1 Households' problem

Denoting by S the aggregate state vector, we can write the recursive problem of the household in the following manner:

$$V(S,b,n) = \max_{c_N, c_T, \mathrm{I}_N, d} u(c_A, d, n) + \beta\, \mathbb{E}\left\{V(S',b',n') \mid \theta\right\}, \tag{1}$$

subject to

$$p(S)\mathrm{I}_N c_N + c_T + b' = (1+r)b + w(S)n + \pi_N(S) + \pi_T(S), \tag{2}$$

$$\mathrm{I}_N = d\ \Psi^d[Q^g(S)], \tag{3}$$

$$n' = (1-\lambda)n + \Phi^w[Q^e(S)](1-n), \tag{4}$$

$$S' = G(S). \tag{5}$$

The value of an additional worker in a representative households with wage w is

$$\widetilde{V}_n(w,S) = w u_{c_T}(S) - \varsigma + \beta\big(1-\lambda-\Phi^w[Q^e(S)]\big)\mathbb{E}\{V_n(S') \mid \theta\}, \tag{6}$$

2 Firms in the tradable goods sector

$$\Omega^T(S,k,n) = \max_{i,v} F^T(k,n) - w(S)n - i - v\kappa + \mathbb{E}\left\{\frac{\Omega^T(S',k',n')}{1+r} \mid \theta\right\}, \tag{7}$$

subject to

$$k' = (1-\delta)k + i, \tag{8}$$

$$n' = (1-\lambda)n + \Phi^f[Q^e(S)]v, \tag{9}$$

$$S' = G(S). \tag{10}$$

The first order condition is

$$\frac{\kappa}{\Phi^f[Q^e(S)]} = \mathbb{E}\left\{\frac{\Omega^T_n(S,k',n')}{1+r} \mid \theta\right\} \tag{11}$$

Using the envelop condition, the value of an additional worker with wage w is

$$\widetilde{\Omega}^T_n(w,S) = F^T_n(S) - w + \frac{(1-\lambda)}{1+r}\mathbb{E}\{\Omega^T_n(S') \mid \theta\}, \tag{12}$$

Alternatively, we have

$$\widetilde{\Omega}_n^T(w,S) = F_n^T(S) - w + (1-\lambda)\frac{\kappa}{\Phi^f[Q^e(S)]}. \tag{13}$$

3 Nash bargaining

To simply the exposition, we only consider the tradable firms and assume that all the workers are hired by the tradable firms. The Nash bargaining problem becomes

$$w(S) = \max_w \left[\widetilde{V}_n(w,S)\right]^{\varphi}\left[\widetilde{\Omega}_n^T(w,S)\right]^{1-\varphi}, \tag{14}$$

where φ is the bargaining power of households Taking the derivative with respect to w yields the first order condition

$$\varphi u_{c_T}(S)\widetilde{\Omega}_n^T(w,S) = (1-\varphi)\widetilde{V}_n(w,S). \tag{15}$$

The first order condition imply that in equilibrium,

$$\varphi u_{c_T}(S)\Omega_n^T(S) = (1-\varphi)V_n(S). \tag{16}$$

Using Equation (16), we can replace $V_n(S')$ by $\Omega_n^T(S')$ and rewrite Equation (6) as

$$\widetilde{V}_n(w,S) = wu_{c_T}(S) - \varsigma + \beta\big(1-\lambda-\Phi^w[Q^e(S)]\big)\mathbb{E}\left\{\frac{\varphi}{1-\varphi}u_{c_T}(S')\Omega_n^T(S') \mid \theta\right\}, \tag{17}$$

Then Equation (15) becomes

$$\frac{\varphi}{1-\varphi}\Omega_n^T(w,S) = w - \frac{\varsigma}{u_{c_T}(S)} + \beta\big(1-\lambda-\Phi^w[Q^e(S)]\big)\mathbb{E}\left\{\frac{\varphi}{1-\varphi}\frac{u_{c_T}(S')}{u_{c_T}(S)}\Omega_n^T(S') \mid \theta\right\} \tag{18}$$

If we use Equation (13), we have

$$\begin{aligned}&\frac{\varphi}{1-\varphi}\left(F_n^T(S) - w + (1-\lambda)\frac{\kappa}{\Phi^f[Q^e(S)]}\right) = \\ &w - \frac{\varsigma}{u_{c_T}(S)} + \beta\big(1-\lambda-\Phi^w[Q^e(S)]\big)\mathbb{E}\left\{\frac{\varphi}{1-\varphi}\frac{u_{c_T}(S')}{u_{c_T}(S)}\left(F_n^T(S') - w(S') + (1-\lambda)\frac{\kappa}{\Phi^f[Q^e(S')]}\right) \mid \theta\right\}\end{aligned} \tag{19}$$

We have already derived the condition that the wage w has to satisfied. Note that we do not need to approximate any value function.

In our model, firms discount future profit using the world interest rate r. If they discount future using $\beta\frac{u_{c_T}(S')}{u_{c_T}(S)}$, then Equation (11) and (12) become

$$\frac{\kappa}{\Phi^f[Q^e(S)]} = \beta\mathbb{E}\left\{\frac{u_{c_T}(S')}{u_{c_T}(S)}\Omega_n^T(S,k',n') \mid \theta\right\} \tag{20}$$

$$\widetilde{\Omega}_n^T(w,S) = F_n^T(S) - w + \beta(1-\lambda)\mathbb{E}\left\{\frac{u_{c_T}(S')}{u_{c_T}(S)}\Omega_n^T(S') \mid \theta\right\}, \tag{21}$$

The wage function (18) now becomes

$$w = \varphi(F_n^T(S) + \kappa Q(S)) + (1-\varphi)\frac{\varsigma}{u_{c_T}(S)} \tag{22}$$

Here, the wage only depends on the current state variables.