{Z}_2 \times \mathbb{Z}_2}(X) \oplus KR^{[H],*}_{\mathbb{Z}_2 \times \mathbb{Z}_2}(X) \tag{9.51}$$

Especially $KR^{[H],*}_{\mathbb{Z}_2 \times \mathbb{Z}_2}(X)$ enjoys the same properties as the other (untwisted) K–groups, e.g. Bott periodicity.

**The case $\mathbf{X} = \{\mathbf{pt}\}$**

Because of eq. 9.51 we can now reduce the computation of the twisted equivariant Real K–theory to the untwisted case, however for more difficult group actions. But this is tractable in the case of $X = \{pt\}$ where everything reduces to representation theory. Obviously we have

$$KR^i_{\mathbb{Z}_2\times\mathbb{Z}_2}(\{pt\}) = KO^i_{\mathbb{Z}_2}(\{pt\}) = KO^i(\{pt\}) \oplus KO^i(\{pt\} \tag{9.52}$$

The difficult part is to compute $KR^*_{D_8}(\{pt\})$, for this we need to know the Real irreducible representations of $D_8$.

Now in general for $g \in G$ a Real representation $\rho : G \to \widetilde{\mathrm{End}}(\mathbb{C}^n)$ is (by choosing a basis) either a complex matrix (if $\theta(g) = 0$) or combination of complex conjugation and a complex matrix (if $\theta(g) = 1$). Denote complex conjugation by

$$\Omega : \mathbb{C}^n \to \mathbb{C}^n, \quad (z_1, \ldots, z_n) \mapsto (\bar{z}_1, \ldots, \bar{z}_n) \tag{9.53}$$

then the Real representations of $D_8$ are listed in table 9.1. Now we have to

| $i$ | $\rho_i(s)$ | $\rho_i(g)$ | $\rho_i(\tau)$ |
|---|---|---|---|
| 1 | $\begin{pmatrix}1\end{pmatrix}$ | $\begin{pmatrix}1\end{pmatrix}$ | $\begin{pmatrix}1\end{pmatrix}\Omega$ |
| 2 | $\begin{pmatrix}1\end{pmatrix}$ | $\begin{pmatrix}-1\end{pmatrix}$ | $\begin{pmatrix}1\end{pmatrix}\Omega$ |
| 3 | $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ | $\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$ | $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Omega$ |

Table 9.1: Real representations of $D_8$

decompose the Real representation ring

$$R_R(G) = A_G \oplus B_G \oplus C_G \tag{9.54}$$

corresponding to commuting fields $\mathbb{R}, \mathbb{C}, \mathbb{H}$. Obviously the first 2 representations in table 9.1 have commuting field $\mathbb{R}$. The matrices of the 2–dimensional representation commute with

$$F_3 \overset{\text{def}}{=} \mathbb{R}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \mathbb{R}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \simeq \mathbb{C} \tag{9.55}$$

So $A_G = \mathbb{Z}^2, B_G = \mathbb{Z}, C_G = 0$ and from [9] Prop. 8.1 we can read off

$$\begin{aligned} KR^*_{D_8}(\{pt\}) =& A_G \otimes KR^*(\{pt\}) \oplus B_G \otimes K^*(\{pt\}) \oplus C_G \otimes KH^*(\{pt\}) \\ =& \oplus_1^2 KO^*(\{pt\}) \oplus K^*(\{pt\}) \end{aligned} \tag{9.56}$$

and by comparing the factors in eq. 9.51 we learn that

$$KR^{[H],i}_{\mathbb{Z}_2\times\mathbb{Z}_2}(\{pt\}) = K^i(\{pt\}) = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases} \tag{9.57}$$

## 9.7 Comparison with Orientifolds

All those (twisted) equivariant Real K–theories are physically interesting because they classify the D–brane charges in orientifolds. Especially the K–groups considered so far are necessary ingredients to understand the $\Omega \times \mathcal{I}_4$ orientifold of Type IIB. This means that spacetime is

$$X = \mathbb{R} \times \mathbb{R}^5 \times \mathbb{R}^4 \tag{9.58}$$

where

- the first factor $\mathbb{R}$ is time.
- the second factor $\mathbb{R}^5$ is flat euclidean.
- the third factor $\mathbb{R}^4$ is flat euclidean $\mathbb{R}^4$ with $\mathbb{Z}_2$ group action

$$\mathcal{I}_4 : \mathbb{R}^4 \to \mathbb{R}^4,\ (x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, -x_3, -x_4) \tag{9.59}$$

and $\Omega$ acts pointwise as complex conjugation on the vector bundles. So the naive guess for the corresponding K–theory is Real equivariant K–theory $KR_{\mathbb{Z}_2\times\mathbb{Z}_2}(X)$.

However there is a subtle possibility for a sign choice in the twisted sector yielding two consistent $\Omega \times \mathcal{I}_4$ orientifolds: the GP model [31] and the BZDP model [11, 24]. The possible D–brane charges were analyzed from the boundary state point of view in [51] and the authors found perfect agreement between the BZDP D–brane charges and $KR_{\mathbb{Z}_2\times\mathbb{Z}_2}(\mathbb{R}^{p,q}) = KO_{\mathbb{Z}_2}(\mathbb{R}^{p,q})$, where $\mathbb{R}^{p,q}$ is the transverse space of a given hyperplane in $X$.

However in the GP model they found different charges. For example while in BZDP the spacetime filling branes carried twisted and untwisted R–R charge (corresponding to $KO_{\mathbb{Z}_2}(\{pt\}) = \mathbb{Z}^2$) in the GP model there is only untwisted R–R charge, so the relevant charge group is $\mathbb{Z}$.

The resolution for this puzzle is that the BZDP orientifold is described by untwisted Real equivariant K–theory while the GP orientifold is described by the twisted version with the twist as in eq. 9.49. I computed $KR^{[H]}_{\mathbb{Z}_2\times\mathbb{Z}_2}(\{pt\}) = \mathbb{Z}$ in the previous section and this agrees with the physical prediction of [51].

# Part II

# Superconformal Field Theories for Exceptional Holonomy Manifolds

# Chapter 10

# Holonomy

## 10.1 Introduction

The unique maximal dimensional supergravity theory lives in 11 dimensions (see [22, 49]) and its field content consists of the metric $g_{\mu\nu}$, a 3-form potential $C_{\mu\nu\rho}$ and a spin $\frac{3}{2}$ field $\psi_\mu$. Because of its uniqueness and since it encompasses various other interesting supergravities via dimensional reduction it is of considerable interest. Today it is widely believed to be itself the low energy limit of *M-theory*, the hypothetical theory that unifies all consistent string theories.

Now the Lagrangian is

$$\begin{aligned}
\mathcal{L} =& \frac{1}{4}eR - \frac{1}{2}e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho - \frac{1}{192}eG^2- && (10.1)\\
&- \frac{1}{192}e\Big(\bar{\psi}_\mu\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\bar{\psi}^\alpha\Gamma^{\beta\gamma}\psi^\delta\Big)G_{\alpha\beta\gamma\delta}+ && (10.2)\\
&+ \frac{1}{288^2}G\wedge G\wedge C + \text{4-fermion terms} && (10.3)
\end{aligned}$$

and it is supersymmetric with respect to the variations

$$\begin{aligned}
\delta_\epsilon e^a_\mu &= \bar{\epsilon}\Gamma^a\psi_\mu && (10.4)\\
\delta_\epsilon C_{\mu\nu\rho} &= -3\bar{\epsilon}\Gamma_{[\mu\nu}\psi_{\rho]} && (10.5)\\
\delta_\epsilon\psi_\mu &= \nabla_\mu\epsilon + \frac{1}{288}\Big(\Gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\delta^\alpha_\mu\Gamma^{\beta\gamma\delta}\Big)\epsilon G_{\alpha\beta\gamma\delta} + \text{3-fermion terms} && (10.6)
\end{aligned}$$

With the ansatz $\psi_\mu = 0$, $G = \mathrm{d}C = 0$ one is lead to the following condition for a unbroken supersymmetry:

$$\nabla_\mu\epsilon = 0 \qquad (10.7)$$

So we need a constant spinor $\epsilon$. Now supersymmetry is — although theoretically attractive — at odds with the real world. Phenomenologically one would like as few supersymmetries as possible, the remaining supersymmetry being broken around the electroweak scale. And of course one would like preferably a 4 dimensional theory, or at least one with 4 macroscopic dimensions.

So we are lead to an ansatz for spacetime of the form

$$M^{11} = \mathbb{R}^{3,1} \times X^7 \tag{10.8}$$

with exactly one constant spinor on $M^{11}$ (and therefore on $X$ because of the product ansatz). M-theory compactified on $M^{11}$ should then yield a $N = 1$, $d = 4$ low energy theory.

Of course given any smooth manifold it is without further knowledge very hard to decide whether there exists some metric on it which allows for exactly one spinor, this would amount to solving a complicated PDE. Fortunately there is the following very useful result, see [62, 40]: A 7 manifold with exactly one constant spinor has $\mathrm{Hol}(g) = G_2$. So we just have to find a 7 dimensional manifold with holonomy $G_2$.

## 10.2 $G_2$ Holonomy

So what is this holonomy, and what is $G_2$? Well $G_2$ is the Lie group

$$G_2 : \quad \bullet\!\equiv\!\bullet \tag{10.9}$$

Now the Dynkin diagram alone is not a very useful description, we have to understand how $G_2$ is a subgroup of $SO(7)$. This is via the following construction:

Let $e_1, \ldots, e_7$ be a dual basis for $\mathbb{R}^7$, $e_{ijk} \stackrel{\text{def}}{=} e_i \wedge e_j \wedge e_k$ and

$$\varphi_0 = e_{123} + e_{145} + e_{167} + e_{246} - e_{257} - e_{347} - e_{356} \ \in \ \Lambda^3\mathbb{R}^7 \tag{10.10}$$

Then $SO(7)$ acts on $\mathbb{R}^7$ as rotations and therefore on $\Lambda^3\mathbb{R}^7$. The group $G_2$ is the subgroup of $SO(7)$ that leaves $\varphi_0$ invariant.

Now suppose that you are given a Riemannian manifold. From the metric you can determine the Levi-Civita connection, so there is a well defined parallel transport. But of course the parallel transport depends on the path taken: In general f you take one frame at a point and transport it via two different paths to a second point the frames will differ by a $GL(7, \mathbb{R})$ coordinate transformation, see figure 10.1. However the Levi-Civita the connection

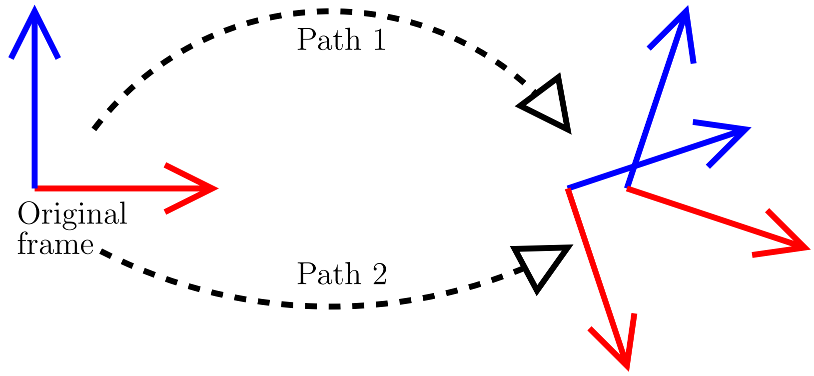


Figure 10.1: Parallel transport by different paths

is metric compatible and thus preserves lengths and angles. So the ambiguity under parallel transport along different paths is actually only within $SO(7)$. Now the holonomy of a Riemannian manifold $\mathrm{Hol}(g)$ is the subgroup of $SO(7)$ that one can actually realize by parallel transport.

But $\varphi_0$ is $G_2$–invariant, and therefore its image under parallel transport unique. So it defines $\varphi_x$ at each point $x \in X$. So on a $G_2$ manifold there exists a 3-form $\varphi \in \Omega^3(X)$ with

$$\nabla\varphi = 0 \tag{10.11}$$

since by definition it does not change under parallel transport.

Conversely let $\varphi \in \Omega^3(X)$ such that at each point there is a frame $e_1, \ \ldots, e_7 \in T_x^*X$ such that $\varphi_x = \varphi_0$. Then one can define a metric $g_x = e_1^2 + \cdots + e_7^2$. If with respect to that metric $\nabla\varphi = 0$ then $\mathrm{Hol}_g(X) \subset G_2$.

The following theorem is useful to actually check that the 3-form is constant:

**Theorem 29 (Salamon).** *For such a 3–form the following is equivalent:*

1. $\nabla\varphi = 0$

2. $\mathrm{d}\varphi = \mathrm{d} * \varphi = 0$

## 10.3 A class of compact $G_2$ manifolds

A useful way to construct many examples of $G_2$ manifolds is to start with a 7-manifold with even smaller holonomy, and then mod out a discrete symmetry which enlarges the holonomy group to $G_2$. Especially we will use the following construction of [39]: Take $Y$ a Calabi–Yau threefold with an antiholomorphic involution $y \mapsto \bar{y}$. Then let

$$X = \frac{\left(Y \times S^1\right)}{\mathbb{Z}_2} \tag{10.12}$$

with the $\mathbb{Z}_2$ group action

$$\sigma : Y \times S^1 \to Y \times S^1, \quad (y,t) \mapsto (\bar{y}, -t) \tag{10.13}$$

This quotient is in general a $G_2$ orbifold: Choose the phase of the holomorphic $(3,0)$-form $\Omega \in H^{3,0}(Y)$ such that

$$\begin{aligned} &\sigma^* \operatorname{Re}\Omega = \operatorname{Re}\Omega \\ &\sigma^* \operatorname{Im}\Omega = -\operatorname{Im}\Omega \end{aligned} \tag{10.14}$$

then the following 3-form has the desired properties:

$$\varphi = \operatorname{Re}\Omega + \omega \wedge \mathrm{d}t \tag{10.15}$$

It is obviously closed $\mathrm{d}\varphi = 0$, and it is also coclosed:

$$*\varphi = \operatorname{Im}\Omega \wedge \mathrm{d}t + \frac{1}{3!}\omega \wedge \omega \quad \Rightarrow \quad \mathrm{d} * \varphi = 0 \tag{10.16}$$

Furthermore one can choose coordinates $z_j = x_j + iy_j$ at a point $y \in Y$ such that

$$\Omega = \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 \qquad \omega = \frac{1}{2i}\sum_{k=1}^{3} \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_k \tag{10.17}$$

Plugging this into the definition of $\varphi$ one calculates that it is locally of the form eq. 10.10.

Of course $X = (Y \times S^1)/\mathbb{Z}_2$ has orbifold singularities where the $\mathbb{Z}_2$ group action has fixed points. From the definition it is clear that the fixed point set consists of two copies of the real subset $Y_{\mathbb{R}} \subset Y$. For example take the Fermat quintic

$$Y = \left\{ [z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{C}\mathrm{P}^4 : \sum z_i^5 = 0 \right\} \tag{10.18}$$

with the ordinary involution $z_i \mapsto \bar{z}_i$. Then the real subset is

$$Y_{\mathbb{R}} = \left\{ [x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{R}\mathrm{P}^4 : \sum x_i^5 = 0 \right\} \tag{10.19}$$

Topologically this is (since $x \mapsto x^5$ is bijective)

$$\begin{aligned} Y_{\mathbb{R}} &= \Big\{(x_0, \ldots, : x_4) \in S^4 : \sum x_i^5 = 0\Big\} \Big/ \mathbb{Z}_2 \\ &= \Big\{(x_0, \ldots, : x_4) \in S^4 : \sum x_i = 0\Big\} \Big/ \mathbb{Z}_2 \\ &= S^3 \big/ \mathbb{Z}_2 = \mathbb{R}\mathrm{P}^3 \end{aligned} \tag{10.20}$$

So in the corresponding $G_2$ orbifold there are two $\mathbb{R}\mathrm{P}^3$'s with transverse space $\mathbb{R}^4/\{\pm 1\}$. The natural question is whether one can resolve these orbifold singularities within $G_2$ holonomy. However this seems to be impossible by the following argument, see [46]:

The obvious guess is to replace the transverse $A_1$ singularity by Eguchi–Hansen spaces. However the moduli space of $G_2$ manifolds is $b_3$ dimensional and thus the resolution must introduce an additional 3-cycle. But if you cut out $\mathbb{R}\mathrm{P}^3 \times \mathbb{R}^4/\mathbb{Z}_2$ (which has $b_3 = 1$) and glue in $\mathbb{R}\mathrm{P}^3 \times \mathcal{O}_{\mathbb{C}\mathrm{P}^1}(-2)$ (with $b_3 = 1$) you will not change $b_3$.

Of course this is only a physical argument rather than a mathematical proof since nothing is known about the moduli space of *singular* $G_2$ manifolds.

Another possibility is to try to apply an "M-theory flop", see [8]. However this only conserves the difficulty:

$$\mathbb{R}\mathrm{P}^3 \times \mathbb{R}^4/\mathbb{Z}_2 \qquad \leftrightarrow \qquad \mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}\mathrm{P}^3 \tag{10.21}$$

But even if we cannot resolve the orbifold singularity we expect the resulting string – or M-theory compactification to be well-defined.

# Chapter 11

# Gepner Models for $G_2$ Manifolds

## 11.1 Gepner Models

So far we only used classical geometry to study spacetime. This is certainly valid near the large volume limit, but we expect quantum corrections at small volume. The problem with M-theory is of course that we do not have the microscopic description, so we cannot make any predictions from first principles. So the idea is to study string theory instead, where one knows the quantum description via CFT's (this was joint work with R. Blumenhagen [12]).

The idea behind Gepner models is to use especially simple rational conformal field theories as building blocks, for example the unitary models of the $N = 2$ super Virasoro algebra. They are classified and form a discrete series with central charge

$$c = \frac{3k}{k+2} \qquad k \in \mathbb{Z}_{>} \tag{11.1}$$

At each level $k$ there are finitely many highest weight representations with

$$\begin{aligned} &\text{Conformal dimension} \quad && h = \frac{l(l+2)-m^2}{4(k+2)} + \frac{s^2}{2} \\ &U(1) \text{ charge} && q = -\frac{m}{k+2} + \frac{s}{2} \end{aligned} \tag{11.2}$$

The indices $(l, m, s)$ range over

$$\begin{aligned}
&0 \geq l \geq k \\
&0 \geq |m-s| \geq l \\
&s = \begin{cases} 0,2 & \text{NS sector} \\ \pm 1 & \text{R sector} \end{cases} \\
&l+m+s = 0 \mod 2
\end{aligned} \tag{11.3}$$

Now a Gepner model is a tensor product of such minimal models to archive the desired central charge $c = 9$ corresponding to Calabi–Yau threefold. For example tensoring 5 copies of the $k = 3$ model yields $c = 9$, and this Gepner model has been identified as a special point in the moduli space of the quintic. The remaining 4 noncompact directions are described (in lightcone gauge) by two free bosons and their superpartners.

The unitary models can be explicitly realized by a coset construction as the product of parafermions and a free $U(1)$:

$$\frac{SU(2)_k}{U(1)} \times U(1) \tag{11.4}$$

From the coset one can then determine the characters

$$\chi^l_{m,s} = \sum_{j=1}^{k} C^l_{m-(4j+s)}(\tau)\Theta_{2m-(k+2)(4j+s),2k(k+2)}\left(\tau, \frac{z}{k+2}\right) \tag{11.5}$$

where the string functions $C^l_m$ are defined by ($q = e^{2\pi i\tau}$):

$$C^l_m(\tau) = \eta(\tau)^{-3} \sum_{\substack{(x,y)\in\mathbb{R}^2 \\ -|x|<|y|\leq|x| \\ (x,y) \text{ or } (\frac{1}{2}-x,\frac{1}{2}-y) \\ \in(\frac{l+1}{2(k+2)},\frac{m}{2k})+\mathbb{Z}^2}} \operatorname{sign}(x) q^{(k+2)x^2-ky^2} \tag{11.6}$$

## 11.2 The Calculation

We want to compute the orbifold of the Gepner model $\times S^1$ by the involution $\sigma$. For that we need to know how the complex conjugation acts on the Gepner model.

Now in the usual identification $(k = 3)^5 \leftrightarrow$ Fermat quintic we identify chiral fields $(l_i, m_i, s_i) = (1, 1, 0)$ with homogeneous coordinates $z_i$ and

$(l_i, m_i, s_i) = (1, -1, 0)$ with the conjugates $\overline{z}_i$. So we guess that complex conjugation on the quintic is $U(1)$ charge conjugation in the Gepner model:

$$\sigma(l, m, s) = (l, -m, -s) \tag{11.7}$$

The goal now is to compute the partition function. For this we need to know the traces over the Hilbert space with $\sigma$ insertion

$$\chi^l_{m,s}(\sigma) \stackrel{\text{def}}{=} \mathrm{Tr}_{\mathcal{H}^l_{m,s}} \left(\sigma e^{2\pi i \tau L_0}\right) \tag{11.8}$$

We have to guess these in a case by case study. In the end there will be consistency checks form the modular transformation.

**The $\mathbf{k = 1}$ case**

Here $c = 1$ and the parafermionic part is trivial: all states are generated by the $j_m$.

The only nontrivial trace with the $\sigma$ insertion is for the highest weight representation (HWR) $(l, m, s) = (0, 0, 0)$:

$$\chi^0_{0,0}(\sigma) = \mathrm{Tr}_{\mathcal{H}^0_{0,0}} \left(\sigma e^{2\pi i \tau L_0}\right) = \frac{q^{-\frac{1}{24}}}{\prod(1 + q^n)} = \sqrt{\frac{2\eta}{\theta_2}} \tag{11.9}$$

**The $\mathbf{k = 3}$ case**

For a nonvanishing trace $\chi^l_{m,s}(\sigma)$ the HWR $(l, m, s)$ must be mapped to itself, this happens only for the HWR $(l, 0, 0)$. There

$$\chi^l_{0,0} = \sum_{j=1}^{k} C^l_{4j}(\tau)\Theta_{-20j,30}\left(\tau, \frac{z}{5}\right) \tag{11.10}$$

Charge conjugation maps $\Theta_{a,b} \to \Theta_{-a,b}$, so the involution acts non–trivially only on

$$C^l_0(\tau)\Theta_{0,30}\left(\tau, \frac{z}{5}\right) \tag{11.11}$$

Only the ground state in $\Theta_{0,30}$ is not exchanged with another state.

How does $\sigma$ act on $C^l_0$? To answer this we use the parafermions ($c = \frac{9}{5} = \frac{4}{5} + 1$):

$$C^l_0(\tau) = \frac{1}{\eta(\tau)}\kappa^l_0(\tau) \tag{11.12}$$

with $\kappa_0^l$ the characters of the parafermions.

The parafermionic part is the $k = 5$ Virasoro unitary model, their characters can be written

$$\kappa_0^0 = \chi_0 + \chi_3 \qquad \kappa_0^2 = \chi_{\frac{2}{5}} + \chi_{\frac{7}{5}} \tag{11.13}$$

We want to interpret this field of conformal dimension 3. This must be

$$G^+_{-\frac{3}{2}} G^-_{-\frac{3}{2}} \left|0\right\rangle \tag{11.14}$$

but how does $\sigma$ act on it? For this recall the $N = 2$ super Virasoro algebra:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12}\, n(n^2-1)\, \delta_{m+n,0} \tag{11.15}$$
$$[L_m, j_n] = -n\, j_{m+n} \tag{11.16}$$
$$[L_m, G^\pm_r] = \left(\frac{m}{2} - r\right)\, G^\pm_{m+r} \tag{11.17}$$
$$[j_m, j_n] = \frac{c}{3}\, n\delta_{m+n,0} \tag{11.18}$$
$$[j_m, G^\pm_r] = \pm G^\pm_{m+r} \tag{11.19}$$
$$\{G^+_r, G^-_s\} = 2L_{r+s} + (r-s)\, j_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \tag{11.20}$$
$$\{G^+_r, G^+_s\} = \{G^-_r, G^-_s\} = 0. \tag{11.21}$$

where we identified the involution $\sigma$ with charge conjugation, i.e. the automorphism

$$L_m \to L_m \qquad j_m \to -j_m \qquad G^+_r \leftrightarrow G^-_r \tag{11.22}$$

So the $\sigma$ action must be

$$G^+_{-\frac{3}{2}} G^-_{-\frac{3}{2}} \left|0\right\rangle \ \xrightarrow{\sigma}\ -G^+_{-\frac{3}{2}} G^-_{-\frac{3}{2}} \left|0\right\rangle \tag{11.23}$$

and we find

$$\begin{aligned} \chi^0_{0,0}(\sigma) &= \sqrt{\frac{2\eta}{\theta_2}}\, (\chi_0 - \chi_3) \\ \chi^2_{0,0}(\sigma) &= \sqrt{\frac{2\eta}{\theta_2}}\, (\chi_{\frac{2}{5}} - \chi_{\frac{7}{5}}) \end{aligned} \tag{11.24}$$

**The untwisted sector**

It remains to determine how the involution acts on the $S^1$, which in the CFT description is one free fermion. For that we have to decompose $SO(2)_1$ into $SO(1)_1 \times SO(1)_1$. The characters are

$$\begin{aligned} O_1 =& \frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}} + \sqrt{\frac{\theta_4}{\eta}}\right) \\ V_1 =& \frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}} - \sqrt{\frac{\theta_4}{\eta}}\right) \\ S_1 =& \sqrt{\frac{\theta_2}{\eta}} \end{aligned} \tag{11.25}$$

Now the characters of the $SO(2)_1$ CFT are the orbits under the simple current $J = V_1 V_1$:

$$\begin{array}{lll} O_2 = O_1 O_1 + V_1 V_1 & \xrightarrow{\sigma} & O_1 O_1 - V_1 V_1 \\ V_2 = O_1 V_1 + V_1 O_1 & \xrightarrow{\sigma} & O_1 V_1 - V_1 O_1 \\ S_2 = S_1 S_1 & \xrightarrow{\sigma} & C_2 \\ C_2 = S_1 S_1 & \xrightarrow{\sigma} & S_2 \end{array} \tag{11.26}$$

So $\sigma$ exchanges the massless chiral, anti–chiral states:

$$\left[(h = \frac{1}{2}, q = 1) \otimes O_2\right]_L \times \left[(h = \frac{1}{2}, q = \pm 1) \otimes O_2\right]_R \tag{11.27}$$

$$\updownarrow \sigma \tag{11.28}$$

$$\left[(h = \frac{1}{2}, q = -1) \otimes \sigma O_2\right]_L \times \left[(h = \frac{1}{2}, q = \mp 1) \otimes \sigma O_2\right]_R \tag{11.29}$$

and there are $h_{11} + h_{21} + 1$ chiral multiplets.

**The twisted sector**

Of course much more interesting is the twisted sector, and especially we are interested in the massless spectrum. For that we need to compute the trace with $\sigma$ insertion, ${}^{\sigma}\square_{1}$. The rest then follows from modular transformations:

$${}^{1}\square_{\sigma} = S\left({}^{\sigma}\square_{1}\right) \qquad {}^{\sigma}\square_{\sigma} = T\left({}^{1}\square_{\sigma}\right) \tag{11.30}$$

We have to satisfy consistency conditions in ${}^{1}\square_{\sigma}$ because the twisted partition function must have an interpretation as sum over states:

- Level matching: $h_L - h_R \in \frac{1}{2}\mathbb{Z}$
- Non–negative integer coefficients
- No Tachyons

The partition function in the twisted sector is then

$$\frac{1}{2}\left( {}^{1}\underset{\sigma}{\square} + {}^{\sigma}\underset{\sigma}{\square} \right) \tag{11.31}$$

Putting everything together we find for the $(k=3)^5$ model:

$$\begin{aligned} {}^{\sigma}\underset{1}{\square} =& \frac{2}{|\eta|^2} \left| \sqrt{\frac{\eta}{\theta_2}} \right|^2 \left| O_1 V_1 - V_1 O_1 \right|^2 \sum_{n=0}^{5} \binom{5}{n} \left| \Big(\chi^0_{0,0}(\sigma)\Big)^n \Big(\chi^2_{0,0}(\sigma)\Big)^{5-n} \right|^2 \\ =& \frac{2^6}{|\eta|^2} \left| \sqrt{\frac{\eta}{\theta_2}} \right|^{12} \left| O_1 V_1 - V_1 O_1 \right|^2 \times \\ & \times \sum_{n=0}^{5} \binom{5}{n} \left| \Big(\chi_0 - \chi_3\Big)^n \Big(\chi_{\frac{2}{5}} - \chi_{\frac{7}{5}}\Big)^{5-n} \right|^2 \end{aligned} \tag{11.32}$$

and after a modular transformation

$$\begin{aligned} {}^{1}\underset{\sigma}{\square} = S\left( {}^{\sigma}\underset{1}{\square} \right) = & \frac{2^5}{|\eta|^2} \left| \sqrt{\frac{\eta}{\theta_4}} \right|^2 \left| S_1(O_1+V_1) - (O_1+V_1)S_1 \right|^2 \times \\ & \times \sum_{n=0}^{5} \binom{5}{n} \left| \Big(\chi_{\frac{1}{40}} - \chi_{\frac{21}{40}}\Big)^n \Big(\chi_{\frac{1}{8}} - \chi_{\frac{13}{8}}\Big)^{5-n} \right|^2 \end{aligned} \tag{11.33}$$

To determine the massless spectrum we have to expand in $q$, the lowest power is

$$q^{-\frac{1}{24}} \sqrt{q^{\frac{1}{24}}}\, q^{\frac{1}{8}} q^{-\frac{1}{24}} q^{5(\frac{1}{40} - \frac{4}{5}\frac{1}{24})} = q^{\frac{1}{16}} \tag{11.34}$$

So the ground state energy is $E = \frac{1}{16} > 0$, there are no massless states in the twisted sector.

We were able to guess the $\sigma$–action for $k = 1, 2, 3, 6$. The result is This is very strange, it should be precisely the other way round: If there is a singularity there are massless states corresponding to its resolution. And on a smooth manifold there should not be any massless twisted sector states.

In fact this is not a coincidence. Shortly after our paper Eguchi & Sugawara [25] showed that this is so for all $k$: There are massless states in the twisted sector if and only if all $k_i \in 2\mathbb{Z}$. But if all $k_i$ are even then the $\sigma$ action is free on the Calabi–Yau manifold and thus the quotient a smooth manifold.

| CFT | Geometry |
|---|---|
| $(k=3)^5$<br>No massless twisted sector states | $\left(\mathbb{CP}_4[5]\times S^1\right)/\mathbb{Z}_2$<br>Orbifold singularity |
| $(k=6)^4$<br>Massless twisted sector states | $\left(\mathbb{CP}_{1,1,1,1,4}[8]\times S^1\right)/\mathbb{Z}_2$<br>Smooth manifold |
| $(k=2)^3(k=6)^2$<br>Massless twisted sector states | $\left(\mathbb{CP}_{1,1,2,2,2}[8]\times S^1\right)/\mathbb{Z}_2$<br>Smooth manifold |

Table 11.1: Comparison Geometry ↔ CFT

## 11.3 The Resolution of the Puzzle

Our explanation for this mismatch between CFT and geometric description is the following: The Gepner model corresponds to a point in the moduli space with non–trivial NS–NS two–form flux, e.g. for the quintic

$$B + iJ = \frac{1}{2} + \frac{i}{2}\cot\left(\frac{\pi}{5}\right) \tag{11.35}$$

But $B$ is projected out by $\sigma$ so it is no longer a continuous parameter but can only assume discrete values (Compare to Type I).

So we propose that the SCFT and the geometric (large volume) point are on different components of the moduli space. The components differ at least by discrete values of the $B$–field.

Geometrically the discrete $B$–flux should come from a flat but nontrivial . So if the $G_2$ quotient is a smooth manifold then we expect $H^3(X,\mathbb{Z})_{\text{Tor}} \neq 0$. We can try to check this by computing the cohomology of $X = (Y\times S^1)/\mathbb{Z}_2$. In general we cannot say how $\sigma$ acts on $H^{1,1}(Y)$, only the Kähler form of course is anti–invariant. For simplicity assume that $\sigma$ acts as $-1$ on all of $H^{1,1}(Y)$, then by simply counting invariant forms we find

$$\begin{aligned} b_1 =& 0 \\ b_2 =& 0 \\ b_3 =& 1 + h_{21} + h_{11} \end{aligned} \tag{11.36}$$

Now to say something about the integer cohomology we will use the following

**Theorem 30 (Cartan–Leray Spectral Sequence).** *Let $Y = W/G$ be the quotient by a freely acting discrete group. Then there is a spectral sequence with*

$$E^2_{p,q} = H_p\Big(G, H_q(W;\mathbb{Z})\Big)$$

*converging to $H_i(Y;\mathbb{Z})$*

where the $H_p(G,H)$ is *group homology* (see [17]) and the arguments $G$, $H$ are groups with $G$ acting on $H$. And note that the CLSS computes homology, so we are interested in $H^3(X,\mathbb{Z})_{\mathrm{Tor}} = H_2(X,\mathbb{Z})_{\mathrm{Tor}}$.

Now the $E^2_{p,q}$ tableau is (with arrows reversed since the CLSS is a homology spectral sequence):

$$E^2_{p,q} = \begin{array}{r|ccccc}
q{=}3 & \mathbb{Z}^{h_{11}+h_{21}+1} & 0 & 0 & 0 & 0 \\
q{=}2 & \mathbb{Z}_2^{h_{11}} & 0 & \mathbb{Z}_2^{h_{11}} & 0 & \mathbb{Z}_2^{h_{11}} \\
q{=}1 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\
q{=}0 & \mathbb{Z} & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 \\
\hline
 & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4
\end{array} \qquad d_2,\ d_3 \tag{11.37}$$

So depending on $d_2$ and $d_3$ we find

$$H_2(X;\mathbb{Z}) = \mathbb{Z}_2^{h_{11}} \text{ or } \mathbb{Z}_2^{h_{11}-1} \text{ or } \mathbb{Z}_2^{h_{11}-2} \tag{11.38}$$

In fact we can exclude the last possibility; I will show this in the remainder of this section. For that we think of $X = (Y\times S^1)/\mathbb{Z}_2$ as a $S^1$ bundle over $Y/\mathbb{Z}_2$. This bundle is the bundle with orientable total space on the non–orientable base. Then again using the CLSS and Poincarè duality with $\mathbb{Z}_2$ coefficients we can determine the homology of $Y/\mathbb{Z}_2$ to be

$$H_p(Y/\mathbb{Z}_2;\mathbb{Z}) = \begin{cases}
0 & p=6 \\
\mathbb{Z}_2 & p=5 \\
\mathbb{Z}^{h^{11}} & p=4 \\
\mathbb{Z}^{h^{21}+1}[\oplus\mathbb{Z}_2] & p=3 \\
\mathbb{Z}_2^{h^{11}-1}[\oplus\mathbb{Z}_2] & p=2 \\
\mathbb{Z}_2 & p=1 \\
\mathbb{Z} & p=0
\end{cases} \tag{11.39}$$

The ambiguity is now less since there are less $\mathbb{Z}_2$'s around. The cohomology of the total space $X$ of the $S^1$ bundle is then determined by the general Leray–Serre spectral sequence (not theorem 14 but for arbitrary base):

$$E_2^{p,q} = \begin{array}{r|ccccccc}
q{=}1 & 0 & \mathbb{Z}_2 & \mathbb{Z}^{h^{11}} & \mathbb{Z}^{h^{21}+1}[\oplus\mathbb{Z}_2] & \mathbb{Z}_2^{h^{11}-1}[\oplus\mathbb{Z}_2] & \mathbb{Z}_2 & \mathbb{Z} \\
q{=}0 & \mathbb{Z} & 0 & \mathbb{Z}_2 & \mathbb{Z}^{h^{21}+1}\oplus\mathbb{Z}_2^{h^{11}-1}[\oplus\mathbb{Z}_2] & \mathbb{Z}^{h^{11}}[\oplus\mathbb{Z}_2] & 0 & \mathbb{Z}_2 \\
\hline
 & p{=}0 & p{=}1 & p{=}2 & p{=}3 & p{=}4 & p{=}5 & p{=}6
\end{array} \qquad d_2 \tag{11.40}$$

The $d_2$ in the above tableau has to vanish because the $\mathbb{Z}_2$ at $(p,q) = (6,0)$ has to survive to $H^6(X;\mathbb{Z}) = \mathbb{Z}_2^2$. Therefore

$$H^3(X;\mathbb{Z})_{\mathrm{Tor}} = H^5(X;\mathbb{Z})_{\mathrm{Tor}} = \mathbb{Z}_2^{h^{11}-1} \text{ or } \mathbb{Z}_2^{h^{11}} \tag{11.41}$$

# Chapter 12

# The Spin(7) case

## 12.1 Noncompact Spin(7) Manifolds

In 8 dimensions a manifold with exactly one parallel spinor has holonomy Spin(7) $\subset SO(8)$. Similarly to the $G_2$ case one can reformulate this as the existence of a certain form, only this time it is a 4-form and it looks locally like

$$\begin{aligned}\varphi_0 = -e_{1234} + e_{1256} + e_{1278} - e_{1367} - e_{1358} - e_{1468} + e_{1457} + \\ + e_{2368} - e_{2357} - e_{2467} - e_{2458} - e_{3456} - e_{3478} + e_{5678}\end{aligned} \tag{12.1}$$

and has to satisfy

$$\varphi = *\varphi \qquad \mathrm{d}\varphi = 0 \tag{12.2}$$

Nobody has ever found an explicit metric with Spin(7) or $G_2$ holonomy on a compact manifold. So this is probably as hard as finding an explicit Calabi–Yau metric. However one can find such metrics with exceptional holonomy on noncompact spaces, the first one was given in [20, 21].

The largest class are the cones over Allof–Wallach spaces, that is cones over the coset $SU(3)/U(1)$. In the notation of [36] the exterior algebra on the base is

$$
\begin{aligned}
\mathrm{d}\sigma_1 &= -\frac{1}{2}\lambda \wedge \sigma_2 - \frac{3}{2}Q \wedge \sigma_2 - \nu_1 \wedge \Sigma_2 - \nu_2 \wedge \Sigma_1 \\
\mathrm{d}\sigma_2 &= \frac{1}{2}\lambda \wedge \sigma_1 + \frac{3}{2}Q \wedge \sigma_1 + \nu_1 \wedge \Sigma_1 - \nu_2 \wedge \Sigma_2 \\
\mathrm{d}\Sigma_1 &= \frac{1}{2}\lambda \wedge \Sigma_2 - \frac{3}{2}Q \wedge \Sigma_2 - \nu_1 \wedge \sigma_2 + \nu_2 \wedge \sigma_1 \\
\mathrm{d}\Sigma_2 &= -\frac{1}{2}\lambda \wedge \Sigma_1 + \frac{3}{2}Q \wedge \Sigma_1 + \nu_1 \wedge \sigma_1 + \nu_2 \wedge \sigma_2 \\
\mathrm{d}\nu_1 &= -\lambda \wedge \nu_2 + \sigma_1 \wedge \Sigma_2 - \sigma_2 \wedge \Sigma_1 \\
\mathrm{d}\nu_2 &= +\lambda \wedge \nu_1 + \sigma_1 \wedge \Sigma_1 + \sigma_2 \wedge \Sigma_2 \\
\mathrm{d}\lambda &= 2\Big(2\nu_1 \wedge \nu_2 + \sigma_1 \wedge \sigma_2 - \Sigma_1 \wedge \Sigma_2\Big)
\end{aligned}
\tag{12.3}
$$

The best known ansatz for the vielbeine is

$$
\begin{array}{ll}
e_1 = \mathrm{d}r & e_2 = f(r)\,\lambda \\
e_3 = a(r)\,\sigma_1 & e_4 = a(r)\,\sigma_2 \\
e_5 = c_1(r)\,\nu_1 & e_6 = c_2(r)\,\nu_2 \\
e_7 = b(r)\,\Sigma_1 & e_8 = b(r)\,\Sigma_2
\end{array}
\tag{12.4}
$$

Smooth solutions were classified in [41], see also [23]. By the ansatz the 4-form $\varphi$ is automatically selfdual and the remaining condition $\mathrm{d}\varphi = 0$ yields the following set of differential equations for the coefficients:

$$
\begin{aligned}
\frac{\mathrm{d}a(r)}{\mathrm{d}r} &= -\frac{f(r)}{a(r)} + \frac{c_1(r)+c_2(r)}{2b(r)} + \\
&\quad + \left(\frac{1}{2}b(r) - \frac{a(r)^2}{2b(r)}\right)\left(\frac{1}{c_1(r)} + \frac{1}{c_2(r)}\right) \\
\frac{\mathrm{d}b(r)}{\mathrm{d}r} &= -\frac{f(r)}{b(r)} + \frac{c_1(r)+c_2(r)}{2a(r)} + \\
&\quad + \left(\frac{1}{2}a(r) - \frac{b(r)^2}{2a(r)}\right)\left(\frac{1}{c_1(r)} + \frac{1}{c_2(r)}\right) \\
\frac{\mathrm{d}c_1(r)}{\mathrm{d}r} &= \frac{a(r)}{b(r)} + \frac{b(r)}{a(r)} - \frac{c_1(r)^2}{a(r)b(r)} - \frac{c_1(r)^2}{2c_2(r)f(r)} + \frac{c_2(r)}{2f(r)} + \frac{2f(r)}{c_2(r)} \\
\frac{\mathrm{d}c_2(r)}{\mathrm{d}r} &= \frac{a(r)}{b(r)} + \frac{b(r)}{a(r)} - \frac{c_2(r)^2}{a(r)b(r)} - \frac{c_2(r)^2}{2c_1(r)f(r)} + \frac{c_1(r)}{2f(r)} + \frac{2f(r)}{c_1(r)} \\
\frac{\mathrm{d}f(r)}{\mathrm{d}r} &= f(r)^2\left(\frac{1}{a(r)^2} + \frac{1}{b(r)^2}\right) + \frac{c_1(r)}{2c_2(r)} + \frac{c_2(r)}{2c_1(r)} - \frac{2f(r)^2}{c_1(r)c_2(r)} - 1
\end{aligned}
\tag{12.5}
$$

For those I was able to find the novel analytic solution

$$
\begin{aligned}
a(r) =& - b(r) = -r\sqrt{\frac{\sqrt{1+xr^2}}{1+\sqrt{1+xr^2}}} \\
f(r) =& -\frac{1}{2}r \\
c_1(r) =& \frac{r}{\sqrt{1+xr^2}} \\
c_2(r) =& 2f(r)
\end{aligned}
\tag{12.6}
$$

It has the nice feature of a finite "M-theory circle" far away from the tip of the cone, with radius parametrized by $x \in \mathbb{R}_{\geq}$. Unfortunately it is not smooth at the origin and therefore only of limited interest.

## 12.2 Spin(7) Gepner Models

Of course we are really interested in compact Spin(7) manifolds, noncompact spaces as in the previous section can only serve as local models. One specific construction for a compact Spin(7) manifold is to start with a Calabi–Yau fourfold $Y$ and then divide by an antiholomorphic involution $\sigma : Y \to Y$, see [40]. If $\sigma$ has fixed points this introduces of course orbifold singularities, and if $\sigma$ acts freely then the holonomy is not the full Spin(7). Still physics is well defined and we expect minimal supersymmetry corresponding to one constant spinor.

Now having identified the $\sigma$ action on the constituents of Gepner models we can of course apply the same knowledge here, this was carried out in joint work with R. Blumenhagen [13]. For example we have the following correspondence

$$
(k=2)^2(k=6)^4 \text{ Gepner model } \leftrightarrow \; Y = \mathbb{C}\mathrm{P}_{1,1,1,1,2,2}[8] \tag{12.7}
$$

with the nontrivial hodge numbers $h_{31} = 443$, $h_{11} = 1$, $h_{21} = 0$ and $h_{22} = 1820$. The analogous computation to the $G_2$ case yields the numbers of chiral and antichiral massless bosons and fermions in the untwisted sector

$$
\begin{aligned}
n^u_{\psi^+} &= 2\,(h_{31} + h_{11}) \\
n^u_{\psi^-} &= 2\,h_{21} \\
n^u_{\phi^+} &= 2\,(h_{31} + h_{11}) \\
n^u_{\phi^-} &= 3\,h_{31} + 3\,h_{11} - h_{21} + 25
\end{aligned}
\tag{12.8}
$$

and in the twisted sector

$$\begin{aligned} n^{tw}_{\phi^+} =&2 \\ n^{tw}_{\phi^-} =&2 \\ n^{tw}_{\psi^+} =&2 \\ n^{tw}_{\psi^-} =&0 \end{aligned} \tag{12.9}$$

So far everything is rather similar to the $G_2$ case. However there is one new feature in this string theory compactification to 2 dimensions: There is a gravitational anomaly

$$I = 2 + \frac{n_{\phi^+} - n_{\phi^-}}{12} + \frac{n_{\psi^+} - n_{\psi^-}}{24} \tag{12.10}$$

and the contribution of the untwisted and the twisted states cancels:

$$I = I^u + \left( \frac{n^{tw}_{\phi^+} - n^{tw}_{\phi^-}}{12} + \frac{n^{tw}_{\psi^+} - n^{tw}_{\psi^-}}{24} \right) = -\frac{1}{12} + \frac{1}{12} = 0 \tag{12.11}$$

This is a nice check on the computation. In fact one can turn the argument around: The untwisted sector states eq. 12.8 are generic for all Gepner models with all $k$ even. Then to cancel the anomaly there must be twisted states.

So again we see that there are massless twisted sector states if the geometric action is free, in contrast to our naive expectation. This time it follows already from anomaly cancellation and does not require sophisticated analysis of the CFT.

# Summary

My primary interests are string compactifications with and without background fields. In this thesis I have described various results which hopefully improve our understanding of the effects. Especially the following had not appeared before in the literature:

- An example of a Calabi–Yau manifold with K–theory torsion, section 6.6.3.
- The complete $KO_{\mathbb{Z}_2}(\mathbb{R}^{p,q})$ groups, section 8.5.
- The twisted Real equivariant K–theory, section 9.6.
- The Gepner models for $G_2$ and Spin(7) manifolds, chapter 11 and section 12.2.
- The Spin(7) cone metric in section 12.1.

To make this accessible to non-experts I tried to start with an introduction to K–theory so that hopefully every reader would benefit. This of course entails an ever increasing pace. Furthermore I tried to combine mathematical and physical language and I hope that physicists can accept the occasional theorem and mathematicians live with the utter absence of any categorial terms.

# Index

# Symbol Index

# Bibliography


[1] Alejandro Adem and Yongbin Ruan. Twisted orbifold K–theory. *math.AT/0107168*, 2001.

[2] Anton Yu. Alekseev and Volker Schomerus. D-branes in the wzw model. *Phys.Rev.*, D60(061901), 1999. hep-th/9812193.

[3] Michael Francis Atiyah. Vector bundles and the Künneth formula. *Topology*, 1:245–248, 1962.

[4] Michael Francis Atiyah. K-theory and Reality. *Quart. J. Math, Oxford*, 17:367–86, 1966.

[5] Michael Francis Atiyah. K–theory past and present. *math.KT/0012213*, 2000.

[6] Michael Francis Atiyah, Raul Bott, and Arnold Shapiro. Clifford modules. *Topology*, 3(1):3–38, 1964.

[7] Michael Francis Atiyah and F. Hirzebruch. Vector bundles and homogenous spaces. *Proceedings of Symposia in Pure Mathematics*, 3:7–38, 1961.

[8] Michael Francis Atiyah, Juan Maldacena, and Cumrun Vafa. An M-theory flop as a large N duality. *J.Math.Phys.*, 42:3209–3220, 2001. hep-th/0011256.

[9] Michael Francis Atiyah and Graeme B. Segal. Equivariant K-theory and completion. *J. Diff. Geom.*, 3:1–18, 1969.

[10] Oren Bergman, Eric G. Gimon, and Petr Hořava. Brane transfer operations and T–duality of non–BPS states. *JHEP*, 04(010), 1999. hep-th/9902160.

[11] Julie D. Blum and Alberto Zaffaroni. An orientifold from F theory. *Phys.Rev. D54*, 1996. hep-th/9601038.

[12] Ralph Blumenhagen and Volker Braun. Superconformal field theories for compact $G_2$ manifolds. *JHEP*, 0112(006), 2001. hep-th/0110232.

[13] Ralph Blumenhagen and Volker Braun. Superconformal field theories for compact manifolds with Spin(7) holonomy. *JHEP*, 0112(013), 2001. hep-th/0111048.

[14] Raul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology*, volume 82 of *Graduate Texts in Mathematics*. Springer, 1982.

[15] Peter Bouwknegt and Varghese Mathai. D–branes, B–fields and twisted K–Theory. *JHEP*, 0003(007), 2000. hep-th/0002023.

[16] Volker Braun. K-theory torsion. *hep-th/0005103*, 2000.

[17] Kenneth S. Brown. *Cohomology of Groups*, volume 87 of *Graduate Texts in Mathematics*. Springer, 1982.

[18] Ilka Brunner and Jacques Distler. Torsion D–branes in nongeometrical phases. *hep-th/0102018*, 2001.

[19] Ilka Brunner, Jacques Distler, and Rahul Mahajan. Return of the torsion D–branes. *hep-th/0106262*, 2001.

[20] Robert L. Bryant. Metrics with exceptional holonomy. *Ann.Math*, 126:525–576, 1987.

[21] Robert L. Bryant and Simon M. Salamon. On the construction of some complete metrics with exceptional holonomy. *Duke Math.J.*, 58(3):829–850, 1989.

[22] Eugène Cremmer, Bernard Julia, and Joël Scherk. Supergravity theory in 11 dimensions. *Phys.Lett. B*, 76:409–412, 1978.

[23] Gottfried Curio, Boris Körs, and Dieter Lüst. Fluxes and branes in Type II vacua and M–theory geometry with $G_2$ and Spin(7) holonomy. *hep-th/0111165*, 2001.

[24] Atish Dabholkar and Jaemo Park. A note on orientifolds and F–theory. *Phys.Lett. B394*, 1997. hep-th/9607041.

[25] Tohru Eguchi and Yuji Sugawara. String theory on $G_2$ manifolds based on Gepner construction. *hep-th/0111012*, 2001.

[26] Samuel Eilenberg and Saunders MacLane. Relations between homology and homotopy groups of spaces. *Ann. Math.*, 46:480–509, 1945.

[27] Stefan Fredenhagen and Volker Schomerus. Brane dynamics in CFT backgrounds. *hep-th/0104043*, 2001.

[28] Stefan Fredenhagen and Volker Schomerus. Branes on group manifolds, gluon condensates, and twisted K–theory. *JHEP*, 0104(007), 2001. hep-th/0012164.

[29] Daniel S. Freed and Edward Witten. Anomalies in string theory with D–branes. *hep-th/9907189*, 1999.

[30] Thomas Friedrich. *Vorlesungen über K–Theorie*. Teubner Verlag, 1978.

[31] Eric G. Gimon and Joseph Polchinski. Consistency conditions for orientifolds and D–manifolds. *Phys.Rev. D*, 54:1667–1676, 1996. hep-th/9601038.

[32] P. S. Green. A cohomology theory based upon self–conjugacies of complex vector bundles. *Bull. Amer. Math. Soc.*, 70:522, 1964.

[33] Phillip Griffith and Joseph Harris. *Principles of Algebraic Geometry*. John Whiley & Sons, Inc., 1978.

[34] Phillip Griffiths and John Adams. *Topics in Algebraic and Analytic Geometry*, volume 13 of *Mathematical Notes*. Princeton University Press, 1974.

[35] Sergei Gukov. K–theory, Reality, and orientifolds. *Commun.Math.Phys.*, 210:621–639, 2000. hep-th/9901042.

[36] Sergei Gukov and James Sparks. M-theory on Spin(7) manifolds. *Nucl.Phys. B*, 625:3–69, 2002.

[37] Robin Hartshorne. *Algebraic Geometry*, volume 54 of *Graduate Texts in Mathematics*. Springer, 1977.

[38] Nigel Hitchin. Lectures on special lagrangian submanifolds. *math.DG/9907034*, 1999. Lectures given at the ICTP School on Differential Geometry April 1999.

[39] Dominic D. Joyce. Compact Riemannian 7-manifolds with holonomy $G_2$: II. *J.Diff.Geom.*, 43:329–375, 1996.

[40] Dominic D. Joyce. *Compact Manifolds with Special Holonomy*. Oxford University Press, 2000.

[41] Hiroaki Kanno and Yukinori Yasui. On Spin(7) holonomy metric based on $SU(3)/U(1)$: II. *hep-th/0111198*, 2001.

[42] Anton Kapustin. D–branes in a topologically nontrivial B–field. *Adv.Theor.Math.Phys.*, 4:127–154, 2000. hep-th/9909089.

[43] Max Karoubi. Algèbres de Clifford et K–théorie. *Ann. Scient. Éc. Norm. Sup.*, 4(1):161–270, 1968.

[44] Max Karoubi. *K–Theory*. Springer, 1978.

[45] Max Karoubi, R. Gordon, Peter Löffler, and M. Zisman. *Séminaire Heidelberg–Saarbrücken–Strasbourg sur la K–Théorie*, volume 136 of *Lecture Notes in Mathematics*. Springer, 1970.

[46] Peter Kaste, Alexandros Kehagias, and Herve Partouche. Phases of supersymmetric gauge theories from M-theory on $G_2$ manifolds. *JHEP*, 0105(058), 2001. hep-th/0104124.

[47] Juan Maldacena, Gregory Moore, and Nathan Seiberg. D–brane instantons and K–theory charges. *JHEP*, 0111(062), 2001. hep-th/0108100.

[48] John McCleary. *A User's Guide to Spectral Sequences*. Cambridge University Press, 1985.

[49] André Miemiec. *Branes between Geometry and Gauge Theory*. PhD thesis, Humboldt University, 2000. http://dochost.rz.hu-berlin.de/dissertationen/miemiec-andre-2000-07-10/.

[50] Ruben Minasian and Gregory Moore. K–theory and Ramond–Ramond charge. *JHEP*, 9711(002), 1997. hep-th/9710230.

[51] Norma Quiroz and Bogdan Stefański, jr. Dirichlet branes on orientifolds. *hep-th/0110041*, 2001.

[52] Jonathan Rosenberg. Homological invariants of extensions of $C^*$–algebras. In *Proceedings of Symposia in Pure Mathematics Volume 38 Part 1*, pages 35–75, 1982.

[53] Jonathan Rosenberg. Continous–trace algebras from the bundle theoretic point of view. *J.Austral.Math.Soc. (Series A)*, 47:368–381, 1989.

[54] Graeme Segal. private correspondence with B. Stefański, jr. 2002.

[55] Ashoke Sen. SO(32) spinors of type I and other solitons on brane-antibrane pair. *JHEP*, 9809(023), 1998. hep-th/9808141.

[56] Ashoke Sen. Descent relations among bosonic D–branes. *Int.J.Mod.Phys.*, A14:4061–4078, 1999. hep-th/9902105.

[57] Eric R. Sharpe. D-branes, derived categories, and Grothendieck groups. *Nucl. Phys. B*, 561:433–450, 1999. hep-th/9902116.

[58] Eric R. Sharpe. Discrete torsion and gerbes i. *hep-th/9909108*, 1999.

[59] Robert M. Switzer. *Algebraic Topology – Homotopy and Homology*, volume 212 of *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*. Springer, 1975.

[60] Ulrike Tillmann. private communication. 2000.

[61] Cumrun Vafa. Modular invariance and discrete torsion on orbifolds. *Nucl.Phys. B*, 273:592–606, 1986.

[62] McKenzie Y. Wang. Parallel spinors and parallel forms. *Annals of Global Analysis and Geometry*, 7:59–68, 1989.

[63] N. E. Wegge-Olsen. *K–theory and $C^*$–algebras*. Oxford University Press, 1993.

[64] Edward Witten. D–branes and K–theory. *JHEP*, 9812(019), 1998. hep-th/9810188.

[65] Edward Witten. Overview of K–theory applied to strings. *Int.J.Mod.Phys.*, A16:693–706, 2001. hep-th/0007175.

[66] Edward Witten and Gregory Moore. Self-duality, Ramond–Ramond fields, and K-theory. *JHEP*, 0005(032), 2000. hep-th/9912279.

# Lebenslauf

| | |
|---|---|
| Name: | Volker Friedrich Braun |
| 09/1994–08/1997 | Studium an der Julius–Maximilians–Universität Würzburg<br>in der Fachrichtung Physik |
| 09/1997–12/1998 | Studium an der University of Texas at Austin<br>in der Fachrichtung Physik |
| 19.12.1998 | Master of Arts, University of Texas at Austin |
| 04/1999–08/2002 | Studium an der Humboldt Universität Berlin<br>in der Fachrichtung Physik |
| 07/1999–08/2002 | Doktorand<br>Lehrstuhl Prof. Lüst,<br>Institut für Physik |

# Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfaßt zu haben und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Volker Friedrich Braun
24. April 2002