Econ 8108 Macroeconomics First Year Session IV Spring 2015*

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Contents

*This is the evolution of class notes by many students over the years, both from Penn and Minnesota.

1. Introduction

A model is an artificial economy. Description of a model's environment may include specifying the agents' preferences and endowment, technology available, information structure as well as property rights. Neoclassical Growth Model becomes one of the workhorses of modern macroeconomics because it delivers some fundamental properties of modern economy, summarized by, among others, Kaldor:

1. Output per capita has grown at a roughly constant rate (2%).

2. The capital-output ratio (where capital is measured using the perpetual inventory method based on past consumption foregone) has remained roughly constant.

3. The capital-labor ratio has grown at a roughly constant rate equal to the growth rate of output.

4. The wage rate has grown at a roughly constant rate equal to the growth rate of output.

5. The real interest rate has been stationary and, during long periods, roughly constant.

6. Labor income as a share of output has remained roughly constant (0.66).

7. Hours worked per capita have been roughly constant.

Equilibrium can be defined as a prediction of what will happen and therefore it is a mapping from environments to outcomes (allocations, prices, etc.). One equilibrium concept that we will deal with is Competitive Equilibrium[1] Characterizing the equilibrium, however, usually involves finding solutions to a system of infinite number of equations. There are generally two ways of getting around this. First, invoke the welfare theorem to solve for the allocation first and then find the equilibrium prices associated with it. The first way sometimes may not work due to, say, presence of externality. So the second way is to look at Recursive Competitive equilibrium, where equilibrium objects are functions instead of variables.

2. Review: Neoclassical Growth Model

We review briefly the basic neoclassical growth model.

2.1. The Neoclassical Growth Model Without Uncertainty

The commodity space is

$$\mathcal{L} = \{(l_1, l_2, l_3) : l_i = (l_{it})_{t=0}^{\infty}\ l_{it} \in \mathbb{R},\ \sup_t |l_{it}| < \infty,\ i = 1, 2, 3\}.$$

The consumption possibility set is

$$\begin{aligned} X(\overline{k}_0) = \{x \in \mathcal{L} :\ & \exists (c_t, k_{t+1})_{t=0}^{\infty} \text{ s.th. } \forall t = 0, 1, ... \\ & c_t, k_{t+1} \geq 0,\ x_{1t} + (1-\delta)k_t = c_t + k_{t+1},\ -k_{t+1} \leq x_{2t} \leq 0,\ -1 \leq x_{3t} \leq 0,\ k_0 = \overline{k}_0\}. \end{aligned}$$

[1] Arrow-Debreu or Valuation Equilibrium.

The production possibility set is $Y = \prod_t Y_t$, where

$$Y_t = \{(y_{1t}, y_{2t}, y_{3t}) \in \mathbb{R}^3 : \ 0 \le y_{1t} \le F(-y_{2t}, -y_{3t})\}.$$

Definition 1 *An Arrow-Debreu equilibrium is $(x^*, y^*) \in X \times Y$, and a continuous linear functional ν^* such that*

1. $x^* \in \arg\max_{x \in X, \nu^*(x) \le 0} \sum_{t=0}^{\infty} \beta^t u(c_t(x), -x_{3t})$,
2. $y^* \in \arg\max_{y \in Y} \nu^*(y)$,
3. *and* $x^* = y^*$.

Now, let's look at the one-sector growth model's Social Planner's Problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, -x_{3t}) \qquad (SPP)$$
$$\text{s.t.}$$
$$c_t + k_{t+1} - (1-\delta)k_t = x_{1t}$$
$$0 \le x_{2t} \le k_t$$
$$0 \le x_{3t} \le 1$$
$$0 \le y_{1t} \le F(-y_{2t}, -y_{3t})$$
$$x = y$$
$$k_0 \text{ given.}$$

Suppose we know that a solution in sequence form exists for (SPP) and is unique. Clearly stating sufficient assumptions on utility and production function, show that (SPP) has a unique solution.

Two important theorems show the relationship between CE allocations and Pareto optimal allocations:

Theorem 1 *Suppose that for all $x \in X$ there exists a sequence $(x_k)_{k=0}^{\infty}$, such that for all $k \ge 0$, $x_k \in X$ and $U(x_k) > U(x)$. If (x^*, y^*, ν^*) is an Arrow-Debreu equilibrium then (x^*, y^*) is Pareto efficient allocation.*

Theorem 2 *If X is convex, preferences are convex, U is continuous, Y is convex and has an interior point, then for any Pareto efficient allocation (x^*, y^*) there exists a continuous linear functional ν such* *that (x^*, y^*, ν) is a quasiequilibrium, that is (a) for all $x \in X$ such that $U(x) \geq U(x^*)$ it implies $\nu(x) \geq \nu(x^*)$ and (b) for all $y \in Y$, $\nu(y) \leq \nu(y^*)$.*

Note that at the very basis of the CE definition and welfare theorems there is an implicit assumption of perfect commitment and perfect enforcement. Note also that the FWT implicitly assumes there is no externality or public goods (achieves this implicit assumption by defining a consumer's utility function only on his own consumption set but no other points in the commodity space).

From the First Welfare Theorem, we know that if a Competitive Equilibrium exits, it is Pareto Optimal. Moreover, if the assumptions of the Second Welfare Theorem are satisfied and if the SPP has unique a solution then the competitive equilibrium allocations is unique and they are the same as the PO allocations. Prices can be constructed using this allocations and first order conditions.

Show that

$$\frac{v_{2t}}{v_{1t}} = F_k(k_t, l_t) \text{ and } \frac{v_{3t}}{v_{1t}} = F_l(k_t, l_t).$$

One shortcoming of the AD equilibrium is that all trade occurs at the beginning of time. This assumption is unrealistic. Modern Economics is based on sequential markets. Therefore we define another equilibrium concept, Sequence of Markets Equilibrium (SME). We can easily show that SME is equivalent to ADE. Therefore all of our results still hold and SME is the right problem to solve.

Define a Sequential Markets Equilibrium (SME) for this economy. Prove that the objects we get from the AD equilibrium satisfy SME conditions and that the converse is also true. We should first show that a CE exists and therefore coincides with the unique solution of (SPP).

Note that the (SPP) problem is hard to solve, since we are dealing with infinite number of choice variables. We have already established the fact that this SPP problem is equivalent to the following dynamic problem:

$$\begin{aligned} v(k) = \max_{c,k'} \quad & u(c) + \beta v(k') \qquad (RSPP) \\ \text{s.t. } \; & c + k' = f(k). \end{aligned}$$

We have seen that this problem is easier to solve.

What happens when the welfare theorems fail? In this case the solutions to the social planners problem and the CE do not coincide and so we cannot use the theorems we have developed for dynamic programming to solve the problem. As we will see in this course, in this case we can work with Recursive Competitive Equilibria. In general, we can prove that the solution to the RCE coincides with a sequential markets problem but not the other way around (for example when we have multiple equilibria). However, in all the models we see in this course, this equivalence will hold.

2.2. A Comment on the Welfare Theorems

Situations in which the welfare theorems would not hold include externalities, public goods, situations in which agents are not price takers (e.g. monopolies), some legal systems or lacking of markets which rule out certain contracts which appears complete contract or search frictions. In all of these situation finding equilibrium through SPP is no longer valid. Therefore, in these situations, as mentioned before, it is better to define the problem in recursive way and find the allocation using the tools of Dynamic Programming.

3. Recursive Competitive Equilibrium

3.1. A Simple Example

What we have so far is that we have established the equivalence between allocation of the SPP problem which gives the unique Pareto optima (which is same as allocation of AD competitive equilibrium and allocation of SME). Therefore we can solve for the very complicated equilibrium allocation by solving the relatively easier Dynamic Programming problem of social planner. One handicap of this approach is that in a lot of environments, the equilibrium is not Pareto Optimal and hence, not a solution of a social planner's problem, e.g. when you have taxes or externalities. Therefore, the above recursive problem would not be the right problem to solve. In some of these situations we can still write the problem in sequence form. However, we would lose the powerful computational techniques of dynamic programming. in order to resolve this issue we will define Recursive Competitive Equilibrium equivalent to SME that we can always solve for.

In order to write the decentralized household problem recursively, we need to use some equilibrium conditions so that the household knows what prices are as a function of some economy-wide aggregate state variable. We know that if capital is K_t and there is 1 unit of labor, then $w(K) = F_n(K, 1)$ and $R(K) = F_k(K, 1)$. Therefore, for the households to know prices they need to know aggregate capital.

Now, a household who is deciding about how much to consume and how much to work has to know the whole sequence of future prices, in order to make his decision. This means that he needs to know the path of aggregate capital. Therefore, if he believes that aggregate capital changes according to $K' = G(K)$, knowing aggregate capital today, he would be able to project aggregate capital path for the future and therefore the path for prices. So, we can write the household problem given function $G(\cdot)$ as follows:

$$\begin{aligned} \Omega(K, a; G) = \max_{c,a'} \quad & u(c) + \beta\Omega(K', a'; G) \qquad\qquad (RCE) \\ \text{s.t.} \quad & c + a' = w(K) + R(K)a \\ & K' = G(K), \\ & c \geq 0 \end{aligned}$$

The above problem is the problem of a household that sees K in the economy, has a belief G, and carries a units of assets from past. The solution of this problem yields policy functions $c(K, a; G), a'(K, a; G)$ and a value function $\Omega(z, K, a; G)$. The functions $w(K), R(K)$ are obtained from the firm's FOCs (below).

$$\begin{aligned} & u_c[c(K, a; G)] = \beta\Omega_a[G(K), a'(K, a; G); G] \\ & \Omega_a[K, a; G] = (1 + r)u_c[c(K, a; G)] \end{aligned}$$

Now we can define the Recursive Competitive Equilibrium.

Definition 2 *A Recursive Competitive Equilibrium with arbitrary expectations G is a set of functions*[2] Ω, $g : \mathcal{A} \times \mathcal{K} \to \mathbb{R}$, R, w, $H : \mathcal{K} \to \mathbb{R}_+$ *such that:*

1. *given* G; Ω, g *solves the household problem in (RCE),*
2. $K' = H(K; G) = g(K, K; G)$ *(representative agent condition),*
3. $w(K) = F_n(K, 1)$,
4. *and* $R(K) = F_k(K, 1)$.

[2] We could add the policy function for consumption $g_c(K, a; G)$.

We define another notion of equilibrium where the expectations of the households are consistent with what happens in the economy:

Definition 3 *A Rational Expectations (Recursive) Equilibrium is a set of functions* Ω, g, R, w, G^*, *such that:*

1. $\Omega(K, a; G^*), g(K, a; G^*)$ *solves HH problem in (RCE),*
2. $G^*(K) = g(K, K; G^*) = K'$,
3. $w(K) = F_n(K, 1)$,
4. *and* $R(K) = F_k(K, 1)$.

What this means is that in a REE, households optimize given what they believe is going to happen in the future and what happens in the aggregate is consistent with the household's decision. The proof that every REE can be used to construct a SME is left as an exercise. The reverse turns out not to be true. Notice that in REE, function G projects next period's capital. In fact, if we construct an equilibrium path based on REE, once a level of capital is reached in some period, next period capital is uniquely pinned down by the transition function. If we have multiplicity of SME, this would imply that we cannot construct the function G since one value of capital today could imply more than one value for capital tomorrow. We will focus on REE unless expressed otherwise.

3.2. Economy with Leisure

We may extend the previous framework to the elastic labor supply case. Note that aggregate employment level is not a state variable but is instead predicted by aggregate states. The households problem is as follows:

$$\begin{aligned}
\Omega(K, a; G, H) = \max_{c, a', n} \quad & u(c, n) + \beta\Omega(K', a'; G, H) \qquad\qquad (RCE) \\
\text{s.t.} \quad & c + a' = w(K, N)n + R(K, N)a \\
& K' = G(K), \\
& N = H(K)
\end{aligned}$$

with solution $a^*(K, a; G, H), n^*(K, a; G, H)$.

We may thus define an recursive competitive equilibrium with rational expectation to be a collection of functions (Ω, a^*, n^*), (G, H), (R, w), such that:

1. Given $(G, H), (R, w), (\Omega, a^*, n^*)$ solves HH problem,
2. $G(K) = a^*(K, K; G, H)$,
3. $H(K) = n^*(K, K; G, H)$,
4. $w(K, N) = F_n(K, N)$,
5. and $R(K, N) = F_k(K, N)$.

Note that Condition 1 is the optimality condition. Conditions 2 and 3 are imposed because of rational expectation. Conditions 4 and 5 are marginal pricing equations.

3.3. Economy with Externalities

Let's consider an economy where the production function is given by $F(K, k, n)$, where the derivative of F with respect to its first argument is negative; i.e. aggregate capital has a negative externality on output, that firms do not internalize when making capital-labor decisions.

In this case, prices are given by $w(K) = F_3(K, K, 1)$ and $R(K) = F_2(K, K, 1)$.

Define the RCE for this economy. Show that the equilibrium allocation is not Pareto optimal in this economy.

3.3.1. Government Taxation

Next, assume government taxes capital, at a constant rate τ, and rebates the revenues in a lump-sum fashion to the households. In this case, household's problem in recursive form becomes:

$$\begin{aligned}
\Omega(K, a) = \max_{c, a'} \quad & u(c) + \beta\Omega(K', a') \\
\text{s.t.} \quad & c + a' = w(K) + R(K)(1-\tau)a + T \\
& K' = G(K).
\end{aligned}$$

In this case, the definition of a RCE becomes a collection of functions, Ω, g, G, W, R, such that:

1. $w(K) = F_3(K, K, 1)$ and $R(K) = F_2(K, K, 1)$,
2. given other functions, Ω solves households functional equation, with g as the associated policy function,
3. $g(K, K) = G(K)$,
4. and $T = \tau R(K)K$.

Suppose government uses the revenues to finance a public project, P, that gives utility to households according to the utility function $U(c, P)$. Write household's problem in recursive form. Define the RCE.

Let's consider an economy where the consumer cares about aggregate consumption in additions to his own, in particular a utility function taking the form $u(c, c/C)$.

$$\begin{aligned}\Omega(K, a) = \max_{c,a'} \quad & u(c, c/C) + \beta\Omega(K', a') \\ \text{s.t.} \quad & c + a' = w(K) + R(K)a \\ & K' = G(K), \\ & C = H(K)\end{aligned}$$

Show that the solution for the household's problem is not Pareto optimal (it is different from the social planner's solution).

In this case an RCE is a collection of functions, Ω, g, H, G, W, R, such that:

1. $w(K) = F_2(K, 1), R(K) = F_1(K, 1)$,
2. given other functions, Ω, g solves households problem,
3. $g(K, K) = G(K)$,
4. and $H(K) = W(K) + R(K)K - G(K)$.

Another possibility could be that the consumer cares about aggregate consumption from the previous period, $u(c, C_{-1})$, or "catching up with the Joneses". In that case we would have an additional state variable C_{-1}, since this information becomes relevant to the consumer when he is solving his problem. Other externalities could appear in how the consumer enjoys leisure, in the production function, etc. To write it out more explicitly:

$$\begin{aligned}
\Omega(K, a, C^-) = \max_{c,a'} \quad & u(c, C^-) + \beta\Omega(K', a', C^{-\prime}) \\
\text{s.t.} \quad & c + a' = w(K) + R(K)a \\
& K' = G\left(K, C^-\right), \\
& C^{-\prime} = H\left(K, C^-\right)
\end{aligned}$$

A RCE is Ω, g, H, G, W, R such that

1. $w(K) = F_2(K, 1), R(K) = F_1(K, 1)$.
2. Given other functions, Ω, g solves the household's problem.
3. $g\left(K, K, C^-\right) = G\left(K, C^-\right)$.
4. $H\left(K, C^-\right) = W\left(K\right) + R\left(K\right)K - G\left(K, C^-\right)$.

3.4. Adding Uncertainty

3.4.1. Markov Processes

In this part, we want to focus on stochastic economies where there is a productivity shock affecting the economy. The stochastic process for productivity that we are assuming is a first order Markov Process that takes on finite number of values in the set $Z = \{z^1 < \cdots < z^{n_z}\}$. A first order Markov process implies

$$\Pr(z_{t+1} = z^i | h_t) = \Gamma_{ij}, \qquad z_t(h_t) = z^j$$

where h_t is the history of previous shocks. Γ is a Markov matrix with the property that the elements of its columns sum to 1.

Let μ be a probability distribution over initial states, i.e.

$$\sum_i \mu_i = 1$$

and $\mu_i \geq 0 \; \forall i = 1, ..., n_z$.

Next periods the probability distribution can be found by the formula: $\mu' = \Gamma^T \mu$.

If Γ is "nice" then $\exists$ a unique μ^* s.t. $\mu^* = \Gamma^T \mu^*$ and $\mu^* = \lim_{m \to \infty} (\Gamma^T)^m \mu_0$, $\forall \mu_0 \in \Delta^i$.

Γ induces the following probability distribution conditional on z_0 on $h_t = \{z^0, z^1, ..., z^t\}$:

$\Pi(\{z^0, z_1\}) = \Gamma_i$ for $z^0 = z_i$.

$\Pi(\{z^0, z_1, z_2\}) = \Gamma^T \Gamma_i$ for $z^0 = z_i$.

Then, $\Pi(h_t)$ is the probability of history h_t conditional on z^0. The expected value of z' is $\sum_{z'} \Gamma_{zz'} z'$ and $\sum_{z'} \Gamma_{zz'} = 1$.

3.4.2. Problem of the Social Planner

Let productivity affect the production function in a multiplicative fashion; i.e. technology is $zF(K, N)$, where z is the shock that follows a Markov chain on a finite state-space. The problem of the social planner problem (SPP) in sequence form is

$$\begin{aligned} \max_{\{c_t(z^t), k_{t+1}(z^t)\} \in X(z^t)} \quad & \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \pi(z^t) u(c_t(z^t)) \\ s.t. \quad & c_t(z^t) + k_{t+1}(z^t) = z_t F(k_t(z^{t-1}), 1), \end{aligned}$$

where z_t is the realization of shock in period t, and z^t is the history of shocks up to (and including) time t.

Therefore, we can formulate the stochastic SPP in a recursive fashion as

$$
\begin{aligned}
V(z_i, K) = \max_{c,K'} \quad & \left\{ u(c) + \beta \sum_j \Gamma_{ji} V(z_j, K') \right\} \\
s.t. \quad & c + K' = z_i F(K, 1),
\end{aligned}
$$

where Γ is the Markov transition matrix. The solution to this problem gives us a policy function of the form $K' = G(z, K)$.

In a decentralized economy, Arrow-Debreu equilibrium can be defined by:

$$
\begin{aligned}
\max_{\{c_t(z^t), k_{t+1}(z^t), x_{1t}(z^t), x_{2t}(z^t), x_{3t}(z^t)\} \in X(z^t)} \quad & \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \pi(z^t) u(c_t(z^t)) \\
s.t. \quad & \sum_{t=0}^{\infty} \sum_{z^t} p_t(z^t) x_t(z^t) \leq 0,
\end{aligned}
$$

where $X(z^t)$ is the consumption feasibility set after history z^t has occurred. Note that we are assuming the markets are dynamically complete; i.e. there is complete set of securities for every possible history that can appear.

By the same procedure as before, SME can be written in the following way:

$$
\begin{aligned}
\max_{\{c_t(z^t), b_{t+1}(z^t, z_{t+1}), k_{t+1}(h_t)\}} \quad & \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \pi(z^t) u(c_t(z^t)) \\
s.t. \quad & c_t(z^t) + k_{t+1}(z^t) + \sum_{z_{t+1}} b_{t+1}(z^t, z_{t+1}) q_t(z^t, z_{t+1}) \\
& = k_t(z^{t-1}) R_t(z^t) + w_t(z^t) + b_t(z^{t-1}, z_t) \\
& b_{t+1}\left(z^t, z_{t+1}\right) \geq -B.
\end{aligned}
$$

To replicate the AD equilibrium, here, we have introduced Arrow securities to allow agents to trade with each other against possible future shocks.

However, in equilibrium and when there is no heterogeneity, there will be no trade. Moreover, we have two ways of delivering the goods specified in an Arrow security contract: after production and before production. In an after production setting, the goods will be delivered after production takes place and can only be consumed or saved for the next period. This is the above setting. It is also possible to allow the consumer to rent the Arrow security income as capital to firms, which will be the before production setting.

An important condition which must hold true in the *before production setting* is the no-arbitrage condition:

$$\sum_{z_{t+1}} q_t(z^t, z_{t+1}) = 1$$

Describe the AD problem, in particular the consumption possibility set X and the production set Y.

Every equilibrium achieved in AD settings can also be achieved in a SM setting, by the relation where

$$q_t(z^t, z_{t+1}) = p_{1t+1}(z^t, z_{t+1})/p_{1t}(z^t),$$

$$R_t(z^t) = -p_{2t}(z^t)/p_{1t}(z^t),$$

and

$$w_t(z^t) = -p_{3t}(z^t)/p_{1t}(z^t).$$

Check that from the FOC's, the same allocations result in the two settings.

The problem above state contingent goods are delivered in terms of consumption goods. Instead of this assume they are delivered in terms of capital goods. Show that the same allocation would be achieved in both settings.

3.4.3. Recursive Competitive Equilibrium

Assume that households can accumulate state contingent capital, as in the sequential market case. We can write a household's problem in recursive form as:

$$\begin{aligned} V(K, z, a) = \max_{c, a'(z')} \quad & \left\{ u(c) + \beta \sum_{z'} \Gamma_{zz'} V(K', z', a'(z')) \right\} \\ s.t. \quad & c + \sum_{z'} a'(z') q_{z'}(K, z) = w(K, z) + aR(K, z) \\ & K' = G(K, z). \end{aligned}$$

Write the first order conditions for this problem, given prices and the law of motion for aggregate capital.

Solving this problem gives policy function $g(K, z, a, z')$. So, a RCE in this case is a collection of functions V, g, G, w, and R, so that

1. given G, w, and R, V solves household's functional equation, with g as the associated policy function,
2. $g(K, z, K, z') = G(K, z)$, for all z',
3. $w(K, z) = zF_n(K, 1)$ and $R(K, z) = zF_k(K, 1)$,
4. and $\sum_{z'} q_{z'}(K, z, z') = 1$.

The last condition is known as the no-arbitrage condition (recall that we had this equation in the case of sequential markets as well). To see why this is a necessary equation in the equilibrium, note that an agent can either save in the form of capital, or Arrow securities. However, these two choices must cost the same. This implies Condition 4 above.

Note that in a sequence version of the household problem in SME, in order for households not to achieve infinite consumption, we need a no-Ponzi condition; a condition that prevents Ponzi schemes is

$$\lim_{t\to\infty} \frac{a_t}{\prod_{s=0}^{t} R_s} < \infty.$$

This is the weakest condition that imposes no restrictions on the first order conditions of the household's problem. It is harder to come up with its analogue for the recursive case. One possibility is to assume that a' lies in a compact set $\mathcal{A}$, or a set that is bounded from below.[3]

[3] We must specify $\mathcal{A}$ such that the borrowing constraint implicit in $\mathcal{A}$ is never binding.

3.5. Economy with Government Expenditures

3.5.1. Lump Sum Tax

The government levies each period T units of goods in a lump sum way and spends it in a public good, say fireworks. Assume consumers do not care about medals. The household's problem becomes:

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad & \{u(c) + \beta V(K',a')\} \\ s.t. \quad & c + a' + T = w(K) + aR(K) \\ & K' = G(K). \end{aligned}$$

A solution of this problem are functions $g_a^*(K,a;G,M,T)$ and $\Omega(K,a;G)$ and the equilibrium can be characterized by $G^*(K,M,T) = g_a^*(K,K;G^*,M,T)$ and $M^* = T$ (the government budget constraint is balanced period by period). We will write a complete definition of equilibrium for a version with government debt (below).

3.5.2. Income Tax

We have an economy in which the government levies taxes in order to purchase medals. Medals are goods which provide utility to the consumers (for this example).

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad & \{u(c,M) + \beta V(K',a')\} \\ s.t. \quad & c + a' + T = (1-\tau)[w(K) + aR(K)] \\ & K' = G(K), \end{aligned}$$

given $M = \tau[w(K) + aR(K)]$. Now the First Welfare Theorem is no longer applicable; the CE will not be Pareto optimal anymore (if $\tau > 0$ there will be a wedge, and the efficiency conditions will not be satisfied).

3.5.3. Taxes and Debt

Assume that government can issue debt and use taxes to finance its expenditures, and these expenditures do affect the utility.

A government policy consists of taxes, spending (fireworks) as well as bond issuance. When the aggregate states are K and B, as you will see why, then a government policy (in a recursive world!) is

$$\tau(K,B),\ F(K,B)\ \text{and}\ B'(K,B).$$

For now, we shall assume these values are chosen so that the equilibrium exists. In this environment, debt issued is relevant for the household because it permits him to correctly infer the amount of taxes. Therefore the household needs to form expectations about the future level of debt from the government.

The government budget constraint now satisfies (with taxes on asset income):

$$P(K,B)+R(K)\cdot B=\tau(K,B)\,w(K)+B'(K,B)$$

Where also $B'(K,B)=G^B(\cdot)$. Notice that the household does not care about the composition of his portfolio as long as assets have the same rate of return which is true because of the no arbitrage condition. Therefore, the problem of a household with assets equal to a is given by:

$$\begin{aligned}
V(K,a)=\max_{c,a'}\quad & \{u(c,P(K,B))+\beta V(K',a')\}\\
s.t.\quad & c+a'=w(K,B)+aR(K,B)[1-\tau(K,B)]\\
& K'=G(K,B)\\
& B'=H(K,B).
\end{aligned}$$

Let $g(K,B,a)$ be the policy function associated with this problem. Then, we can define a RCE as follows.

Definition 4 *A* Rational Expectations Recursive Competitive Equilibrium, *given policies $P(K,B)$ and $\tau(K,B)$ is a set of functions V, g, G, H, w, and R, such that*

1. *V and g solve household's functional equation,*
2. *$w(K)=F_2(K,1)$ and $R(K)=F_1(K,1)$,*
3. *$g(K,B,K+B)=G(K,B)+H(K,B)$,*
4. *Government's budget constraint is satisfied*

$$H(K,B)=R(K,B)\,B+P(K,B)-\tau(K,B)\,R(K,B)\,K,$$

5. *and, government debt is bounded; i.e. there exists some $\bar{B}$ so that for all $K \in \left[0, \widetilde{k}\right)$, $B(K, B) \leq \bar{B}$.*

4. Adding Heterogeneity

In the previous section we looked at situations in which recursive competitive equilibria (RCEa) were useful. In particular these were situations in which the welfare theorems failed and so we could not use the standard dynamic programming techniques learned earlier. In this section we look at another way in which RCEa are helpful, in models with heterogeneous agents.

4.1. Heterogeneity in Wealth

First, lets consider a model in which we have two types of households that differ only in the amount of wealth they own. Say there are two types of agents, labeled type 1 and 2, of equal measure of 1/2. Agents are identical other than their initial wealth position and there is no uncertainty in the model. The problem of an agent with wealth a is given by

$$\begin{aligned} V\left(K^1, K^2, a\right) = \max_{c, a'} \quad & \left\{u(c) + \beta V\left(K'^1, K'^1, a'\right)\right\} \\ s.t. \quad & c + a' = R\left(\frac{K^1 + K^2}{2}\right) a + W\left(\frac{K^1 + K^2}{2}\right) \\ & K'^i = G^i\left(K^1, K^2\right), \quad i = 1, 2. \end{aligned}$$

Note that (in general) the decision rules of the two types of agents are not linear (even though they might be *almost* linear); therefore, we cannot add the two states, K^1 and K^2, to write the problem with one aggregate state, in the recursive form.

Definition 5 *A* Rational Expectations Recursive Competitive Equilibrium *is a set of functions V, g, R, w, G^1, and G^2, so that:*

1. *V solves the household's functional equation, with g as the associated policy function,*

2. *w and R are the marginal products of labor and capital, respectively (watch out for arguments!),*

3. *representative agent conditions are satisfied; i.e.*

$$g(K^1, K^2, K^1) = G^1(K^1, K^2),$$

and

$$g(K^1, K^2, K^2) = G^2(K^1, K^2).$$

Note that $G^1(K^1, K^2) = G^2(K^2, K^1)$ (why?).

This is a variation of the simple neoclassical growth model; what does the growth model say about inequality?

In the steady state of a neoclassical growth model, Euler equations for the two types simplify to

$$u'\left(c^1\right) = \beta R u'\left(c^1\right), \text{ and } u'\left(c^2\right) = \beta R u'\left(c^2\right).$$

Therefore, we must have $\beta R = 1$, where

$$R = F_K\left(\frac{K^1 + K^2}{2}, 1\right).$$

Finally, by the household's budget constraint, we must have:

$$k^i R + W = c^i + k^i,$$

where $k^i = K^i$, by representative agent's condition. Therefore, we have three equation, with four unknowns (k^i and c^i's). This means, this theory is silent about the distribution of wealth in the steady state!

4.2. Heterogeneity in Skills

Now, consider a slightly different economy where type i has labor skill ϵ_i. Measures of agents' types, μ^1 and μ^2, satisfy $\mu^1\epsilon_1 + \mu^2\epsilon_2 = 1$ (below we will consider the case where $\mu^1 = \mu^2 = 1/2$).

The question we have to ask ourselves is, would the value functions of two types remain to be the same, as in the previous subsection? The answer turns out to be no!

The problem of the household $i \in \{1,2\}$ can be written as follows:

$$
\begin{aligned}
V^i\left(K^1, K^2, a\right) = \max_{c,a'} \quad & \left\{u(c) + \beta V^i\left(K'^1, K'^1, a'\right)\right\} \\
s.t. \quad & c + a' = R\left(\frac{K^1+K^2}{2}\right) a + W\left(\frac{K^1+K^2}{2}\right)\epsilon_i \\
& K'^j = G^j\left(K^1, K^2\right), \quad j = 1,2.
\end{aligned}
$$

Notice that we have indexed the value function by the agent's type; the reason is that the marginal product of the labor supplied by each of these types is different.

We can rewrite this problem as

$$
\begin{aligned}
V^i(K, \lambda, a) = \max_{c,a'} \quad & \left\{u(c) + \beta V^i(K', \lambda', a')\right\} \\
s.t. \quad & c + a' = R(K)\, a + W(K)\, \epsilon_i \\
& K = G(K, \lambda) \\
& \lambda' = H(K, \lambda),
\end{aligned}
$$

where K is the total capital in this economy, and λ is the share of one type in this wealth (e.g. type 1).

Then, if g^i is the policy function of type i, in the equilibrium, we must have:

$$
G(K, \lambda) = g^1(K, \lambda, \lambda K) + g^2(K, \lambda, (1-\lambda) K),
$$

and

$$
H(K, \lambda) = g^1(K, \lambda, \lambda K) / G(K, \lambda).
$$

4.3. An International Economy Model

In an international economy model the specifications which determine the definition of country is an important one; we can introduce the idea of different locations or geography; countries can be victims of different policies; trade across countries maybe more difficult due to different restrictions.

Here we will see a model with two countries, A and B, such that labor is not mobile between the countries, but with perfect capital markets. Two countries may have different technologies, $F^A(K_A, 1)$ and $F^B(K_B, 1)$; therefore, the resource constraint in this world would be

$$C^A + C^B + K'^A + K'^B = F^A\left(K^A, 1\right) + F^B\left(K^B, 1\right).$$

(Therefore, while the product of the world economy can move freely between the two countries, once installed, it has to be used in that country.)

The first question to ask, as usual, is *what are the appropriate states in this world?* As it is apparent from the resource constraint and production functions, we need the capital in each country. But, also, we need to know who owns this capital. Therefore, we need an additional variable as the aggregate state; we can choose λ, the share of country A in total wealth. But, why not the share of this country in the total capital in each of these countries? Since, at the point of saving, the capital in the two countries is perfect substitute. In other words

$$F_k^A\left(K^A, 1\right) = F_k^B\left(K^B, 1\right) = R^A = R^B,$$

where we have incorporated the depreciation into the production function. This follows from a no arbitrage argument.

As a result, country i's problem can be written as:

$$\begin{aligned} V^i\left(K^A, K^A, \lambda, a\right) = \max_{c,a'} \quad & \left\{u\left(c\right) + \beta V^i\left(K'^A, K'^B, \lambda', a'\right)\right\} \\ s.t. \quad & c + a' = R\left(K^i\right)a + w^i\left(K^i\right) \\ & K'^j = G^j\left(K^A, K^B, \lambda\right), \quad j = A, B \\ & \lambda' = \Gamma\left(K^A, K^B, \lambda\right). \end{aligned}$$

Notice that, given K^A, we know that, from our earlier no-arbitrage argument, that

$$R\left(K^A\right) = F_k^A\left(K^A, 1\right) = F_k^B\left(K^B, 1\right).$$

Therefore, we can infer K^B, and $R\left(K^B\right)$. As a result, we don't need to keep track of K^B any longer.

Moreover, the wage rate is given by

$$w^A\left(K^A\right) = F_n^A\left(K^A, 1\right).$$

Thus, we may rewrite the problem as

$$\begin{aligned}
V^{i}(K,\lambda,a) = \max_{c,a'} \quad & \left\{u(c)+\beta V^{i}(K',\lambda',a')\right\} \\
s.t. \quad & c+a' = R(K)\,a + w^{i}(K) \\
& K' = G(K,\lambda) \\
& \lambda' = \Gamma(K,\lambda),
\end{aligned}$$

where K is the total capital in the world, and R and w^i can be derived as

$$\begin{aligned}
R(K) &= F_k^A(\alpha K,1), \\
w^A(K) &= F_n^A(\alpha K,1), \\
w^B(K) &= F_n^B((1-\alpha)K,1),
\end{aligned}$$

and α is the solution to the following equation:

$$F_k^A(\alpha K,1) = F_k^B((1-\alpha)K,1).$$

Now, we can define the equilibrium as

Definition 6 *A Recursive Competitive Equilibrium for the (world's) economy is a set of functions, V^i, g^i, G^i, and w^i, for $i \in \{A,B\}$, and R, and H, such that the following conditions hold:*

1. *V^i and g^i solve the household's problem in country i ($i \in \{A,B\}$),*
2. *$G(K,\lambda) = g^A(K,\lambda,\lambda K) + g^B(K,\lambda,(1-\lambda)K)$,*
3. *$\Gamma(K,\lambda) = g^A(K,\lambda,\lambda K)/G(K,\lambda)$,*
4. *w^i is equated to the marginal products of labor in each country,*
5. *and, R is equal to the marginal product of capital (in both countries).*

5. Some Other Examples

5.1. A Few Popular Utility Functions

Consider the following three utility forms:

1. $u(c, c^-)$: this function is called *habit formation* utility function; utility is increasing in consumption today, but, decreasing in the deviations from past consumption (e.g. $u(c, c^-) = v(c) - (c - c^-)^2$). In this case, the aggregate states in a standard growth model are K and C^-, and individual states are a and c^-. Is the equilibrium optimum in this case?

2. $u(c, C^-)$; this form is called *catching up with Jones*; there is an externality from the aggregate consumption to the payoff of the agents. Intuitively, in this case, agents care about what their neighbors consume. Aggregate states in this case are K and C^-. But, c^- is no longer an individual state.

3. $u(c, C)$: the last function is called keeping up with Jones. Here, the aggregate state is K; C is no longer a pre-determined variable to appear as a state.

5.2. An Economy with Capital and Land

Consider an economy with with capital and land but without labor; a firm in this economy buys and installs capital. They also own one unit of land, that they use in production, according to the production function $F(K, L)$. In other words, a firm is a "chunk of land of are one", in which firm installs its capital. Share of these firms are traded in a stock market.

A household's problem in this economy is given by:

$$\begin{aligned} V(K, a) = \max_{c, a'} \quad & \{u(c) + \beta V(K', a')\} \\ s.t. \quad & c + a' = R(K). \\ & K' = G(K) \end{aligned}$$

On the other hand, a firm's problem is

$$\Omega(K,k) = \max_{k'} \quad \{F(k,1) - k' + q(K')\,\Omega(K',k')\}$$
$$s.t. \quad K' = G(K).$$

Ω here is the *value of the firm,* measured in units of output, today. Therefore, the value of the firm, tomorrow, must be discounted into units of output today. This is done by a discount factor $q(K)$.

A Recursive Competitive Equilibrium consists of functions, V, Ω, g, h, q, G, and R, so that:

1. V and g solve household's problem,
2. Ω and h solve firm's problem,
3. $G(K) = h(K,K)$, and,
4. $q(G(K))\,\Omega(G(K), G(K)) = g(K, \Omega(K,K))$.

One condition is missing in the definition of the RCE above. Find it! *[Hint: it relates the rate of return on the household's assets to the discount rate of firm's value.]*

6. Asset Pricing: Lucas Tree Model

We now turn to the simplest of all models in term of allocations as they are completely exogenous, *the Lucas tree model*. We want to characterize the properties of prices that are capable of inducing households to consume the endowment.

6.1. The Lucas Tree with Random Endowments

Consider an economy in which the only asset is a tree that gives fruit. The agents problem is

$$V(z,s) = \max_{c,s'} \quad \left\{u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z',s')\right\}$$
$$s.t. \quad c + q(z)\,s' = s\,[q(z) + d(z)],$$

where $q(z)$ is the price of the shares (to the tree), in state z, and $d(z)$ is the dividends associated with state z.

Definition 7 *A Rational Expectations Recursive Competitive Equilibrium is a set of functions, V, g, d, and q, such that*

1. *V and g solves the household's problem,*
2. *$d(z) = z$, and,*
3. *$g(z, 1) = 1$, for all z.*

To explore the problem more, note that the first order conditions for the household's problem imply:

$$u_c(c(z,1)) = \beta \sum_{z'} \Gamma_{zz'} \left[\frac{q(z') + d(z')}{q(z)} \right] u_c(c(z',1)).$$

As a result, if we let $u_c(z)\, u_c(c(z,1))$, we get:

$$q(z)\, u_c(z) = \beta \sum_{z'} \Gamma_{zz'} u_c(z') \left[q(z') + z' \right].$$

Derive the Euler equation for household's problem.

Notice that this is just a system of n equations with unknowns $\{q(z_i)\}_{i=1}^{n}$. We can use the power of matrix algebra to solve it. To do so, let:

$$\mathbf{q} \begin{bmatrix} q(z_1) \\ \vdots \\ q(z_n) \end{bmatrix},$$

and

$$\mathbf{u}_c \begin{bmatrix} u_c(z_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_c(z_n) \end{bmatrix}.$$

Then

$$\mathbf{u}_c . \mathbf{q} = \begin{bmatrix} q(z_1)\, u_c(z_1) \\ \vdots \\ q(z_n)\, u_c(z_n) \end{bmatrix},$$

and

$$\mathbf{u}_c.\mathbf{z} = \begin{bmatrix} z_1 u_c(z_1) \\ \vdots \\ z_n u_c(z_n) \end{bmatrix},$$

Now, rewrite the system above as

$$\mathbf{u}_c\mathbf{q} = \beta\Gamma\mathbf{u}_c\mathbf{z} + \beta\Gamma\mathbf{u}_c\mathbf{q},$$

where Γ is the transition matrix for z, as before. Hence, the price for the shares is given by

$$\mathbf{u}_c\mathbf{q} = (\mathbf{I} - \beta\Gamma)^{-1}\beta\Gamma\mathbf{u}_c\mathbf{z},$$

or

$$\mathbf{q} = \begin{bmatrix} u_c(z_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_c(z_n) \end{bmatrix}^{-1} (\mathbf{I} - \beta\Gamma)^{-1}\beta\Gamma\mathbf{u}_c\mathbf{z}.$$

6.2. Taste Shocks

Consider an economy in which the only asset is a tree that gives fruits. The fruit is constant over time (normalized to 1) but the agent is subject to preference shocks for the fruit each period, $\theta \in \Theta$. The agent's problem in this economy is

$$\begin{aligned} V(\theta, s) = \max_{c,s'} & \quad \left\{ \theta u(c) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V(\theta', s') \right\} \\ s.t. & \quad c + q(\theta) s' = s\left[q(\theta) + d(\theta)\right]. \end{aligned}$$

The equilibrium is defined as before; the only difference is that, now, we must have $d(\theta) = 1$. What does it mean that the output of the economy is constant, and fixed at one, but, the tastes in this output changes? In this settings, the function of the price is to convince agents keep their consumption constant.

All the analysis follows through, once we write the FOC's characterizing price, $q(\theta)$, and state contingent prices $p(\theta, \theta')$.

6.3. Asset Pricing

Consider our simple model of Lucas tree with fluctuating output; what is the definition of an asset in this economy? It is "a claim to a chunk of fruit, sometime in the future".

If an asset, a, promises an amount of fruit equal to $a_t(z^t)$ after history $z^t = (z_0, z_1, \dots, z_t)$ of shocks, after a set of (possible) histories in H, the price of such an entitlement in date $t = 0$ is given by:

$$p(a) = \sum_t \sum_{z^t \in H} q_t^0(z^t)\, a_t(z^t),$$

where $q_t^0(z^t)$ is the price of one unit of fruit after history z^t, in today's "dollars"; this follows from a no-arbitrage argument. If we have the sate of date $t = 0$ prices, $\{q_t\}$, as functions of histories, we can *replicate any possible asset by a set of state-contingent claims*, and use this formula to price that asset.

To see how we can find prices at date $t = 0$, consider a world in which the agent wants to solve

$$\begin{aligned} \max_{c_t(z^t)} \quad & \left\{ \sum_{t=0}^{\infty} \beta^t \sum_{z^t} \pi_t(z^t)\, u(c_t(z^t)) \right\} \\ s.t. \quad & \sum_{t=0}^{\infty} \sum_{z^t} q_t^0(z^t)\, c_t(z^t) \leq \sum_{t=0}^{\infty} \sum_{h^t} q_t(z^t)\, z_t. \end{aligned}$$

This is the familiar Arrow-Debreu market structure, where the household owns a tree, and the tree can yield $z \in Z$ amount of fruit in each period. The first order condition for this problem implies:

$$q_t^0(z^t) = \beta^t \pi_t(z^t) \frac{u_c(z_t)}{u_c(z_0)}.$$

This enables us to price the good in each history of the world, and price any asset accordingly.

How can we obtain the prices in the recursive framework of the previous subsections?

What happens if we add state contingent shares into our recursive model? Then the agent's problem becomes:

$$V(z,s,b) = \max_{c,s',b'(z')} \quad \left\{ u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z', s', b'(z')) \right\}$$
$$s.t. \quad c + q(z) s' + \sum_{z'} p(z, q') b'(z') = s[q(z) + z] + b.$$

A characterization of p can be written as:

$$p(z, z') u_c(z) = \beta \Gamma_{zz'} u_c(z').$$

We can price *all types* of securities using p and q in this economy.

To see how we can price an asset, consider *the option to sell tomorrow* at price P, if today's shock is z, as an example; the price of such an asset today is

$$\hat{q}(z, P) = \sum_{z'} \max\{P - q(z'), 0\} p(z, z').$$

The option to sell at price P either tomorrow or the day after tomorrow is priced as:

$$\widetilde{q}(z, P) = \sum_{z'} \max\{P - q(z'), \hat{q}(z', P)\} p(z, z').$$

Finally, note that $R(z) = \left[\sum_{z'} p(z, z')\right]^{-1}$ is the risk free rate, given today's shock is z.

7. Endogenous Productivity in a Product Search Model

Let's model the situation where households need to find the fruit before consuming it;[4] assume that households have to find the tree in order to consume the fruit. Finding trees is characterized by a constant returns to scale (increasing) matching function $M(T, D)$,[5] where T is the *number of trees* and D is the *shopping effort,* exerted by households when searching. Thus, the probability that a tree finds a shopper is $M(T, D)/T$; total number of matches, divided by the number of trees. And, the probability that a unit of shopping effort finds a tree is $M(T, D)/D$.

We further assume that M takes the form $D^{\varphi} T^{1-\varphi}$, and denote the probability of finding a tree by

[4] Think of fields in *Tleghn*, full of mushrooms, that are owned by firms; agents have to buy the mushrooms. In addition, they have to search for them as well!

[5] What does the fact that M is constant returns to scale imply?

$\Psi_d(Q)\,Q^{1-\varphi}$, where $1/QD/T$ is the ratio of shoppers per trees, capturing *the market tightness*; the more the number of people searching, the smaller the probability of finding a tree. Then, $\Psi_t(Q)\,Q^{-\varphi}$. Note that, in this economy, the number of trees is constant, and equal to one.[6]

The household's problem can be written as:

$$V(\theta, s) = \max_{c,d,s'} \quad \left\{ u(c, d, \theta) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V(\theta', s') \right\} \tag{1}$$

$$s.t. \quad c = d\Psi_d(Q(\theta)) \tag{2}$$

$$c + P(\theta)\, s' = P(\theta)\,[s(1 + R(\theta))], \tag{3}$$

where P is the price of tree relative to that of consumption, and R is the dividend income (in units of tree). d is the amount of search the individual household exerts to acquire fruit.

If we substitute the constraints in the objective, we get the simplified problem as

$$V(\theta, s) = \max_{d} \{ u(d\Psi_d(Q(\theta)), d, \theta) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V\left(\theta', s[1 + R(\theta)] - \frac{1}{P(\theta)} d\Psi_d(Q(\theta))\right) \right\}. \tag{4}$$

The first order condition for this problem is

$$u_c(d\Psi_d(Q(\theta)), d, \theta) + \frac{u_d(d\Psi_d(Q(\theta)), d, \theta)}{\Psi_d(Q(\theta))} = \beta \sum_{\theta'} \Gamma_{\theta\theta'} V_s\left(\theta', s[1 + R(\theta)] - \frac{1}{P(\theta)} d\Psi_d(Q(\theta))\right) \frac{1}{P(\theta)}. \tag{5}$$

To get rid of V_s, look at the initial household problem, (1). Let the multiplier on the budget constraint

[6] It is easy to find the statements for Ψ_t and Ψ_d, given the Cobb-Douglas matching function;

$$\Psi_d(Q) = \frac{D^{\varphi} T^{1-\varphi}}{D} = \left(\frac{T}{D}\right)^{1-\varphi} = Q^{1-\varphi},$$

$$\Psi_t(Q) = \frac{D^{\varphi} T^{1-\varphi}}{T} = \left(\frac{T}{D}\right)^{-\varphi} = Q^{-\varphi}.$$

The question is, is Cobb-Douglas an appropriate choice for the matching function, or its choice is a matter of simplicity?

be λ. Applying the envelope theorem and writing the first order condition (with respect to c) we get:

$$V_s(\theta, s) = \lambda P(\theta)(1 + R(\theta)),$$
$$u_c(c, d, \theta) + \frac{u_d(c, d, \theta)}{\Psi_d(Q(\theta))} = \lambda,$$

which implies:

$$V_s(\theta, s) = P(\theta)(1 + R(\theta))\left(u_c(c, d, \theta) + \frac{u_d(c, d, \theta)}{\Psi_d(Q(\theta))}\right).$$

Thus we get the Euler equation for household's problem:

$$\begin{aligned} & u_c(d\Psi_d(Q(\theta)), d, \theta) + \frac{u_d(d\Psi_d(Q(\theta)), d, \theta)}{\Psi_d(Q(\theta))} \\ & = \beta \sum_{\theta'} \Gamma_{\theta\theta'} \frac{P(\theta')[1 + R(\theta')]}{P(\theta)} \\ & \qquad \times \left(u_c(d\Psi_d(Q(\theta')), d', \theta') + \frac{u_d(d\Psi_d(Q(\theta')), d', \theta')}{\Psi_d(Q(\theta'))}\right). \end{aligned} \tag{6}$$

This equation has the usual interpretation of an Euler equation; the left hand side is the gains from the consumption of one more unit of fruit today (the marginal utility of consumption plus the marginal disutility of the search required to obtain that one more unit). The right hand side is the marginal utility of saving, to consume the fruit tomorrow (discounted to today's utility terms).

We still need another functional equation to find an equilibrium; note that we have one equation, whereas two variables, P and Q, have to be determined (other objects, C and R, are known functions of P and Q). We now turn to various ways of doing so.

7.1. Competitive Search

Competitive search is a particular search protocol of what is called non-random (or directed) search. To understand this protocol, consider a world consisted of a large number of islands. Each island has a sign that reads two number, W and Q. W is the wage rate on the island, and Q is a measure of market tightness in that island, or the number of workers on the island divided by the number of job opportunities. Both, workers and firms have to decide to go to one island. In an island with higher wage, the worker might be happier, conditioned on finding a job. However, the probability of finding a job might be low on the island, depending on the tightness of the labor market on that island. The same story holds for the job owners, who are searching to hire workers.

In our economy, both firms and workers search for specific markets *indexed* by price P and market tightness Q. Agents can go to any such market provided that is operational. From the point of view of the firm, a pair would be operational if it guarantees enough utility to the household (an amount determined in equilibrium). First, solve the problem of a household given P and Q, and then let the firm choose which particular pair of P and Q gives earns the highest profit. Competitive search is magic. It does not presuppose a particular pricing protocol (wage posting, bargaining) that other search protocols need.

We start by defining a useful object Ω, that tells us the value for a household of facing arbitrary tightness Q and price P today, given V. In particular, Ω is defined as

$$\begin{aligned} \Omega(\theta, s, P, Q) = \max_{c,d,s'} \quad & \left\{ u(c, d, \theta) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V(\theta', s') \right\} \\ s.t. \quad & c = d\Psi_d(Q) \\ & c + Ps' = P\left[s\left(1 + R(\theta)\right)\right]. \end{aligned} \tag{7}$$

Note that Ω is not an equilibrium object.

Of course, from the perspective of a tree, the best is to have maximum price (or, minimum P, since $1/P$ is the price of the fruit in terms of shares), and maximum number of potential buyers (or, highest market tightness). The question is, would this market attract any buyers? The answer is no, as long as there are other trees that offer slightly lower prices in a market that is just as tight.[7]

To formalize this idea, we let $\overline{V}$ denote the value for households, shopping in the most attractive market, yet to be determined. To attract households, trees have to offer combinations of prices and market tightness that provide at least $\overline{V}$. The problem of a tree, which is in fact a static problem, is to find a combination of price and tightness to maximize its profit while satisfying the participation

[7] In other words, such a combination of price and market tightness would not be an equilibrium. To see why, consider a game in which all the trees are offering maximum price in a market with highest tightness. Then, deviating to a new island in which the price is just by a small amount lower would attract all the buyers. This would be a profitable deviation for a tree!

constraint of households:

$$\max_{P,Q} \quad \Pi(P,Q) = \frac{1}{P}\Psi_t(Q) \tag{8}$$

$$s.t. \quad \bar{V} \leq \Omega(\theta, s, P, Q), \tag{9}$$

where Ω is the solution to (7), evaluated at households' optimal shopping effort d^* in response to (θ, s, P, Q),

$$\Omega(\theta, s, P, Q) = u\left(d^*\Psi_d(Q), d^*, \theta\right) + \beta\sum_{\theta'}\Gamma_{\theta\theta'}V\left(\theta', s\left[1+R(\theta)\right] - \frac{1}{P}d^*\Psi_d(Q)\right).$$

Therefore, using $\Psi_t(Q) = Q^{-\varphi}$ and $\Psi_d(Q) = Q^{1-\varphi}$, and the fact that $s = 1$ in equilibrium, we can rewrite the problem of the tree as:

$$\max_{P,Q} \quad \Pi(P,Q) = \frac{1}{P}Q^{-\varphi} \tag{10}$$

$$s.t. \quad \bar{V} \leq u\left(d^*Q^{1-\varphi}, d^*, \theta\right) + \beta\sum_{\theta'}\Gamma_{\theta\theta'}V\left(\theta', 1 + R(\theta) - \frac{1}{P}d^*Q^{1-\varphi}\right) \tag{11}$$

Note that, by Envelope Theorem:[8]

$$\frac{\partial\Omega(\theta, s, P, Q)}{\partial Q} = d^*\left[u_c\left(d^*\Psi_d(Q), d^*, \theta\right) - \frac{1}{P}\beta\sum_{\theta'}\Gamma_{\theta\theta'}V\left(\theta', s\left[1+R(\theta)\right] - \frac{1}{P}d^*\Psi_d(Q)\right)\right]\Psi'_d(Q).$$

So, if we let the multiplier of the household's participation constraint ((11)) be γ, the first order

[8] To see why this is the case, take the derivative of Ω with respect to Q, to get:

$$\begin{aligned}\frac{\partial\Omega}{\partial Q} &= \left[u_c\Psi_d + u_d - \frac{1}{P}\Psi_d\beta\sum_{\theta'}\Gamma_{\theta\theta'}V_s(\theta')\right]\frac{\partial d^*}{\partial Q} + d^*(1-\varphi)Q^{-\varphi}\left[u_c - \frac{1}{P}\beta\sum_{\theta'}\Gamma_{\theta\theta'}V_s(\theta')\right]\\ &= \Psi_d\left[u_c + \frac{u_d}{\Psi_d} - \frac{1}{P}\beta\sum_{\theta'}\Gamma_{\theta\theta'}V_s(\theta')\right]\frac{\partial d^*}{\partial Q} + d^*(1-\varphi)Q^{-\varphi}\left[u_c - \frac{1}{P}\beta\sum_{\theta'}\Gamma_{\theta\theta'}V_s(\theta')\right].\end{aligned}$$

By Equation (5), the first term in this equality is zero. Therefore:

$$\frac{\partial\Omega}{\partial Q} = d^*(1-\varphi)Q^{-\varphi}\left[u_c - \frac{1}{P}\beta\sum_{\theta'}\Gamma_{\theta\theta'}V_s(\theta')\right].$$

condition for Problem (10) becomes:

$$-\frac{1}{P}\varphi Q^{-\varphi-1}+\gamma\left[d^{*}\left(1-\varphi\right)Q^{-\varphi}u_{c}\left(d^{*}Q^{1-\varphi},d^{*},\theta\right)\right.$$
$$\left.-\frac{1}{P}d^{*}\left(1-\varphi\right)Q^{-\varphi}\beta\sum_{\theta'}\Gamma_{\theta\theta'}V_{s}\left(\theta',1+R\left(\theta\right)-\frac{1}{P}d^{*}Q^{1-\varphi}\right)\right]=0.$$

We can rearrange this equation as:

$$\frac{1}{P}\varphi Q^{-\varphi-1}=\gamma d^{*}\left(1-\varphi\right)Q^{-\varphi}\left[u_{c}\left(d^{*}Q^{1-\varphi},d^{*},\theta\right)\right.$$
$$\left.-\frac{1}{P}\beta\sum_{\theta'}\Gamma_{\theta\theta'}V_{s}\left(\theta',1+R\left(\theta\right)-\frac{1}{P}d^{*}Q^{1-\varphi}\right)\right]=0,$$

which can be simplified as:

$$\frac{1}{P}=\gamma\frac{(1-\varphi)}{\varphi}Qd^{*}\left[u_{c}\left(d^{*}\Psi_{d}(Q),d^{*},\theta\right)\right.$$
$$\left.-\frac{1}{P}\beta\sum_{\theta'}\Gamma_{\theta\theta'}V_{s}\left(\theta',1+R\left(\theta\right)-\frac{1}{P}d^{*}\Psi_{d}(Q)\right)\right]. \quad (12)$$

On the other hand, by a similar logic, the first order condition for Problem (10) with respect to P is:

$$-\frac{1}{P^{2}}Q^{-\varphi}+\gamma\frac{1}{P^{2}}d^{*}Q^{1-\varphi}\beta\sum_{\theta'}\Gamma_{\theta\theta'}V_{s}\left(\theta',1+R\left(\theta\right)-\frac{1}{P}d^{*}Q^{1-\varphi}\right)=0,$$

that can be rearranged as:

$$\frac{Q^{-\varphi}}{P^{2}}=\gamma\frac{Q^{1-\varphi}}{P^{2}}d^{*}\beta\sum_{\theta'}\Gamma_{\theta\theta'}V_{s}\left(\theta',1+R\left(\theta\right)-\frac{1}{P}d^{*}Q^{1-\varphi}\right).$$

If we multiply both sides by $Q^{\varphi}P^{2}$, we get:

$$1=\gamma Qd^{*}\beta\sum_{\theta'}\Gamma_{\theta\theta'}V_{s}\left(\theta',1+R\left(\theta\right)-\frac{1}{P}d^{*}\Psi_{d}(Q)\right) \quad (13)$$

Combining the two conditions, (12) and (13) to cancel γ, we have:

$$u_{c}\left(d\Psi_{d}\left(Q\left(\theta\right)\right),d,\theta\right)=\frac{1}{1-\varphi}\beta\sum_{\theta'}\Gamma_{\theta\theta'}V_{s}\left(\theta',1+R\left(\theta\right)-\frac{1}{P\left(\theta\right)}d^{*}Q^{1-\varphi}\right)\frac{1}{P\left(\theta\right)}. \quad (14)$$

If we compare Conditions (5) and (14), we observe they are very similar. One difference is the term u_d in (5). The reason that this terms does not appear in (14) is that households are the ones who have to exert search effort to find the trees in this environment.

Definition 8 *An equilibrium with competitive search consists of c, d, s, P, Q, R, and $\bar{V}$, that satisfy:*

1. *households' shopping constraint, (Condition* (2)*),*
2. *budget constraint (Condition* (3)*),*
3. *Euler equation (Equation* (6)*),*
4. *trees' first order condition (Condition* (14)*),*
5. *households' participation constraint (Condition* (9)*), and,*
6. *market clearing conditions, $s = 1$ and $Q = d$.*

At the end, note that we assume dividends, $R(\theta)$, are paid out in units of the tree. So that, in equilibrium, consumption is given by

$$C(\theta) = P(\theta)\, R(\theta).$$

One way of thinking about competitive search is that, instead of having one attribute which is the price, goods have two attributes; price and the difficulty of getting the good. We will give another interpretation to the model we just considered, based on this perspective, in Appendix 13.

This definition is excessively cumbersome, and we can go to the core of the issue by writing the two functional equations that characterize the equilibrium. Note that In equilibrium, we have $c = Q^{-\varphi}$ and $c = PR$. Putting together the household's Euler (**??**) and the tree's FOC (**??**), we have that $\{Q, P\}$ have to solve

$$\begin{aligned} \varphi u_c(Q^{-\varphi}, Q^{-1}, \theta) &= -\frac{u_d(Q^{-\varphi}, Q^{-1}, \theta)}{Q^{1-\varphi}}, && (15) \\ u_c(Q^{-\varphi}, Q^{-1}, \theta) &= \beta \sum_{\theta'} \Gamma_{\theta\theta'} \frac{P(\theta') + Q(\theta')^{-\varphi}}{P} u_c(\theta'). && (16) \end{aligned}$$

7.1.1. A Note on Dividends

In the previous section, we assume that dividends, $R(\theta)$, are paid out in units of the tree. So that, in equilibrium, consumption is given by

$$C(\theta) = P(\theta) R(\theta).$$

Consider the following budget constraint:

$$c + P(\theta) \cdot s' = s(P(\theta) + H(\theta)) \tag{17}$$

where $H(\theta)$ is the dividend paid out in the form of fruit

Equation (**??**) becomes: :

$$\Omega(\theta, s, P, Q) = \max_{d} \quad u\left[d\Psi_d(Q), d, \theta\right] + \beta \sum_{\theta'} \Gamma_{\theta\theta'} \ V\left(\theta', s + \frac{1}{P}(s \cdot H(\theta) - d\Psi_d(Q))\right)$$

Notice that now:

$$\frac{\partial \Omega}{\partial P} = -\left(\beta \sum_{\theta'} \Gamma_{\theta\theta'} \ V\left(\theta', s + \frac{1}{P}(s \cdot H(\theta) - d\Psi_d(Q))\right)\right) \frac{sH(\theta) - d\Psi_d(Q)}{P^2}$$

Following the procedure in section 6.1, we can derive an analogous condition as (15).

7.1.2. Pareto Optimality

One of the fascinating properties of competitive search is that the resulting equilibrium is optimal. To see this, note that, from the point of view of optimality, there are no dynamic considerations in this model; there are no capital accumulation decisions, etc.

Therefore, we may consider a social planner's problem (in each possible state, θ) as:

$$\begin{aligned} \max_{P,Q} \quad & u(C, D, \theta) \\ s.t. \quad & C = D^{\varphi}. \end{aligned}$$

The first order condition for this problem implies

$$\varphi D^{\varphi-1} u_c(D^{\varphi}, D, \theta) + u_d(D^{\varphi}, D, \theta) = 0. \tag{18}$$

On the other hand, if we combine Equations (5) and (14), we get:

$$u_c(d\Psi_d(Q(\theta)), d, \theta) + \frac{u_d(d\Psi_d(Q(\theta)), d, \theta)}{\Psi_d(Q(\theta))} = (1-\varphi)\, u_c(d\Psi_d(Q(\theta)), d, \theta).$$

If we substitute from the equilibrium conditions for $Q = 1/D$ and $\Psi_d = Q^{1-\varphi}$, we can write this condition as:

$$u_c(D^{\varphi}, D, \theta) + \frac{u_d(D^{\varphi}, D, \theta)}{D^{\varphi-1}} = (1-\varphi)\, u_c(D^{\varphi}, D, \theta),$$

which can be rearranged as Equation (18).

7.2. Random Search and Nash Bargaining

In the case of competitive search, many different markets can exist potentially, and consumers and firms (trees) choose to participate in the best one for them. Here, we consider another kind of market structure; only one market exists, and shoppers meet with trees randomly. After a shopper and a tree form a match, the price is determined via *Nash bargaining*.

Nash bargaining is a bilateral bargaining protocol in which two parties seek to maximize their utility, which we denote by u and v, through a process of negotiation. If we denote the utility of the parties in case of no deal being made by $\bar{u}$ and $\bar{v}$, the outcome of the process will be the solution to the following problem:

$$\max_{x} \; [u(x) - \bar{u}]^{\mu} [v(x) - \bar{v}]^{1-\mu},$$

where $\mu \in [0, 1]$ represents the *bargaining power* of the parties. It can be easily shown that the outcome lies on the Pareto frontier.

Show that the ratio of utilities is equal to $\mu/(1-\mu)$ in the solution.

In some circumstances we can come up with a theory of the bargaining power, μ; an example is when a public randomization device decides which party is making the initial offer. But, in general, there is no theory for μ. In this sense, Nash bargaining is not an appealing theory for determining the outcomes of a bilateral negotiation process.

However, the power of this bargaining protocol is its ability of rationalizing a unique outcome.

Assume, in the economy described in the previous sections, when a consumer and a tree match, they engage in a Nash bargaining process. If the price P is agreed upon in the process, the value of selling a unit of fruit to the firm (the tree) is going to be $1/P$. Note that, since the tree is already found when the bargain is taking plpace, $\Psi_t(Q(\theta))$ does not show up in the return of the tree. If no trade happens, the tree would not loose anything; its outside option would be zero.

Let us consider the shopper's payoffs from bargain; if the price of fruit (in terms of shares) is set at $1/P$, the payoff to the shopper (of one unit of fruit) would be u_c, while, the price of $1/P$ (per unit of fruit in terms of shares) would give him an outside option of buying $1/P$ units of shares, which, yields $[1+R(\theta')]/P$ units of share, next period. Given the price of fruit is $1/p(\theta')$ in state θ', this return can be traded for $P(\theta')[1+R(\theta')]$ units of fruit in the following period, provided that the shopper *searches again* in next period and finds a tree.

Therefore, the value to the shopper, of a price $1/P$ is

$$u_c(c,d,\theta) - \beta \sum_{\theta'} \Gamma_{\theta\theta'} \frac{P(\theta')[1+R(\theta')]}{P} \left[u_c(c',d',\theta') + \frac{u_d(c',d',\theta')}{\Psi_d(Q(\theta'))} \right].$$

As a result, the Nash bargaining problem can be written as

$$\max_P \quad \left(\frac{1}{P}\right)^{1-\mu}$$
$$\times \left[u_c(c,d,\theta) - \beta \sum_{\theta'} \Gamma_{\theta\theta'} \frac{p(\theta')[1+R(\theta')]}{P} \left[u_c(c',d',\theta') + \frac{u_d(c',d',\theta')}{\Psi_d(Q(\theta'))} \right] \right]^{\mu}.$$

where μ is the bargaining power of the shopper.

The first order condition for this problem would be

$$
\begin{aligned}
& -(1-\mu)\frac{1}{P^2}\left(\frac{1}{P}\right)^{-\mu} \\
& \times\left[u_c(c,d,\theta)-\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{p(\theta')[1+R(\theta')]}{P}\left[u_c(c',d',\theta')+\frac{u_d(c',d',\theta')}{\Psi_d(Q(\theta'))}\right]\right]^{\mu} \\
& +\mu\left(\frac{1}{P}\right)^{1-\mu} \\
& \times\left[u_c(c,d,\theta)-\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{p(\theta')[1+R(\theta')]}{P}\left[u_c(c',d',\theta')+\frac{u_d(c',d',\theta')}{\Psi_d(Q(\theta'))}\right]\right]^{\mu-1} \\
& \qquad\qquad \times\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{p(\theta')[1+R(\theta')]}{P^2}\left[u_c(c',d',\theta')+\frac{u_d(c',d',\theta')}{\Psi_d(Q(\theta'))}\right]=0,
\end{aligned}
$$

which can be simplified as

$$
\begin{aligned}
& (1-\mu)\left[u_c(c,d,\theta)-\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{p(\theta')[1+R(\theta')]}{P}\left[u_c(c',d',\theta')+\frac{u_d(c',d',\theta')}{\Psi_d(Q(\theta'))}\right]\right] \\
& \qquad\qquad =\mu\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{p(\theta')[1+R(\theta')]}{P}\left[u_c(c',d',\theta')+\frac{u_d(c',d',\theta')}{\Psi_d(Q(\theta'))}\right].
\end{aligned}
$$

Using the Euler equation in (19), and the equilibrium conditions, this equality can be simplified further as

$$
\mu u_c\left(Q^{-\varphi},Q^{-1},\theta\right)=-\frac{u_d\left(Q^{-\varphi},Q^{-1},\theta\right)}{Q^{1-\varphi}}. \tag{19}
$$

If we compare Equations (19) and (18), we observe that by setting the Nash bargaining parameter μ equal to the goods matching elasticity φ, the solution in the environment with random search (and Nash bargaining) coincides with the one under efficient competitive search.[9]

If we set $\mu=0$, then $u_d=0$. This is because when the shopper has no bargaining power, the tree will obtain all the surplus and leave the shopper with

$$
u_c(c,d,\theta)-\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{p(\theta')[1+R(\theta')]}{P}\left(u_c(\theta')+\frac{u_d(\theta')}{\Psi_d[Q(\theta')]}\right)=0
$$

[9] This condition is referred to as *Hossios condition*. However, in our environment, there is no assumption to ensure this is the case.

The shopping disutility u_d is not compensated, and consumers will not search at the first place. Unless consumers have non-zero bargaining power, a hold-up problem will show up that prevents the household from doing any investment in searching. Consequently, other issues have to be present in environments with price posting which is equivalent to $\mu = 0$.

7.3. Price Posting by a Monopolist

So far, we have considered two different approaches to model the supply side of the economy; competitive search, and random search (with Nash Bargaining). Here, we study another approach; the case that all trees are run by a monopoly that posts the price, but understands that Q is a function of P.

In this environment, there is only one market for the fruit; in this market, the monopolist posts a price P. However, the demand for the fruit, or the market tightness Q, follows from the problem of the household. When choosing the optimal price to maximize its profit, the monopolist takes this demand into account.

To formalize this idea, once again, consider the Euler equation, derived from the first order conditions of the household, as in Equation (5); noting that $\Psi_d(Q) = Q^{1-\varphi}$, and $Q = 1/D$ in equilibrium, we can rewrite this equation as:

$$u_c\left(Q^{-\varphi}, Q^{-1}, \theta\right) + \frac{u_d\left(Q^{-\varphi}, Q^{-1}, \theta\right)}{Q^{1-\varphi}} = \beta \frac{1}{P} \sum_{\theta'} \Gamma_{\theta\theta'} V_s\left(\theta', 1 + R(\theta) - \frac{1}{P} Q^{-\varphi}\right),$$

which is equivalent to

$$\begin{aligned} &u_d\left(Q^{-\varphi}, Q^{-1}, \theta\right) \\ &\quad + Q^{1-\varphi}\left[u_c\left(Q^{-\varphi}, Q^{-1}, \theta\right) - \beta \frac{1}{P(\theta)} \sum_{\theta'} \Gamma_{\theta\theta'} V_s\left(\theta', 1 + R(\theta) - \frac{1}{P(\theta)} Q^{-\varphi}\right)\right] = 0. \quad (20) \end{aligned}$$

This equation, implicitly defines demand and market tightness, as a function of price, given the value function V.

The monopolist chooses the optimal price, taking into account the demand of the household:

$$\begin{aligned} \max_{P(\theta)} \quad & \Pi(P) = \frac{1}{P(\theta)} \Psi_t(Q) \\ s.t. \quad & \Psi_t = Q^{-\varphi} \\ & \textit{Equation } (20). \end{aligned}$$

There are some issues regarding whether the monopolist understands what happens in the future, that we will leave aside.

8. Measure Theory

This section will be a quick review of measure theory to be able to use it in the subsequent sections.

Definition 9 *For a set S, $\mathcal{S}$ is a family of subsets of S, if $B \in \mathcal{S}$ implies $B \subseteq S$ (but not the other way around).*

Definition 10 *A family of subsets of S, $\mathcal{S}$, is called a σ-algebra in S if*

1. *$S, \emptyset \in \mathcal{S}$;*
2. *$A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to complements); and,*
3. *for $\{B_i\}_{i\in\mathbb{N}}$, $B_i \in \mathcal{S}$ for all i implies $\bigcap_{i\in\mathbb{N}} B_i \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to countable intersections).*

1. The power set of S (i.e. all the possible subsets of a set S), is a σ-algebra in S.
2. $\{\emptyset, S\}$ σ-algebra in S.
3. $\left\{\emptyset, S, S_{1/2}, S_{2/2}\right\}$, where $S_{1/2}$ means the lower half of S (imagine S as an closed interval in $\mathbb{R}$), σ-algebra in S.

4. If $S = [0, 1]$, then

$$\mathcal{S} = \left\{ \emptyset, \left[0, \frac{1}{2}\right), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S \right\}$$

is *not* a σ-algebra in S.

Note that, a convention is to

1. use small letters for elements,
2. use capital letters for sets, and
3. use fancy letters for a set of subsets (or families of subsets).

Now, we are ready to define a measure.

Definition 11 *Suppose $\mathcal{S}$ is a σ-algebra in S. A measure is a function $x : \mathcal{S} \to \mathbb{R}_+$, that satosfies*

1. $x(\emptyset) = 0$;
2. $B_1, B_2 \in \mathcal{S}$ *and* $B_1 \cap B_2 = \emptyset$ *implies* $x(B_1 \cup B_2) = x(B_1) + x(B_2)$ *(additivity); and,*
3. $\{B_i\}_{i \in \mathbb{N}} \in \mathcal{S}$ *and* $B_i \cap B_j = \emptyset$*, for all* $i \neq j$*, implies* $x(\cup_i B_i) = \sum_i x(B_i)$ *(countable additivity).*[10]

A set S, a σ-algebra in it, $\mathcal{S}$, and a measure on $\mathcal{S}$, define a measure space, $(S, \mathcal{S}, x)$.

Definition 12 *Borel σ-algebra is a σ-algebra generated by the family of all open sets (generated by a topology).*

Since a Borel σ-algebra contains all the subsets generated by the intervals, you can recognize any subset of a set using Borel σ-algebra. In other words, Borel σ-algebra corresponds to complete information.

Definition 13 *A probability (measure) is a measure with the property that $x(S) = 1$.*

[10] Countable additivity means that the measure of the union of countable disjoint sets is the sum of the measure of these sets.

Definition 14 *Given a measure space $(S, \mathcal{S}, x)$, a function $f : S \to \mathbb{R}$ is measurable (with respect to the measure space) if, for all $a \in \mathbb{R}$, we have*

$$\{b \in S \mid f(b) \leq a\} \in \mathcal{S}.$$

One way to interpret a σ-algebra is that it describes the information available based on observations; a structure to organize information, and how fine are the information that we receive. Suppose that S is comprised of possible outcomes of a dice throw. If you have no information regarding the outcome of the dice, the only possible sets in your σ-algebra can be $\emptyset$ and S. If you know that the number is even, then the smallest σ-algebra given that information is $\mathcal{S} = \{\emptyset, \{2, 4, 6\}, \{1, 3, 5\}, S\}$. Measurability has a similar interpretation. A function is measurable with respect to a σ-algebra $\mathcal{S}$, if it can be evaluated under the current measure space $(S, \mathcal{S}, x)$.

We can also generalize Markov transition matrix to any measurable space. This is what we do next.

Definition 15 *A function $Q : S \times \mathcal{S} \to [0, 1]$ is a transition probability if*

1. *$Q(\cdot, s)$ is a probability measure for all $s \in S$; and,*
2. *$Q(B, \cdot)$ is a measurable function for all $B \in \mathcal{S}$.*

Intuitively, given $B \in \mathcal{S}$ and $s \in S$, $Q(B, s)$ gives the probability of being in set B tomorrow, given that the state is s today. Consider the following example: a *Markov chain* with transition matrix given by

$$\Gamma = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix},$$

on the set $S = \{1, 2, 3\}$, with the σ-algebra $\mathcal{S} = P(S)$ (where $P(S)$ is the power set of S). If Γ_{ij} denotes the probability of state j happening, given a present state i, then

$$Q(\{1, 2\}, 3) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 .$$

As another example, suppose we are given a measure x on S; x_i gives us the fraction of type i, for $i \in S$. Given the previous transition function, we can calculate the fraction of types tomorrow using the following formulas:

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31}, \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32}, \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33}. \end{aligned}$$

In other words

$$\mathbf{x}' = \Gamma^T \mathbf{x},$$

where $\mathbf{x}^T = (x_1, x_2, x_3)$.

To extend this idea to a general case with a general transition function, we define an *updating operator* as $T(x, Q)$, which is a measure on S with respect to the σ-algebra $\mathcal{S}$, such that

$$\begin{aligned} x'(B) =& T(x, Q)(B) \\ =& \int_S Q(B, s)\, x(ds), \quad \forall B \in \mathcal{S}. \end{aligned}$$

A stationary distribution is a fixed point of T, that is x^* so that

$$x^*(B) = T(x^*, Q)(B), \quad \forall B \in \mathcal{S}.$$

We know that, if Q has nice properties,[11] then a unique stationary distribution exists (for example, we discard *flipping* from one state to another), and

$$x^* = \lim_{n\to\infty} T^n(x_0, Q),$$

for any x_0 in the space of measures on S.

Consider unemployment in a very simple economy (we have an exogenous transition matrix). There are two states; employed and unemployed. The transition matrix is given by

$$\Gamma = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}.$$

Compute the stationary distribution corresponding to this Markov transition matrix.

[11] See Chapter 11 in Stokey and Lucas (1989).

9. Industry Equilibrium

9.1. Preliminaries

Now we are going to study a type of models initiated by **?**. We will abandon the general equilibrium framework from the previous section to study the dynamics of distribution of firms in a partial equilibrium environment.

To motivate things, let's start with the problem of a single firm that produces a good using labor input according to a technology described by the production function f. Let us assume that this function is increasing, strictly concave, with $f(0) = 0$. A firm that hires n units of labor is able to produce $sf(n)$, where s is a productivity parameter. Markets are competitive, in the sense that a firm takes prices as given and chooses n in order to solve

$$\pi(s, p) = \max_{n \geq 0} \{psf(n) - wn\}.$$

The first order condition implies that in the optimum, n^*,

$$psf_n(n^*) = w.$$

Let us denote the solution to this problem as a function $n^*(s, p)$.[12] Given the above assumptions, n^* is an increasing function of s (i.e. more productive firms have more workers), as well as p.

Suppose now there is a mass of firms in the industry, each associated with a productivity parameter $s \in S \subset \mathbb{R}_+$, where

$$S[s, \bar{s}].$$

Let $\mathcal{S}$ denote a σ-algebra on S (Borel σ-algebra for instance). Let x be a measure defined over the space $(S, \mathcal{S})$ that describes the cross sectional distribution of productivity among firms. Then, for any $B \subset S$ with $B \in \mathcal{S}$, $x(B)$ is the mass of firms having productivities in S.

We will use x to define statistics of the industry. For example, at this point, it is convenient to define the aggregate supply of the industry. Since individual supply is just $sf(n^*(s, p))$, the aggregate supply

[12] As we declared in advance, this is a partial equilibrium analysis. Hence, we ignore the dependence of the solution on w to focus on the determination of p.

can be written as[13]

$$Y^S(p) = \int_S sf(n^*(s,p))\,x(ds).$$

Observe that Y^S is a function of the price p; for any price, p, $Y^S(p)$ gives us the supply in this economy.

Search Wikipedia for an index of concentration in an industry, and adopt it for our economy.

Suppose now that the demand of the market is described by some function $Y^D(p)$. Then the equilibrium price, p^*, is determined by the market clearing condition

$$Y^D(p^*) = Y^S(p^*). \tag{21}$$

So far, everything is too simple to be interesting. The ultimate goal here is to understand how the object x is determined by the fundamentals of the industry. Hence, we will be adding tweaks to this basic environment in order to obtain a theory of firms' distribution in a competitive environment. Let's start by allowing firms to die.

9.2. A Simple Dynamic Environment

Consider now a dynamic environment, in which the situation above repeats every period. Firms discount profits at rate r_t, which is exogenously given. In addition, assume that a single firm, in each period, faces a probability δ of disappearing! We will focus on *stationary equilibria*; i.e. equilibria in which the price of the final output p, the rate of return, r, and the productivity of firm, s, stay constant through time.

Notice first that firm's decision problem is still a static problem; we can easily write the value of an incumbent firm as

$$\begin{aligned} V(s,p) &= \sum_{t=0}^{\infty}\left(\frac{1-\delta}{1+r}\right)^t \pi(s,p) \\ &= \left(\frac{1+r}{r+\delta}\right)\pi(s,p) \end{aligned}$$

[13] S in Y^S stands for supply.

Note that we are considering that p is fixed (therefore we can omit it from the expressions above). Observe that every period there is positive mass of firms that die. Therefore, how can this economy be in a stationary equilibrium? To achieve that, we have to assume that there is constant fellow of firms entering the economy in each period, as well.

As before, let x be the measure describing the distribution of firms within the industry. The mass of firms that die is given by $\delta x(S)$. We will allow these firms to be replaced by new entrants. These entrants draw a productivity parameter s from a probability measure γ.

One might ask what keeps these firms out of the market in the first place? If

$$\pi(s, p) = psf(n^*(s, p)) - wn^*(s, p) > 0,$$

which is the case for the firms operating in the market, then all the (potential) firms with productivities in S would want to enter the market!

We can fix this flaw by assuming that there is a fixed entry cost that each firm must pay in order to operate in the market, denoted by c^E. Moreover, we will assume that the entrant has to pay this cost before learning s. Hence the value of a new entrant is given by the following function:

$$V^E(p) = \int_S V(s, p)\gamma(ds) - c^E. \tag{22}$$

Entrants will continue to enter if V^E is greater than 0, and decide not to enter if this value is less than zero. As a result, stationarity occurs when V^E is exactly equal to zero (this is the *free entry* assumption, and we are assuming that there is an infinite number (mass) of prospective firms).

Let's analyze how this environment shapes the distribution of firms in the market. Let x_t be the cross sectional distribution of firms in period t. For any $B \subset S$, portion δ of the firms with productivity $s \in B$ will die, and that will attract some newcomers. Hence, next period's measure of firms on set B will be given by:

$$x_{t+1}(B) = (1-\delta)x_t(B) + m\gamma(B).$$

That is, mass m of firms would enter the market in $t+1$, and only fraction $\gamma(B)$ of them will have productivities in the set B. As you might suspect, this relationship must hold for every $B \in S$. Moreover, since we are interested in stationary equilibria, the previous expression tells us that the cross sectional distribution of firms will be completely determined by γ.

If we let mapping T be defined by

$$Tx(B) = (1-\delta)x(B) + m\gamma(B), \quad \forall B \in S, \tag{23}$$

a stationary distribution of productivity is the fixed point of the mapping T; i.e. x^* with $Tx^* = x^*$, implying:

$$x^*(B;m) = \frac{m}{\delta}\gamma(B), \quad \forall B \in S.$$

Now, note that the demand and supply relation in (21) takes the form:

$$y^d(p^*(m)) = \int_S sf(n^*(s,p))\,dx^*(s;m), \tag{24}$$

whose solution, $p^*(m)$, is continuous function under regularity conditions stated in Stokey and Lucas (1989).

We have two equations, (22) and (24), and two unknowns, p and m. Thus, we can defined the equilibrium as:

Definition 16 *A stationary distribution for this environment consists of functions p^*, x^*, and m^*, that satisfy:*

1. $y^d(p^*(m)) = \int_S sf(n^*(s,p))\,dx^*(s;m)$;
2. $\int_s V(s,p)\gamma(ds) - c_E = 0$; *and,*
3. $x^*(B) = (1-\delta)x^*(B) + m^*\gamma(B), \quad \forall B \in S.$

9.3. Introducing Exit Decisions

We want to introduce more (economic) content by making the exit of firms endogenous (a decision of the firm). One way to do so is to assume that the productivity of the firms follow a Markov process governed by a transition function, Γ. This would change the mapping T in Equation (23), as:

$$Tx(B) = (1-\delta)\int_S \Gamma(s,B)\,x(ds) + m\gamma(B), \quad \forall B \in S.$$

But, this wouldn't add much economic content to our environment; firms still do not make any (interesting) decision. To change this, let's introduce cost of operation into the model; suppose firms have to pay c^v each period in order to stay in the market. In this case, when s is low, the firm's profit might not cover its cost of operation. So, the firm might decide to leave the market. However, firm has already paid (a sink cost of) c^E, and, since s changes according to a Markov process, prospects of future profits might deter the firm from quitting. Therefore, negative profit in one period does not imply immediately that the firm's optimal choice is to leave the market.

By adding such a minor change, the solution will have a reservation productivity property under some conditions (to be discussed below). In words, there will be a minimum productivity, $s^* \in S$, above which it is profitable for the firm to stay in the market.

To see this, note that the value of a firm with productivity $s \in S$ in a period is given by

$$V(s,p) = \max \left\{ 0, \pi(s,p) + \frac{1}{(1+r)} \int_S \Gamma(s, ds') V(s', p) - c^v \right\}.$$

Show that the firm's decision takes the form of a reservation productivity strategy, in which, for some $s^* \in S$, $s < s^*$ implies that the firm would leave the market.

In this case, the transition of the distribution of productivities on S will be:

$$x'(B) = m\gamma([s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}]) x(ds), \quad \forall B \in \mathcal{S}.$$

A stationary distribution of the firms in this economy, x^*, is the fixed point of this equation.

How productive does a firm have to be, to be in the top 10% largest firms in this economy? The answer to this question is the solution to the following equation, $\hat{s}$:

$$\frac{\int_{\hat{s}}^{\bar{s}} x^*(ds)}{\int_{s^*}^{\bar{s}} x^*(ds)} = 0.1 .$$

Then, the fraction of the labor force in the top 10% largest firms in this economy, is

$$\frac{\int_{\hat{s}}^{\bar{s}} n^*(s,p) x^*(ds)}{\int_{s^*}^{\bar{s}} n^*(s,p) x^*(ds)}.$$

Compute the average growth rate of the smallest one third of the firms.

What would be the fraction of firms in the top 10% largest firms in the economy that remain in the top 10% in next period?

(??) To see that this will be the case you should prove that the profit before variable cost function $\pi(s, p)$ is increasing in s. Hence the productivity threshold is given by the s^* that satisfies the following condition:

$$\pi(s^*, p) = c_v$$

for an equilibrium price p. Now instead of considering γ as the probability measure describing the distribution of productivities among entrants, you must consider $\widehat{\gamma}$ defined as follows

$$\widehat{\gamma}(B) = \frac{\gamma(B \cap [s^*, \bar{s}])}{\gamma([s^*, \bar{s}])}$$

for any $B \in S$.

One might suspect that this is an *ad hoc* way to introduce the exit decision. To make the things more concrete and easier to compute, we will assume that s is a Markov process. To facilitate the exposition, let's make S finite and assume s has transition matrix Γ. Assume further that Γ is regular enough so that it has a stationary distribution γ. For the moment we will not put any additional structure on Γ.

The operation cost c^v is such that the exit decision is meaningful. Let's analyze the problem from the perspective of the firm's manager. He has now two things to decide. First, he asks himself the question "Should I stay or should I go?". Second, conditional on staying, he has to decide how much labor to hire. Importantly, notice that this second decision is still a static decision. Later, we will introduce adjustment cost that will make this decision a dynamic one.

Let $\phi(s, p)$ be the value of the firm before having decided whether to stay or to go. Let $V(s, p)$ be the value of the firm that has already decided to stay. $V(s, p)$ satisfies

$$V(s, p) = \max_n \left\{ spf(n) - n - c^v + \frac{1}{1+r} \sum_{s' \in S} \Gamma_{ss'} \phi(s', p) \right\}$$

Each morning the firm chooses d in order to solve

$$\phi(s,p) = \max_{d\in\{0,1\}} dV(s,p)$$

Let $d^*(s,p)$ be the optimal decision to this problem. Then $d^*(s,p) = 1$ means that the firm stays in the market. One can alternatively write:

$$\phi(s,p) = \max_{d\in\{0,1\}} d\left[\pi(s,p) - c^v + \frac{1}{1+r}\sum_{s'\in S}\Gamma_{ss'}\phi(s',p)\right]$$

or even

$$\phi(s,p) = \max\left[\pi(s,p) - c^v + \frac{1}{1+r}\sum_{s'\in S}\Gamma_{ss'}\phi(s',p),0\right] \tag{25}$$

All these are valid. Additionally, one can easily add minor changes to make the exit decision more interesting. For example, things like scrap value or liquidation costs will affect the second argument of the max operator above, which so far is just zero.

What about $d^*(s,p)$? Given a price, this decision rule can take only finitely many values. Moreover, if we could ensure that this decision is of the form "*stay only if the productivity is high enough and go otherwise*" then the rule can be summarized by a unique number $s^* \in S$. Without doubt, that would be very convenient, but we don't have enough structure to ensure that such is the case. Because, although the ordering of s (lower s are ordered before higher s) gives us that the value of s today is bigger than value of smaller s', depending on the Markov chain, on the other hand, the value of productivity level s tomorrow may be lower than the value of s' (note $s' < s$) tomorrow. Therefore we need some additional regularity conditions.

In order to get a cutoff rule for the exit decision, we need to add an assumption about the transition matrix Γ. Let the notation $\Gamma(s)$ indicate the probability distribution over next period state conditional on being on state s today. You can think of it as being just a column of the transition matrix. Take s and $\widehat{s}$. We will say that the matrix Γ displays *first order stochastic dominance* (*FOSD*) if $s > \widehat{s}$ implies $\sum_{s'\leq b}\Gamma(s' \mid s) \leq \sum_{s'\leq b}\Gamma(s' \mid \widehat{s})$ for any $b \in S$. It turns out that *FOSD* is a sufficient condition for having a cutoff rule. You can prove that by using the same kind of dynamic programming tricks that we have used in the first semester for obtaining the reservation wage property in search problems. Try it as an exercise. Also note that this is just a sufficient condition.

Finally, we need to mention something about potential entrants. Since we will assume that they have to pay the cost c_E before learning their s, they can leave the industry even before producing anything. That requires us to be careful when we describe industry dynamics.
Now the law of motion becomes;

$$x'(B) = m\gamma(B \cap [s^*, \bar{s}]) + \int_S \sum_{s'} \mathbf{1}_{\{s' \in B \cap [s^*, \bar{s}]\}} \Gamma(s, s') x(ds)$$

9.4. Stationary Equilibrium

Now that we have all the ingredients in the table, let's define the equilibrium formally.

Definition 17 *A stationary equilibrium for this environment consists of a list of functions (ϕ, n^*, d^*), a price p^* and a measure x^* such that*

1. *Given p^*, the functions ϕ, n^*, d^* solve the problem of the incumbent firm*
2. $V^E(p^*) = 0$
3. *For any $B \in \mathcal{B}_S$ (assuming we have a cut-off rule with s^* is cut-off in stationary distribution)*[14]

$$x^*(B) = m\gamma(B \cap [s^*, \bar{s}]) + \int_S \sum_{s'} \mathbf{1}_{\{s' \in B \cap [s^*, \bar{s}]\}} \Gamma(s, s') x(ds) \tag{26}$$

You can think of condition (2) as a "*no money left over the table*" condition, which ensures additional entrants find unprofitable to participate in the industry.

We can use this model to compute interesting statistics. For example the average output of the firm

[14] If we do not have such cut-off rule we have to define

$$x^*(B) = \int_S \sum_{s' \in S} \Gamma_{ss'} \mathbf{1}_{\{s' \in B\}} \mathbf{1}_{\{d(s', p^*) = 1\}} x^*(ds) + \mu^* \int_S \mathbf{1}_{\{s \in B\}} \mathbf{1}_{\{d(s, p^*) = 1\}} \gamma(ds)$$

where

$$\mu^* = \int_S \sum_{s' \in S} \Gamma_{ss'} \mathbf{1}_{\{d(s', p^*) = 0\}} x^*(ds)$$

is given by

$$\frac{Y}{N} = \frac{\sum sf(n^*(s))x^*(ds)}{\sum x^*(ds)}$$

Next, suppose that we want to compute the share out output produced by the top 1% of firms. To do this we first need to compute $\widetilde{s}$ such that

$$\frac{\sum_{\widetilde{s}}^{\bar{s}} x^*(ds)}{N} = .01$$

where N is the total measure of firms. Then the share output produced by these firms is given by

$$\frac{\sum_{\widetilde{s}}^{\bar{s}} sf(n^*(s))x^*(ds)}{\sum_{\underline{s}}^{\bar{s}} sf(n^*(s))x^*(ds)}$$

Suppose now that we want to compute the fraction of firms who are in the top 1% two periods in a row. This is given by

$$\sum_{s \geq \widetilde{s}} \sum_{s' \geq \widetilde{s}} \Gamma_{ss'} x^*(ds)$$

We can use this model to compute a variety of other statistics include the Gini coefficient.

9.5. Adjustment Costs

To end with this section it is useful to think about environments in which firm's productive decisions are no longer static. A simple way of introducing dynamics is by adding adjustment costs.

We will consider labor adjustment costs.[15] Consider a firm that enters period t with n_{t-1} units of labor, hired in the previous period. We can consider three specifications for the adjustment costs, due to hiring n_t units of labor in t, $c(n_t, n_{t-1})$:

- *Convex Adjustment Costs*: if the firm wants to vary the units of labor, it has to pay $\alpha (n_t - n_{t-1})^2$ units of the numeraire good. The cost here depends on the size of the adjustment.
- *Training Costs or Hiring Costs*: if the firm wants to increase labor, it has to pay $\alpha [n_t - (1-\delta) n_{t-1}]^2$

[15] These costs work pretty much like capital adjustment costs, as one might suspect.

units of the numeraire good, only if $n_t > n_{t-1}$; we can write this as

$$\max\left\{0, \alpha\left[n_t - (1-\delta)\, n_{t-1}\right]^2\right\} = \mathbf{1}_{\{n_t > n_{t-1}\}} \alpha\left[n_t - (1-\delta)\, n_{t-1}\right]^2,$$

where $\mathbf{1}$ is the indicator function, and δ measures the exogenous attrition of workers in each period.

- *Firing Costs.*

The recursive formulation of the firm's problem would be:

$$V(s, n_-) = \max_{n \geq 0} \left\{ -c(0, n_-),\ sf(n) - wn - c(n, n_-) + \frac{1}{(1+r)} \sum_{s' \in S} \Gamma_{ss'} V(s', n) \right\}, \qquad (27)$$

where c gives the specified cost of adjusting n_- to n.

Write the first order conditions for the problem in (27).

Define the recursive competitive equilibrium for this economy.

Another example of labor adjustment costs is when the firm has to post vacancies to attract labor. As an example of such case, suppose the firm faces a firing cost according to the function c. The firm also pays a cost κ to post vacancies, and after posting vacancies, it takes one period for the workers to be hired. How can we write the problem of firms in this environment?

10. Incomplete Market Models

10.1. A Farmer's Problem

Consider the following problem of a farmer:

$$V(s,a) = \max_{c,a'} \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \right\} \tag{28}$$
$$\begin{aligned} s.t. \quad & c + qa' = a + s \\ & c \geq 0 \\ & a' \geq 0, \end{aligned}$$

where a is his holding of coconuts, which can only take positive values, c is his consumption, and s is amount of coconuts that he produces. s follows a Markov chain, taking values in a finite set S. q is the price of coconuts. Note that, the constraint on the holding of coconuts tomorrow, is a constraint imposed by nature; nature allows the farmer to store coconuts at rate $1/q$, but, it does not allow him to *transfer coconuts* from tomorrow to today.

We are going to consider this problem in the context of a partial equilibrium, where q is given. The first question is, what can be said about q?

Assume there are no shocks in the economy, so that s is a fixed number. Then, we could write the problem of the farmer as:

$$V(a) = \max_{c,a'\geq 0} \left\{ u(a+s-qa') + \beta V(a') \right\}.$$

If u is assumed to be logarithmic, the first order condition for this problem implies:

$$\frac{c'}{c} \geq \frac{\beta}{q},$$

with equality if $a' > 0$. Therefore, if $q > \beta$ (i.e. nature is more stingy than farmer's patience), then $c' < c$, and the farmer dis-saves (at least, as long as $a' > 0$). But, when $q < \beta$, consumption grows without bound. This is the reason we put this assumption on the model, in what follows.

A crucial assumption for generating a bounded asset space is $\beta/q < 1$, stating that agents are sufficiently impatient so they tend to consume more and decumulate their asset when they get richer and far away from the non-negativity constraint, $a' \geq 0$. However, this does not mean that, when faced with a possibility of very low consumption (potentially when $a' = 0$), agents would not save, even though the rate of return, $1/q$, is smaller than the rate of impatience.

The first order condition for farmer's problem (28) is given by:

$$u_c\left(c\left(s,a\right)\right) \geq \frac{\beta}{q}\sum_{s'}\Gamma_{ss'}u_c\left(c\left(s',a'\left(s,a\right)\right)\right),$$

with equality when $a'\left(s,a\right) > 0$, where $c\left(\cdot\right)$ and $a'\left(\cdot\right)$ are policy functions from the farmer's problem. Notice that $a'\left(s,a\right) = 0$ is possible, if we assume appropriate shock structure in the economy. Specifically, it depends on the value of $s_{\min}\min_{s_i \in S}\ s_i$.

The solution to the problem of the farmer, for a given value of q, implies a distribution of coconut holdings in each period. This distribution, together with the Markov chain describing the evolution of s, can be summed together as a single probability measure for the distribution of shocks and assets (coconut holdings) over the product space $E = S \times \mathbb{R}_+$, and its σ-algebra, B, which we denote by X. The evolution of this probability measure is given by:

$$X'\left(B\right) = \sum_{s' \in B_s}\Gamma_{ss'}\int \mathbf{1}_{a'(s,a)\in B_a}dX\left(s,a\right), \quad \forall B \in B, \tag{29}$$

where B_s and B_a are the S-section and $\mathbb{R}$-section of B (projections of B on S and $\mathbb{R}_+$), respectively, and $\mathbf{1}$ is the indicator function. Let $\widetilde{T}\left(\Gamma,a',\cdot\right)$ be the mapping associated with (29) (the adjoint operator), so that:

$$X'\left(B\right) = \widetilde{T}\left(\Gamma,a',X\right)\left(B\right), \quad \forall B \in B.$$

Define $\widetilde{T}^n\left(\Gamma,a',\cdot\right)$ as:

$$\widetilde{T}^n\left(\Gamma,a',X\right) = \widetilde{T}^{n-1}\left(\Gamma,a',\widetilde{T}\left(\Gamma,a',X\right)\right).$$

Then, we have the following theorem.

Theorem 3 *Under some conditions on $\widetilde{T}(\Gamma, a', \cdot)$, there is a unique probability measure X^*, so that:*

$$X^*(B) = \lim_{n\to\infty} \widetilde{T}(\Gamma, a', X_0)(B), \quad \forall B \in B,$$

for all initial probability measures X_0 on (E, B).

A condition that makes things considerably easier for this theorem to hold is that E is a compact set. Then, we can use Theorem (12.12) in Stokey and Lucas (1989), to show this result holds. Given that S is finite, this is equivalent to a compact support for the distribution of asset holdings. We discuss this in detail in Appendix 14.

10.2. Huggett (1993)'s Economy

Now we modify the farmer's problem in (28) a little bit:

$$\begin{aligned} V(s,a) = \max_{c,a'} \quad & \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \right\} \\ s.t. \quad & c + qa' = a + s \\ & c \geq 0 \\ & a' \geq a, \end{aligned} \tag{30}$$

where $a < 0$, so now farmers can borrow and lend with each other, but with a borrowing limit. We continue to make the same assumption on q; i.e. $\beta/q < 1$. Solving this problem gives the policy function $a'(s,a)$. It is easy to extend the analysis in the last section to show that there is an upper bound of the asset space, which we denote by $\bar{a}$, so that for any $a \in A[a, \bar{a}]$, $a'(s,a) \in A$, for any $s \in S$.

One possibility for a is what we call the natural borrowing constraint. This is the constraint that ensures the agent can pay back his debt for sure, no matter what the nature reveals (whatever sequence of idiosyncratic shocks is realized). This is given by

$$a^n - \frac{s_{\min}}{\left(\frac{1}{q} - 1\right)}.$$

If we impose this constraint on (30), even when the farmer receives an infinite sequence of bad shocks, he can pay back his debt by setting his consumption equal to zero, forever.

But, what makes this problem more interesting is to tighten this borrowing constraint; the natural borrowing constraint is very unlikely to be hit. One way to restrict borrowing is no borrowing at all, as in the previous section. Another case is to choose $0 > a > a^n$. This is the case that we consider in this section.

Now suppose there is a (unit) mass of farmers with distribution function $X(\cdot)$, where $X(D, B)$ denotes fraction of people with shock $s \in D$ and $a \in B$, where D in an element of the power set of S , $P(S)$ (which, when S is finite, is the natural σ-algebra over S) and B is a Borel subset of A, $B \in A$. Then the distribution of farmers tomorrow is given by:

$$X'(S', B') = \int_{A \times S} \mathbf{1}_{\{a'(s,a) \in B'\}} \sum_{s' \in S'} \Gamma_{ss'} dX(s, a), \tag{31}$$

for any $S' \in P(S)$ and $B' \in A$.

Implicitly this defines an operator T such that $T(X) = X'$. If T is *sufficiently nice*, then there exits a unique X^* such that $X^* = T(X^*)$ and $X^* = \lim_{n\to\infty} T^n(X_0)$, for any initial distribution over the product space $S \times A$, X_0. Note that the decision rule is obtained given q. Hence, the resulting stationary distribution X^* also depends on q. So, let us denote it by $X^*(q)$.

To determine the equilibrium value of q, in a general equilibrium setting, consider the following variable (as a function of q):

$$\int_{A \times S} adX^*(q).$$

This expression give us the average asset holdings, given the price, q.

We want to determine an endogenous q, so that the asset market clears; we assume that there is no storage technology so that asset supply is 0 in equilibrium. Hence, price q should be such that asset demand equals asset supply; i.e.

$$\int_{A \times S} adX^*(q) = 0.$$

In this sense, equilibrium price q is the price that generates the stationary distribution that clears the asset market.

We can show that a solution exists by invoking intermediate value theorem, by showing that the following three conditions are satisfied (note that $q \in [\beta, \infty]$):

1. $\int_{A\times S} adX^* (q)$ is a continuous function of q;
2. $\lim_{q\to\beta} \int_{A\times S} adX^* (q) \to \infty$; (Intuitively, as $q \to \beta$, interest rate increases. Hence, agents would like to save more. Together with precautionary savings motive, they accumulate unbounded quantities of assets in the stationary equilibrium, in this case.) and,
3. $\lim_{q\to\infty} \int_{A\times S} adX^* (q) < 0$. (This is also intuitive; as $q \to \infty$, interest rate converges to 0. Hence, everyone would rather borrow.)

10.3. Aiyagari (1994)'s Economy

In an Aiyagari-94 Economy, there is physical capital. In this sense, the average asset holdings in the economy must be equal to the average (physical) capital. So, if we keep denoting the stationary distribution of assets by X^*, we must have:

$$\int_{A\times S} adX^* (q) = K,$$

where A is the support of the distribution of wealth. (It is not difficult that this set is compact.)

On the other hand, the shocks affect the labor income; we can think of these shocks as fluctuations in the employment status of individuals. Thus, the problem of an individual in this economy can be written as:

$$\begin{aligned} V(s, a) = \max_{c,a'} \quad & \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \right\} \\ s.t. \quad & c + a' = (1 + r) a + ws \\ & c \geq 0 \\ & a' \geq a, \end{aligned} \tag{32}$$

where, here, r is the return to savings, and w is the wage rate. Therefore,

$$\int_{A\times S} sdX^* (q)$$

gives the average labor in this economy. If we decide to think of agents supplying one unit of labor, then, we may think of the expression as determining the *effective labor supply*.

We assume the standard constant returns to scale production technology for the firm, of the form:

$$F(K, L) = AK^{1-\alpha}L^{\alpha},$$

with the rate of depreciation δ. Hence:

$$\begin{aligned} r =& F_k(K, L) - \delta \\ =& (1-\alpha) A \left(\frac{K}{L}\right)^{-\alpha} \\ & r\left(\frac{K}{L}\right), \end{aligned}$$

and

$$\begin{aligned} w =& F_l(K, L) \\ =& \alpha A \left(\frac{K}{L}\right)^{1-\alpha} \\ & w\left(\frac{K}{L}\right). \end{aligned}$$

In terms of Huggett (1993), q, the price of assets, is given by

$$q = \frac{1}{(1+r)} = \frac{1}{[1 + F_k(K, L) - \delta]}.$$

Therefore, prices faced by agents are all functions of the capital-labor ratio. As a result, we may write the stationary distribution of assets as a function of capital-labor ratio as well, $X^*\left(\frac{K}{L}\right)$. Thus, the equilibrium condition becomes:

$$\frac{K}{L} = \frac{\int_{A\times S} a dX^*\left(\frac{K}{L}\right)}{\int_{A\times S} s dX^*\left(\frac{K}{L}\right)}.$$

Using this condition, one can solve for the equilibrium capital-labor ratio.

10.3.1. Policy

(??) In Aiyagari (1994) [or Huggett (1993)] economy, model parameters can be summarized by $\theta = \{u, \beta, s, \Gamma\}$. In stationary equilibrium, value function $v(s, a; \theta)$ as well as $X^*(\theta)$ can be obtained, where $X^*(\theta)$ is a mapping from model parameters to stationary distribution of agent's asset holding and shocks. Suppose now there is a policy change that shifts θ to $\hat{\theta} = \{u, \beta, s, \hat{\Gamma}\}$. Associated with this new environment there is a new value function $v\left(s, a; \hat{\theta}\right)$ and $X^*\left(\hat{\theta}\right)$. Define $\eta(s, a)$ to be the solution of:

$$v\left(s, a + \eta(s, a), \hat{\theta}\right) = v(s, a, \theta)$$

which is the transfer payment necessary to the households so that they are indifferent between living in the old environment and in the new one. Hence total payment needed to compensate households for this policy change is given by:

$$\int_{A \times S} \eta(s, a)\, dX^*(\theta)$$

Notice that the changes do not take place when the government is trying to compensate the households. Hence we use the original stationary distribution associated with θ to aggregate the households.

If $\int_{A\times S} v(s, a)\, dX^*\left(\hat{\theta}\right) > \int_{A\times S} v(s, a)\, dX^*(\theta)$, does this necessarily mean that households are willing to accept this policy change? The answer is not necessarily because the economy may well spend a long time in the transition path to the new steady state, during which there may be severe welfare loss.

11. Models with Growth

So far, we have seen the *neoclassical growth model* as our benchmark model, and built on it for the analysis of more interesting economic questions. One peculiar characteristic of our benchmark model, unlike its name suggests, is the lack of growth (after reaching the steady state), whereas, many interesting questions in economics are related to the cross-country differences of growth rates. To see why this is the case, consider the standard neoclassical technology:

$$F(K, N) = AK^{\theta_1} L^{\theta_2},$$

for some $\theta_1, \theta_2 \geq 0$. We already know that the only possible case that is consistent with the notion of competitive equilibrium is that $\theta_1 + \theta_2 = 1$. However, this implies a decreasing marginal rate of product for capital. Given a fixed quantity for labor supply, in the presence of depreciation, this implies a maximum sustainable capital stock, and puts a limit on the sustainable growth; economy converges to some steady-state, without exhibiting any balanced growth.

We will try to cover some models that will allow for growth, so that we will be able to attempt to answer such questions.

11.1. Exogenous Growth

We know that in our standard neoclassical growth model there are basically two ways of growth; one in which everything grows, which is not necessarily a per-capita growth, and the other is per-capita growth. We will be presenting one model for each of these categories. The title *exogenous growth* refers to the structure of models in which the growth rate is determined exogenously, and is not an outcome of the model. The simplest one of these is one in which the determinant of the growth rate is population.

11.1.1. Population Growth

Suppose the population of the economy grows at rate γ, and we have the classical constant returns to scale technology in capital and labor inputs:

$$Y_t = AF\left(K_t, N_t\right),$$

where $N_t = N_0 \gamma^t$.

Note that our economy is no longer stationary, but as we will see, within the exogenous growth framework, we can make these economies look like stationary ones by re-normalizing the variables. Thus, at the end of the day, it will only be a mathematical twist on our standard growth model. Once we do that, we will be looking for the counterpart of a steady state that we have in our stationary economies, the *balanced growth path*, in which all the variables will be growing at constant, but not necessarily equal, rates.

One question is how to model population growth in our representative agent model. One way to do so is to assume that there is a constant proportion of immigration to our economy from outside. But, this has to assume the immigrants are identical to our existing agents (specially in terms of ownership of capital), which is a bit problematic. The other way could be to assume growing dynasties which preserves the representative agent nature of our economy. In this sense, in the same spirit as the neoclassical growth model, we can assume there is a single representative agent in the economy, at the beginning of time, owning $k_0 = K_0$ units of physical capital, and $n_0 = N_0$ units of labor, that he supplies inelastically in the labor market; all small letters represent per-capita variables. However, this agent is the head of a dynasty, whose population grows at rate γ. The representative agent cares about the utility of all his descendants, and discounts future utility at rate β. In particular, his utility is given by (we can think of this as the average utility of the whole dynasty):

$$\sum_{t=0}^{\infty} \beta^t N_t U(c_t),$$

where c_t is the per-capita consumption of the dynasty at date t, and N_t is its population. Thus, the dynasty maximizes the average utility of all its members at each date.

In this setting, we can write the problem of the dynasty as:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t N_t U(c_t) \\ s.t. \quad & N_t c_t + X_t \leq r_t K_t + w_t N_t \\ & K_{t+1} = (1-\delta) K_t + X_t, \end{aligned}$$

where X represents the aggregate investment, r is the return on capital, and w is wage rate. Given the standard production function, in the equilibrium:

$$\begin{aligned} w_t &= AF_n(K_t, N_t), \\ r_t &= AF_k(K_t, N_t). \end{aligned}$$

Since F is assumed to be homogenous, we have:

$$r_t K_t + w_t N_t = AF(K_t, N_t).$$

Therefore, we may write household's problem as:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t N_t U(c_t) \\ s.t. \quad & N_t c_t + X_t \leq AF(K_t, \gamma^t N_0) \\ & K_{t+1} = (1-\delta) K_t + X_t. \end{aligned}$$

Next, let us divide both sides of the budget constraint by N_t, to get:

$$c_t + \frac{X_t}{N_t} \leq \frac{1}{N_t} AF(K_t, N_t) = AF\left(\frac{K_t}{N_t}, 1\right).$$

Replacing by per-capita variables, we may write this as:

$$c_t + x_t \leq AF(k_t, 1).$$

On the other hand, by dividing the law of motion of capital by N_t, we have:

$$\frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} = \gamma k_{t+1} = (1-\delta) k_t + x_t.$$

In summary, we may write household's problem as:

$$\begin{aligned} \max \quad & N_0 \sum_{t=0}^{\infty} (\gamma\beta)^t U(c_t) \\ s.t. \quad & c_t + \gamma k_{t+1} \leq (1-\delta) k_t + AF(k_t, 1). \end{aligned}$$

This is, now, a familiar neoclassical growth model, in per-capita terms, where nothing grows! We know how to solve this problem, e.g. by using dynamic programming techniques, and we know, under fairly general conditions, the solution converges to a steady state, where all per-capita terms remain constant thereafter. Therefore, the aggregate variables will grow at the rate of population growth, γ. That is the economy converges to a balanced growth path, without per-capita growth.

11.1.2. Labor Augmenting Productivity Growth

Next, instead of a constant growth of population, suppose population remains constant, which we normalize to one. However, labor force becomes constantly more productive (e.g. as a result of learning by doing). This is known as *labor augmenting* productivity growth. If we let γ denote the growth rate of labor productivity, a technology that captures this idea is given by:

$$Y_t = AF\left(K_t, \gamma^t N_t\right),$$

where N_t is aggregate labor demand, and F is the standard homogenous production function.

Like the case in previous section, we may write the homogenous production function as:

$$AF\left(K_t, \gamma^t N_t\right) = A\left[K_t F_K\left(K_t, \gamma^t N_t\right) + \gamma^t N_t F_N\left(K_t, \gamma^t N_t\right)\right].$$

Noting that the right hand side of this equality is the labor plus capital share of output, we can write a representative household's problem in this economy as:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t U\left(c_t\right) \\ s.t. \quad & c_t + x_t \leq AF\left(k_t, \gamma^t n_t\right) \\ & k_{t+1} = \left(1-\delta\right) k_t + x_t \\ & n_t \leq 1, \end{aligned}$$

where we have written all the variables in per-capita terms. Using the fact that the production function is homogenous of degree one, we can follow a similar argument as in the previous section to write the budget constraint and law of motion of capital as:

$$\begin{aligned} & \frac{c_t}{\gamma^t} + \frac{x_t}{\gamma^t} \leq AF\left(\frac{k_t}{\gamma^t}, 1\right), \\ & \gamma \frac{k_{t+1}}{\gamma^{t+1}} = \left(1-\delta\right)\frac{k_t}{\gamma^t} + \frac{x_t}{\gamma^t}. \end{aligned}$$

If we define $\hat{c}_t c_t/\gamma^t$, $\hat{x}_t x_t/\gamma^t$, and $\hat{k}_t k_t/\gamma^t$ as consumption, investment, and capital *per unit of effective* *labor* - or *per productivity* - (note that $n_t = 1$, for all $t \geq 0$), we can write household's problem as:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t U\left(\gamma^t \hat{c}_t\right) \\ s.t. \quad & \hat{c}_t + \gamma \hat{k}_{t+1} \leq AF\left(\hat{k}_t, 1\right) + (1-\delta)\,\hat{k}_t. \end{aligned}$$

At this point, this does not look like a problem that we know how to solve; the difficulty is that, in general, utility can grow unbounded, even when $\hat{c}$ converges to a bounded value. To deal with this, let us assume a constant relative risk aversion utility form. Then, we can write preference as:

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t U\left(\gamma^t \hat{c}_t\right) &= \sum_{t=0}^{\infty} \beta^t \frac{\left(\gamma^t \hat{c}_t\right)^{1-\sigma} - 1}{(1-\sigma)} \\ &= \sum_{t=0}^{\infty} \beta^t \frac{\left(\gamma^{1-\sigma}\right)^t \hat{c}_t^{1-\sigma} - 1}{(1-\sigma)} \\ &= \sum_{t=0}^{\infty} \left(\beta \gamma^{1-\sigma}\right)^t \frac{\hat{c}_t^{1-\sigma} - \left(\gamma^{\sigma-1}\right)^t}{(1-\sigma)}. \end{aligned}$$

An affine transformation of this function would lead us to write the problem as:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \left(\beta \gamma^{1-\sigma}\right)^t U\left(\hat{c}_t\right) \\ s.t. \quad & \hat{c}_t + \gamma \hat{k}_{t+1} \leq AF\left(\hat{k}_t, 1\right) + (1-\delta)\,\hat{k}_t. \end{aligned}$$

This is a problem we are familiar with; when $\beta\gamma^{1-\sigma} \in (0,1)$, it is a neoclassical growth model, where are the variables converge to constant values. This implies the allocation, in effective labor terms, converges, while the per-capita terms grow at the same rate as the rate of growth of labor productivity, γ. Also note that constant relative risk aversion assumption on the utility form is crucial for this result.

11.2. Endogenous Growth

So far, we have considered models where the growth rate has been determined exogenously. Next, we will look into models in which the growth rate is chosen by the model itself.

We do know that for a fixed amount of labor, the curvature of our technology limits the growth due to diminishing marginal return on capital, and, with depreciation, there is an upper limit on capital accumulation. So if our economy is to experience sustainable growth for a long period of time, we either give up the curvature assumption on our technology, or we have to be able to shift our production function upwards. Given a fixed amount of labor, this shift is possible either by an increasing (total factor) productivity parameter or increasing labor productivity. The simplest of the models that incorporate this is the AK model, where the technology is linear in capital stock, so that the diminishing marginal returns on capital does not set in.

11.2.1. AK Model

As we said, when there is curvature in the production function (as a function of capital), there is no long-run growth. When the production function is linear in capital, there is a balanced growth path, but there is no transitional dynamics. From this simple observation, we can see that, in order to get long-run growth, we need a model that behaves similar to the AK growth model. One way to achieve this is to assume there is human capital, and production function exhibits constant returns to scale with respect to the physical and human capital. When investment in human capital is done using consumption goods, this economy behaves as an AK economy.[16] We are going to see more about this in the following section.

However, in this section, let us assume that the technology is given by an AK production function of the form

$$F(K) = AK,$$

for some productivity parameter A.

One way to achieve this production function is to assume the technology is given by

$$F(K, k, n) = Ak^{\theta} (Kn)^{1-\theta},$$

where K captures the effect of capital accumulation on the productivity of the labor force. If this effect is directly observable by the firms, we can assume a single producer in the economy with the production function

$$F(K, N) = \left(AN^{1-\theta}\right) K.$$

[16] Note that this is not true when time enters as a factor in the human capital accumulation.

The difficulty with this interpretation of the AK technology is that, it is far fetched that the firms can observe the effect of capital on the productivity of labor; i.e. it is more relevant to observe this effect as an externality. We are going to talk more about this in the following sections.

The planning problem in this environment is:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t U(c_t) \\ s.t. \quad & c_t + k_{t+1} = Ak_t \\ & c_t, k_{t+1} \geq 0. \end{aligned} \tag{33}$$

This problem will give rise to the following Euler equation:

$$U_c(c_t) = \beta U_c(c_{t+1}) A.$$

In case of constant relative risk aversion utility,

$$U(c) = \frac{c^{1-\sigma}}{(1-\sigma)}.$$

by substituting into the Euler equation, we get:

$$c_t^{-\sigma} = \beta A c_{t+1}^{-\sigma},$$

implying that:

$$\frac{c_{t+1}}{c_t} = (\beta A)^{\frac{1}{\sigma}}.$$

Therefore, on the balanced growth path, this model gives rise to a growth rate of consumption of $\gamma_c = (\beta A)^{\frac{1}{\sigma}}$.

We emphasize again that this model does not have a transition phase; we are on the balanced growth path from the beginning! Moreover, Problem (33) is a planning problem. Nevertheless, it is easy to decentralize the resulting allocations (there are no externalities in the economy).

11.2.2. Models with Human Capital

Let us assume that production technology is such that we have another production factor, namely human capital. Then followings are the feasibility condition and laws of motion for the physical and human capital:

$$\begin{aligned} C_t + I_t^K + \left[I_t^H\right] &= AK_t^\theta \left(H_t n_t^1\right)^{1-\theta} \\ K_{t+1} &= (1-\delta) K_t + I_t^K \\ \left[n_t^1 + n_t^2 = 1\right] & \\ C_t, K_{t+1} &\geq 0 \end{aligned}$$

Equations and variables in brackets appear when they are needed in the following specifications of law of motion for capital. Law of motion for the capital can be different for different interpretations of human capital formation. If we think human capital is built *with bricks* (time is not required to build up human capital), then

$$H_{t+1} = (1-\delta_h) H_t + I_t^H.$$

Under this specification, one can show that the model behaves very much like an AK model.

If we think human cpital is formed by learning by doing, then we can define individual human capital formation in two ways:

$$h_{t+1} = g\left(n_t^2, h_t\right),$$

or

$$h_{t+1} = g\left(n_t^2, H_t\right),$$

where in the second formulation, the aggregate human capital is the determinant of rate of growth of capital accumulation.

Lucas defined a human capital model with the specification of schooling and inelastic labor supply. Now that there is no limit to the accumulation of human capital, sustainable growth on a balanced growth path is feasible. Furthermore, an analysis of the characterization of the balanced growth path will indicate that this model indeed has transitional dynamics. So, unlike the AK model, if we start off the optimal growth path, economy can adjust and will converge to it by responding to prices in a decentralized setting. If one defines a learning by doing model, s/he can see there is a natural limit to the growth of human capital and such an economy might not have a balanced growth path. The key ingredient of endogenous growth with labor is then the reproducibility of the human capital without such a limit.

11.2.3. Romer (1986)'s Growth Model with Externalities

Next, let us consider a growth model with externality (Romer-86 was the first one to come up with this idea). We have seen in the AK model that the growth rate is determined solely by model primitives. Though endogenized, growth still is not directly or indirectly determined by the agents' choices. In Lucas' human capital model, the growth rate is determined by the choice of agents, specifically by the optimal ratio of human and physical capital. The source of growth in Lucas' model is reproducibility of human capital.

In the next model, we introduces the notion of externality generated by the aggregate capital stock to go through the problem of diminishing marginal returns to aggregate capital. In this model, the source of growth will be the aggregate capital accumulation, which is possible with a linear aggregate technology in capital as we saw in the AK model. The firms in our model will not be aware of this externality, will have the usual constant returns to scale technology, and observe total factor productivity as the main source of growth. As usual with externalities, the equilibrium outcome will not be optimal.

Suppose, each firm has the following technology:

$$y_t = A_t k_t^{\theta} \left(K_t n_t\right)^{1-\theta}.$$

We can write this technology as follows

$$y_t = \bar{A}_t k_t^{\theta} n_t^{1-\theta}$$

where $\bar{A}_t A K_t^{1-\theta}$. Then, firms' problem becomes:

$$\max \ \left(A K_t^{1-\theta}\right) k_t^{\theta} n_t^{1-\theta} - R_t k_t - w_t n_t,$$

giving rise to the following price functions:

$$R_t = \bar{A}_t \theta k_t^{\theta-1} n_t^{1-\theta},$$
$$w_t = \bar{A}_t (1-\theta) k_t^{\theta} n_t^{-\theta}.$$

By substituting these in the household's first order conditions, we get the following Euler equation:

$$u_c(c_t) = \beta\theta A u_c(c_{t+1}).$$

Assuming constant relative risk aversion utility form, we have:

$$\frac{c_{t+1}}{c_t} = (\beta\theta A)^{\frac{1}{\sigma}}.$$

Therefore, the economy grows on a balanced growth path.

By comparing these results with those in Section 11.2.1, we see that the resulting equilibrium in this economy is not optimal, naturally, because of the presence of externality.

11.2.4. Monopolistic Competition, Endogenous Growth, and R&D

Consider the following economy due to the highly cited model of endogenous growth of Romer (1990); there are three sectors in the economy: a final good sector, an intermediate goods sector, and an R&D sector. Final goods are produced using labor (as we will see there is only one wage, since there is only one type of labor) and intermediate goods according to the production function

$$N_{1,t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha}\, di$$

where $x(i)$ denotes the utilization of intermediate good of variety $i \in [0, A_t]$.[17] Note that marginal contribution of each variety is decreasing (since $\alpha < 1$), however, an increase in the number of varieties would increase the output. We will assume that the final good producers operate in a competitive market.

If the price of all varieties are the same, what is the optimal choice of input vector for a producer?

[17] The function that aggregates consumption of intermediate goods is often referred as Dixit-Stiglitz aggregator.

What if they do not have the same amount? Would a firm decide not to use a variety in the production?

Intermediate producers are monopolists that have access to a differentiated technology of the form:

$$x(i) = \frac{k(i)}{\eta}.$$

Therefore, they can end up charging a mark up above the marginal cost for their product. This is the main force behind research and development in this economy; developer of a new variety is the sole proprietor of the blue print that allows him earn profit. It is easy to observe that the aggregate demand of capital from the intermediate sector is $\int_0^{A_t} \eta x(i)\, di$.

The R&D sector in the economy is characterized by a flow of intermediate goods in each period; a new good is a new variety of the intermediate good. The flow of the new intermediate goods is created by using labor, according to the following production technology:

$$\frac{A_{t+1}}{A_t} = 1 + \xi N_{2,t}.$$

Notice that, after some manipulation, one can express growth in the stock of intermediate goods as follows:

$$A_{t+1} - A_t = A_t \xi N_{2,t}. \tag{34}$$

Hence, the flow of new intermediate goods is determined by the current stock of them in the economy. This type of externality in the model is the key propeller in the model. This assumption provides us with a constant returns to scale technology in the R&D sector. In what follows, we will assume that the inventors act as price takers in the economy.

The reason we see A_t on the right hand side of (34) as an externality is that it is indeed used as an input in the process of R&D, while, it is not paid for. Thus, inventors, in a sense, do not pay the investors of the previous varieties, while using their inventions. They only pay for the labor they hire. Perhaps, the basic idea of this production function might be traced back to *Isaac Newton*'s quote: "*If I have seen further, it is only by standing on the shoulders of giants*".

The preferences of the consumers are represented by the following utility function:

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

and their budget constraint in period t is give by:

$$c_t + k_{t+1} \leq r_t k_t + w_t + (1-\delta)\, k_t.$$

In this economy, GDP, in terms of gross product, is given by:

$$Y_t = W_t + r_t K_t + \delta K_t + \pi_t,$$

where π_t in the net profits. On the other hand, in terms of expenditures, GDP is:

$$Y_t = C_t + I_t = C_t + K_{t+1} - (1-\delta)\, K_t + p_t\,(A_{t+1} - A_t)\,,$$

where p_t is the price of new inventions, and the last two terms, $K_{t+1} - (1-\delta)\, K_t$ and $P_t\,(A_{t+1} - A_t)$, are the investment in physical capital and R&D. At last, in terms of value added, it is given by:

$$Y_t = N_t^{\alpha} \int_0^{A_t} X_t\,(i)^{1-\alpha}\, di + p_t\,(A_{t+1} - A_t)\,.$$

Certainly, this is not a model that one can map to the data. Instead it has been carefully crafted to deliver what is desired and it provides an interesting insight in thinking about endogenous growth.

Solving the Model Let's first consider the problem of a final good producer; in every period, he chooses N_{1t} and $x_t\,(i)$, for every $i \in [0, A_t]$, in order to solve:

$$\max\ N_{1,t}^{\alpha} \int_0^{A_t} x_t\,(i)^{1-\alpha}\, di - w_t N_{1,t} - \int_0^{A_t} q_t\,(i)\, x_t\,(i)\, di,$$

where $q_t\,(i)$ is the price of variety i in period t. First order conditions for this problem are:

1. $N_{1,t}$: $\alpha N_{1,t}^{\alpha-1} \int_0^{A_t} x_t\,(i)^{1-\alpha}\, di = w_t$; and,
2. $x_t\,(i)$: $(1-\alpha)\, N_{1,t}^{\alpha} x_t\,(i)^{-\alpha} = q_t\,(i)$, for all $i \in [0, A_t]$.

From the second condition, one obtains:

$$x_t\,(i) = \left(\frac{(1-\alpha)}{q_t\,(i)}\right)^{\frac{1}{\alpha}} N_{1,t}, \tag{35}$$

which, given N_{1t}, is the *demand function* for variety i, by the final good producer.

Next, let's consider the problem of an intermediate firm; these firms acts as price setters. The reason is the ownership of a differentiated patent, whose sole owner is the intermediate good producer of variety i. In addition, as long as $\alpha < 1$, this variety does not have a perfect substitute, and always demanded in the equilibrium. Therefore, their problem is to choose $q_t(i)$, in order to solve:

$$\begin{aligned} \pi_t(i) = \max \quad & q_t(i)\, x_t(q_t(i)) - r_t \eta x_t(q_t(i)) \\ s.t. \quad & x_t(q_t(i)) = \left(\frac{(1-\alpha)}{q_t(i)}\right)^{\frac{1}{\alpha}} N_{1,t}, \end{aligned}$$

where $x_t(q_t(i))$ is the demand function, substituted from (35). Notice that we have substituted for the technology of the monopolist, $x(i) = k(i)/\eta$. First order condition for this problem, with respect to $q_t(i)$, is:

$$x_t(q_t(i)) + (q_t(i) - r_t\eta)\frac{\partial x_t(q_t(i))}{\partial q_t(i)} = 0,$$

which can be written as

$$\frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1}{\alpha}}} N_{1,t} = \frac{(q_t(i) - r_t\eta)}{\alpha}\frac{(1-\alpha)^{\frac{1}{\alpha}}}{q_t(i)^{\frac{1+\alpha}{\alpha}}} N_{1,t}.$$

Rearranging yields:

$$q_t(i) = \frac{1}{(1-\alpha)} r_t \eta. \tag{36}$$

This is the familiar pricing function of a monopolist; price is marked-up above the marginal cost.

By substituting (36) into (35), we get:

$$x_t(i) = \left[\frac{(1-\alpha)^2}{r_t\eta}\right]^{\frac{1}{\alpha}} N_{1,t}, \tag{37}$$

and the demand for capital services is simply $\eta x_t(i)$. In a symmetric equilibrium, where all the intermediate good producers choose the same pricing rule, we have:

$$\int_0^{A_t} x_t(i)\, di = A_t x_t = \frac{k_t}{\eta},$$

where x_t is the common supply of intermediate goods. Therefore:

$$x_t = \frac{k_t}{\eta A_t}.$$

Moreover, if we let Y_t be the production of the final good; by plugging (37) we get:

$$Y_t = N_{1,t} A_t \left[\frac{(1-\alpha)^2}{r_t \eta} \right]^{\frac{1-\alpha}{\alpha}}. \tag{38}$$

Hence the model displays constant returns to scale in $N_{1,t}$ and A_t.

Let us study the problem of the R&D firms, next; a representative firm in this sector (recall that this is a competitive sector) chooses $N_{2,t}$, in order to solve the following problem:

$$\max_{N_{2,t}} \quad p_t A_{t-1} \xi N_{2,t} - w_t N_{2,t}.$$

The first order condition for this problem implies:

$$p_t = \frac{w_t}{A_{t-1}\xi}.$$

In summary, there are two equations to be solved form; one relating the choice of consumption versus saving (or capital accumulation), and one dividing labor demand for R&D, and that for final good production. Consumption-investment decisions result from solving household's problem in equilibrium, and the corresponding Euler equation:

$$u'(c_t) = \beta u'(c_{t+1}) \left[r_{t+1} + (1-\delta) \right].$$

For determining the labor choices $N_{1,t}$ and $N_{2,t}$, first note that the demand for patterns produced by R&D sector, is from the prospect monopolists. As long as there is positive profit from buying demand, the new monopolists would keep entering markets in a given period. This fact derives the profits of prospect monopolists to zero. So, the lifetime profit of the monopolist, must be equal to the price he pays for the blueprints;

$$p_t = \sum_{s=t}^{\infty} \left(\prod_{\tau=t}^{s} \frac{1}{1+r_\tau} \right) \pi_s.$$

This completes the solution to the model. Notice that output can grow at the same rate as A_t, from Equation (38). In addition, K_t grows at the same rate. As a result, the rate of growth of A_t would be the important aspect of equilibrium. For instance, if A_t grows at rate γ in the long-run, we have a balanced growth path in equilibrium. This growth comes from the externality in the R&D sector. Without that, we cannot get sustained growth in this model. The nice thing about this model is how neat is is in delivering the balanced growth, with just enough structure imposed on the conomy.

12. Some Applications

In this section, we give a few examples on different ways we can use the tools and instruments acquired so far, to study economic situations of interest.

12.1. An Overlapping Generation Model

Imagine an overlapping generation economy in which times is discrete and goes on forever. In each period t, a new generation is born, that live for I periods. During their life-cycle, agents deliver a sequence of effective labor, denoted by $\{\epsilon_i\}_{i=1}^{I}$. Therefore, the total labor in the economy, in a given period t is

$$N_t = \sum_{i \in I} \epsilon_i \mu_{i,t}, \tag{39}$$

where $\mu_{i,t}$ is the total number of people of age i, alive in date t. If we don't have population growth and normalize it to one, we have $\mu_{i,t} = 1$. If we let $A_{i,t}$ denote the asset holdings of agents of age i in period t, the aggregate capital is given by

$$K_t = \sum_{i \in I} A_{i,t}. \tag{40}$$

To model this economy in recursive form, the asset holdings of each generation alive in a certain period, say $A\{A_i\} \in \mathbb{R}^I$, would be the aggregate state. However, if we assume the generations cannot leave bequests, then $A_1 = 0$, and, hence, in practice, the asset holdings of $I-1$ generations would be required as states, $\{A_i\} \in \mathbb{R}^{I-1}$.

Then, the dynamic program for an agent of age i, becomes:

$$
\begin{aligned}
V^{i}(A, a) = \max_{c, a'} \quad & \left\{ u(c) + \beta V^{i+1}(A', a') \right\} \\
s.t. \quad & c + a' = R(K)\, a + \epsilon_i w(K) \\
& A' = G(A),
\end{aligned}
$$

where G is a vector valued function. Assuming a neoclassical production function, the prices are given by:

$$
\begin{aligned}
R(K) &= F_k(K, N), \\
w(K) &= F_n(K, N),
\end{aligned}
$$

where K and N are given in (39) and (40). To solve this problem, we have to note that the value of an agent at the terminal point of his life-cycle is given by:

$$
V^{I+1}(A, a) = 0,
$$

for all a. Therefore $a_{I+1} = 0$ would be the optimal choice. Then, we can solve backward, to pin down the sequence of value functions.

This simple model is supposed to serve as a way of thinking about the problem of agents, that more closely represent their decision making during life-cycle. We can add twists to this model in several ways to study certain economic aspects of human agents' life-cycles.

12.2. An Entrepreneurship Problem

In this section, we will present a simple model to study the idea of entrepreneurship. To this end, consider the farmer's problem studied before one more time:

$$
\begin{aligned}
V(s, a) = \max_{c, a'} \quad & \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \right\} \\
s.t. \quad & c + a' = sw + Ra \\
& c, a' \geq 0.
\end{aligned}
$$

We talked about the resulting policy functions when $S = \left\{ s^1, s^2, \ldots, s^N \right\}$ (see Appendix 14).

Now, imagine that s is a vector $s = (s_1, s_2)$, where $s_1 \in S_1 = \left\{s^1, s^2, \dots, s^N\right\}$, and $s_2 \in S_2 = \left\{0, s^b, s^m, s^g\right\}$. A farmer, now can decide to work as a worker, in which case his problem would be as before,

$$\begin{aligned} V^w(s,a) = \max_{c,a'} & \quad \left\{ u(c) + \beta \sum_{s'} \Gamma^w_{ss'} V(s',a') \right\} \\ s.t. & \quad c + a' = sw + Ra \\ & \quad c, a' \geq 0, \end{aligned}$$

or to be an entrepreneur, in which case his problem becomes:

$$\begin{aligned} V^e(s,a) = \max V_1, \max_{c,a',k} & \quad \left\{ u(c) + \beta \sum_{s'} \Gamma^e_{ss'} V(s',a') \right\} \\ s.t. & \quad c + a' = s_2 f(k) w + R(k,a)(a-k) \\ & \quad k \in \phi[s_2, a]. \end{aligned}$$

In this problem, ϕ captures different frictions that can restrict the investment of an entrepreneur. For instance, ϕ can represented by a *Kiyotaki–Moore collateral constraint* of the form:

$$k \leq \frac{a}{(1-\lambda)}.$$

On the other hand, f captures the production venture; it can be thought of as a static problem of production using a neoclassical technology. R has be made a function of k and a to capture the idea that borrowing and lending can entail different rates of return. We also assume the evolution of s depends on the choice of being a worker or an entrepreneur. Then, the value for an agent who receives shock s, and has asset holdings of a is:

$$V(s,a) = \max \left\{V^w(s,a), V^e(s,a)\right\}.$$

12.3. Aggregate Shocks

In this section, we consider an economy that is subject to both aggregate and idiosyncratic shocks, at the same time; consider Aiyagari (1994)'s economy again, now, with a production function that is subject to an aggregate shock; $zF\left(K, \bar{N}\right)$.

Let X be the distribution of types; then the aggregate capital is given by:

$$K = \int a dX(s, a).$$

$$K' = G(z, K)$$

The question is what are the sufficient statistics for predicting the aggregate capital stock and, consequently, prices tomorrow? Are z and K sufficient detemine capital tomorrow? The answer to these questions is no, in general; this is true if, and only if, the decision rules are linear. Therefore, X, the distribution of types becomes a state variable (even in the stationary equilibrium) for this economy.

Then, the problem of an individual becomes:

$$\begin{aligned} V(z, X, s, a) = \max_{a'} \quad & \left\{ u(c) + \beta \sum_{z', s'} \Pi_{zz'} \Gamma^{w}_{ss'} V(s', a') \right\} \\ s.t. \quad & c + a' = azf_k\left(K, \bar{N}\right) + szf_n\left(K, \bar{N}\right) \\ & K = \int a dX(s, a) \\ & X' = G(z, X) \\ & c, a' \geq 0. \end{aligned}$$

Computationally, this problem is a beast! So, how can we solve it? To provide some idea, we will first consider an economy with *dumb* agents!

Consider an economy in which people are stupid; people believe tomorrow's capital depends only on K, and not X. This, obviously, is not an economy in which expectations are rational. Nevertheless, people's problem in such settings becomes:

$$\begin{aligned} \widetilde{V}(z, X, s, a) = \max_{a'} \quad & \left\{ u(c) + \beta \sum_{z', s'} \Pi_{zz'} \Gamma^{w}_{ss'} V(s', a') \right\} \\ s.t. \quad & c + a' = azf_k\left(K, \bar{N}\right) + szf_n\left(K, \bar{N}\right) \\ & K = \int a dX(s, a) \\ & X' = \widetilde{G}(z, K) \\ & c, a' \geq 0. \end{aligned}$$

Next step is to allow people become slightly smarter; they now can use extra information, like mean and variance of X, to predict X'. Does this economy work better than our *dumb benchmark*? Computationally no! This answer, as stupid as it may sound, has an important message: people actually act linearly in the economy; decision rules are approximately linear. Therefore, we may use Aiyagari-94's results without fear; the approximations are quite reliable!

13. Competitive Search: Another Perspective

In Section 7.1, we saw a model of *competitive search* in the context of a *Lucas tree economy*. In what follows, we will try to give another interpretation to this model. It is important to note that, our goal here is not to introduce a new model; we solely want to get to the same equilibrium concept as in Section 7.1, from a different path.[18]

We start by describing the economy, which is the same economy as in Section 7.1. After that, we set up individuals problem, firm's problem, and define an equilibrium for this economy. We finish by a brief comparison of the results and the ones in Section 7.1.

13.1. The Economy

Consider an economy populated by a continuum of individuals, and a continuum of restaurants, of unit mass. Restaurants are assumed to be ex-post identical in the economy; i.e. they all own identical buildings and equipment, similar number of staff and chefs, etc.

Time is discrete, and, in each period, individuals want to eat food (in one or several restaurants). But, aside from costing money, going to a restaurant, ordering their favorite food, and eating food requires time. Perhaps weirdly, individuals in this world dislike spending (too much) time in the restaurant. However, the level of disutility they get from spending time in the restaurant, depends on the weather outside.

In general, spending d units of time in a restaurant, eating c units of food, while the weather outside is $\theta \in \Theta$, would give an individual an instantaneous utility of $u(c, d, \theta)$. However, as you might expect, the time required to eat c units of food is not independent of the choice of the restaurant, and how crowded it is. To see how this works in our economy, we now turn to the problem of a restaurant.

[18] This section was prepared by *Keyvan Eslami*, at the *University of Minnesota*.

A restaurant manager, in each period, must decide about its marketing strategy; it must choose to be either a crowded restaurant, that serves cheap (though high quality) food, or a classy one, without many costumers, but serving expensive food (with the same quality). We can model this marketing strategy by imagining sub-markets in different parts of the city, that differ in how crowded they are. In any given part of the city, the amount of time spent by the costumers (eating food) per number of restaurants determines the amount of congestion in each restaurant. If we let D be the amount of time individuals spend in a sub-market, and T be the number of restaurants in that sub-market, $1/QD/T$ would be an index of market tightness. Given D and T, the number of transactions (or matches, as it is known) between the restaurants and individuals is given by a Cobb-Douglas matching function:

$$M(T, D) = D^{\varphi} T^{1-\varphi}.$$

Therefore, for a restaurant that chooses the strategy of going to a sub-market with D consumers and T restaurants, the number of customers served is going to be

$$\Psi_t(Q) \frac{D^{\varphi} T^{1-\varphi}}{T} = Q^{-\varphi}.$$

On the other hand, for an individual to go to a sub-market (in search of a restaurant) with D other customers and T restaurants, the average amount of time she has to wait to be served one meal (one unit of consumption good) is

$$\frac{1}{\Psi_d(Q)} \frac{1}{\frac{D^{\varphi} T^{1-\varphi}}{D}} = \frac{1}{Q^{1-\varphi}}.$$

Given the market tightness Q in a sub-market, price of each serving of food is determined by the interaction of demand and supply. Therefore, we may write the price of food as a function of the market tightness. We denote this price by $1/P(Q)$. To complete our environment, we will assume that there is a holding, operating in a perfectly competitive financial market, owning all the restaurants, and individuals own shares of this holding.

13.2. Problem of Individuals

An individual who holds s shares of the holding, when the (aggregate) state of the economy is θ, solves the following recursive problem:[19]

$$V(\theta, s) = \max_{Q,c,d,s'} \left\{ u(c, d, \theta) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V(\theta', s') \right\} \tag{41}$$

$$s.t. \quad d = \frac{c}{\Psi_d(Q)} \tag{42}$$

$$\frac{1}{P(Q)} c + s' = s(1 + R), \tag{43}$$

where she chooses the sub-market in which she plans to eat, Q, the amount she wants to eat, c, the time she has to spend eating in the sub-market, d, and the number of shares she wants to hold in the next period. R is the return or dividends per unit of shares, measured in terms of share. Constraint (42) determines the (minimum) quantity of time the individual has to spend in the sub-market to consume c servings of food, given that the average waiting time for one serving per consumer is $1/\Psi_d(Q)$ in sub-market Q. Constraint (43) is simply the budget constraint of the individual.[20] Noticing that, given the amount of time she spends in the market d, the market tightness Q effectively determines the number of servings c, we can rewrite this problem as:

$$V(\theta, s) = \max_{Q,d} \left\{ u(d\Psi_d(Q), d, \theta) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V\left(\theta', s(1 + R) - \frac{d\Psi_d(Q)}{P(Q)}\right) \right\}. \tag{44}$$

Fist order conditions, for the maximization problem on the right hand side of this functional equation are:

[19] Note that, individual is assumed to be choosing only one sub-market in each period. In principle, this needs not be the case. However, under the assumption that in the equilibrium only one sub-market is operational, relaxing this assumption would not change anything.

[20] Note that, if we divide this constraint by $P(Q)$, we can normalize the price of food to one. In this case, the effective price of each share of the holding, for an individual in a sub-market with tightness Q, in terms of food, would be $P(Q)$.

1.

$$\Psi_d(Q)\,u_c(d\Psi_d(Q),d,\theta)+u_d(d\Psi_d(Q),d,\theta)$$
$$-\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{\Psi_d(Q)}{P(Q)}V_s\left(\theta',s(1+R)-\frac{d\Psi_d(Q)}{P(Q)}\right)=0,$$

and,

2.

$$d\Psi_d'(Q)\,u_c(d\Psi_d(Q),d,\theta)-$$
$$\beta\sum_{\theta'}\Gamma_{\theta\theta'}\left[\frac{d\Psi_d'(Q)\,P(Q)-P'(Q)\,d\Psi_d(Q)}{P^2(Q)}\right]V_s\left(\theta',s(1+R)-\frac{d\Psi_d(Q)}{P(Q)}\right)=0.$$

We can rewrite Condition 1 as:

$$u_c(d\Psi_d(Q),d,\theta)+\frac{u_d(d\Psi_d(Q),d,\theta)}{\Psi_d(Q)}$$
$$=\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{1}{P(Q)}V_s\left(\theta',s(1+R)-\frac{d\Psi_d(Q)}{P(Q)}\right).\quad(45)$$

By the Envelope theorem:

$$V_s(\theta,s)=\lambda P(Q)(1+R),\quad(46)$$

where $\lambda P(Q)$ is the Lagrange multiplier on Constraint (43). The first order condition of Problem (41) with respect to c implies:

$$\lambda=u_c\left(c,\frac{c}{\Psi_d(Q)},\theta\right)+\frac{1}{\Psi_d(Q)}u_d\left(c,\frac{c}{\Psi_d(Q)},\theta\right).$$

If we substitute this into Equation (46), we get:

$$V_s(\theta,s)=\left[u_c(c,d,\theta)+\frac{1}{\Psi_d(Q)}u_d(c,d,\theta)\right]P(Q)(1+R),\quad(47)$$

where the derivatives on the right hand side are evaluated at the optimal values. As a result, we may write (45) as:

$$
\begin{aligned}
&u_c\left(d\Psi_d\left(Q\right), d, \theta\right) + \frac{u_d\left(d\Psi_d\left(Q\right), d, \theta\right)}{\Psi_d\left(Q\right)} \\
&\quad = \beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{P\left(Q'\right)\left(1+R'\right)}{P\left(Q\right)}\left[u_c\left(d'\Psi_d\left(Q'\right), d', \theta'\right) + \frac{1}{\Psi_d\left(Q'\right)}u_d\left(d'\Psi_d\left(Q'\right), d', \theta'\right)\right]. \quad (48)
\end{aligned}
$$

Condition 2 above can also be simplified further as:

$$
\begin{aligned}
u_c\left(d\Psi_d\left(Q\right), d, \theta\right) =& \beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{1}{\Psi_d'\left(Q\right)}\left[\frac{\Psi_d'\left(Q\right)P\left(Q\right) - P'\left(Q\right)\Psi_d\left(Q\right)}{P^2\left(Q\right)}\right] \\
&\times V_s\left(\theta', s\left(1+R\right) - \frac{d\Psi_d\left(Q\right)}{P\left(Q\right)}\right) \\
=& \beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{1}{P\left(Q\right)}\left[1 - \frac{P'\left(Q\right)}{P\left(Q\right)}\frac{\Psi_d\left(Q\right)}{\Psi_d'\left(Q\right)}\right] \\
&\times V_s\left(\theta', s\left(1+R\right) - \frac{d\Psi_d\left(Q\right)}{P\left(Q\right)}\right). \quad (49)
\end{aligned}
$$

13.3. Firms' Problem

A restaurant manager, in any given period, tries to maximize the returns to the holding by choosing the right marketing strategy, Q. Q determines the number of foods the restaurant gets to serve in that sub-market. Note that, by choosing a sub-market Q, the restaurant is effectively choosing the price of its food, $1/P\left(Q\right)$.

Since there is nothing dynamic in the choice of marketing strategy (note that, we are assuming restaurants can choose a different sub-market in each period), we may write the problem of a manager as:

$$
\pi = \max_{Q} \frac{1}{P\left(Q\right)}\Psi_t\left(Q\right). \quad (50)
$$

The first order condition for the optimal choice of Q is

$$
-\frac{1}{P^2\left(Q\right)}\Psi_t\left(Q\right)P'\left(Q\right) + \frac{1}{P\left(Q\right)}\Psi_t'\left(Q\right) = 0,
$$

which can be written as:

$$\Psi_t(Q)\,P'(Q) = P(Q)\,\Psi_t'(Q).$$

This condition determines the price as a function of Q as:

$$\frac{P'(Q)}{P(Q)} = \frac{\Psi_t'(Q)}{\Psi_t(Q)}. \tag{51}$$

13.4. Equilibrium

Before defining the equilibrium concept for this economy, let's simplify the optimality conditions we have derived so far, a bit further; note that, by the choice of a Cobb-Douglas matching function, we have:

$$\begin{aligned}
&\Psi_t'(Q) = -\varphi Q^{-\varphi-1},\\
&\therefore \frac{\Psi_t'(Q)}{\Psi_t(Q)} = -\frac{\varphi Q^{-\varphi-1}}{Q^{-\varphi}} = -\frac{\varphi}{Q},
\end{aligned} \tag{52}$$

and:

$$\begin{aligned}
&\Psi_d'(Q) = (1-\varphi)\,Q^{-\varphi},\\
&\therefore \frac{\Psi_d'(Q)}{\Psi_d(Q)} = \frac{(1-\varphi)\,Q^{-\varphi}}{Q^{1-\varphi}} = \frac{(1-\varphi)}{Q}.
\end{aligned} \tag{53}$$

By Equations (51) and (52), we have:

$$\frac{P'(Q)}{P(Q)} = -\frac{\varphi}{Q}. \tag{54}$$

By substituting Equations (53) and (54) into (49), we get:

$$\begin{aligned}
u_c\left(d\Psi_d(Q), d, \theta\right) =& \beta \sum_{\theta'} \Gamma_{\theta\theta'} \frac{1}{P(Q)} \left[1 + \frac{\varphi}{Q}\frac{Q}{(1-\varphi)}\right] V_s\left(\theta', s(1+R) - \frac{d\Psi_d(Q)}{P(Q)}\right)\\
=& \frac{1}{(1-\varphi)}\beta \sum_{\theta'} \Gamma_{\theta\theta'} V_s\left(\theta', s(1+R) - \frac{d\Psi_d(Q)}{P(Q)}\right)\frac{1}{P(Q)}.
\end{aligned} \tag{55}$$

Substituting from Equation (47), we can write this equality as:

$$u_c\left(d\Psi_d\left(Q\right), d, \theta\right) = \frac{1}{(1-\varphi)}\beta\sum_{\theta'}\Gamma_{\theta\theta'}\frac{P\left(Q'\right)\left(1+R'\right)}{P\left(Q\right)} \times \left[u_c\left(d'\Psi_d\left(Q'\right), d', \theta'\right) + \frac{1}{\Psi_d\left(Q'\right)}u_d\left(d'\Psi_d\left(Q'\right), d', \theta'\right)\right]. \quad (56)$$

Note that, since all the restaurants are assumed to be ex-ante identical, the value of the restaurant in a given period, π in Problem (50), would be the same for all the restaurants; if not, a restaurant manager has a profitable deviation from his marketing strategy, that he is not exhausting. This is what is paid as dividends to individuals, since the holding is operating in a perfectly competitive market.

Now, we are ready to define an equilibrium of this economy.

Definition 18 *An equilibrium with competitive search in this economy consists of a collection of functions,* $c : \Theta \to \mathbb{R}_+$, $d : \Theta \to \mathbb{R}_+$, $s : \Theta \to [0,1]$, $P : \mathbb{R}_+ \to \mathbb{R}_+$, $Q : \Theta \to \mathbb{R}_+$, $\pi : \Theta \to \mathbb{R}_+$, *and* $R : \Theta \to \mathbb{R}_+$, *that satisfy:*

1. *individuals' shopping and budget constraints, Constraints* (42) *and* (43)*,*
2. *individual's Euler equation, Equation* (48)*,*
3. *restaurants' first order condition, Equation* (56)*,*
4. *restaurants' value function, Identity* (50)*, and,*
5. *market clearing conditions;* $s(\theta) = 1$, $Q(\theta) = d(\theta)$, *and* $R(\theta) = \pi(\theta)$, *for all* $\theta \in \Theta$.

13.5. Concluding Remarks

To conclude, we will discuss some of properties that connect our economy to the one in Section 7.1.

First of all, note that the first order conditions of the individual's and firm's problems are exactly the same; this, once more, depicts the fact that these two are the same economies.

Notice that our definition of equilibrium in this section lacks one condition that appears in our previous definition; individual's participation constraint (Constraint (9) in the Section 7.1). The reason is that, in our definition here, we have not added $\bar{V}$ as a part of the equilibrium concept; this variable is no longer a part of firm's problem. The intuition behind this is that, in this new interpretation, by choosing the sub-market Q, restaurants are assumed to be implicitly choosing the price, while the price is assumed to be determined in equilibrium in the interaction of demand and supply. What appears as a constraint in Constraint (9), is to make sure firms are choosing a market that is operational. This condition is dropped here, by making the assumption that by choosing a sub-market Q, price in that market is no longer a choice of the restaurant. This combination of price and market tightness deters restaurants from choosing sub-markets that are not operational in the equilibrium; one attribute of the good, is made a function of the other in equilibrium.

At last, we should note that there is no reason that the equilibrium market tightness Q, and therefore the equilibrium price $P(Q)$, are unique, at this level of generality. In fact, if the individuals are not ex-ante identical, e.g. they hold different quantities of shares, we might end up with different sub-markets being operational in the equilibrium. This might cause us a bit of difficulty, specially in our definition of equilibrium; in this case, Q must be set equal to the measure of people divided by the measure of restaurants in each operational sub-market. Therefore, in our definition of equilibrium, we implicitly assume there is a unique sub-market operational in equilibrium.

14. A Farmer's Problem: Revisited

Consider the following problem of a farmer that we studied in class:

$$\begin{aligned} V(s,a) = \max_{c,a'} \quad & \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \right\} \\ s.t. \quad & c + qa' = a + s \\ & c \geq 0 \\ & a' \geq 0. \end{aligned} \tag{57}$$

As we discussed, we are in particular interested in the case where $\beta/q < 1$. In what follows, we are going to show that, under monotonicity assumption on the Markov chain governing s, the optimal policy associated with (57) implies a finite support for the distribution of asset holding of the farmer, a.[21]

[21] This section was prepared by *Keyvan Eslami*, at the *University of Minnesota*. This section is essentially a slight variation on the proofs found in Huggett (1993). However, he accepts the responsibility for the errors.

Before we start the formal proof, suppose $s_{\min} = 0$, and $\Gamma_{ss_{\min}} > 0$, for all $s \in S$. Then, the agent will optimally always choose $a' > 0$. Otherwise, there is a strictly positive probability that the agent enters tomorrow into state $s_{\min}$, where he has no *cash in hand* ($a' + s_{\min} = 0$) and is forced to consume 0, which is extremely painful to him (e.g. when Inada conditions hold for the instantaneous utility). Hence he will raise his asset holding a' to insure himself against such risk.

If $s_{\min} > 0$, then the above argument no longer holds, and it is indeed possible for the farmer to choose zero assets for tomorrow.

Notice that the borrowing constraint $a' \geq 0$ is affecting agent's asset accumulation decisions, even if he is away from the zero bound, because he has an incentive to ensure against the risk of getting a series of bad shocks to s and is forced to 0 asset holdings. This is what we call *precautionary savings motive*.

Next, we are going to prove that the policy function associated with (57), which we denote by $a'(\cdot)$, is similar to that in Figure 1. We are going to do so, under the following assumption.

Assumption 1 *The Markov chain governing the state s is monotone; i.e. for any $s_1, s_2 \in S$, $s_2 > s_1$ implies $E(s|s_2) \geq E(s|s_1)$.*

It is straightforward to show that, the value function for Problem (57) is concave in a, and bounded. Now, we can state our intended result as the following theorem.

Theorem 4 *Under Assumption 1, when $\beta/q < 1$, there exists some $\hat{a} \geq 0$ so that, for any $a \in [0, \hat{a}]$, $a'(s, a) \in [0, \hat{a}]$, for any realization of s.*

To prove this theorem, we proceed in the following steps. In all the following lemmas, we will assume that the hypotheses of Theorem 4 hold.

Lemma 1 *The policy function for consumption is increasing in a and s;*

$$c_a(a, s) \geq 0 \text{ and } c_s(a, s) \geq 0.$$

By the first order condition, we have:

$$u'(c(s, a)) \geq \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} V_a\left(s', \frac{a + s - c(s, a)}{q}\right),$$

Figure 1: Policy function associated with farmer's problem.

Fig-1.jpg

with equality, when $a+s-c(s,a)>0$.

For the first part of the lemma, suppose a increases, while $c(s,a)$ decreases. Then, by concavity of u, the left hand side of the above equation increases. By concavity of the value function, V, the right hand side of this equation decreases, which is a contradiction.

For the the second part, we claim that $V_a(s,a)$ is a decreasing function of s. To show this is the case, firs consider the mapping T as follows:

$$\begin{aligned} Tv(s,a) = \max_{c,a'} \quad & \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} v(s',a') \right\} \\ s.t. \quad & c + qa' = a + s \\ & c \geq 0 \\ & a' \geq 0. \end{aligned}$$

Suppose $v_a^n(s,a)$ is decreasing in its first argument; i.e. $v_a^n(s_2,a) < v_a^n(s_1,a)$, for all $s_2 > s_1$ and $s_1, s_2 \in S$. We claim that, $v^{n+1} = Tv^n$ inherits the same property. To see why, note that for $a^{n+1}(s,a) = a'$ (where a^{n+1} is the policy function associated with n'th iteration) we must have:

$$u'(a+s-qa') \geq \frac{\beta}{q} \sum_{s'} \Gamma_{ss'} v_a^n(s',a'),$$

with strict equality when $a'>0$. For a fixed value of a', an increase in s leads to a decrease in both sides of this equality, due to the monotonicity assumption of Γ, and the assumption on v_a^n. As a result, we must have

$$u'\left(a+s_2-qa^{n+1}(s_2,a)\right) \leq u'\left(a+s_1-qa^{n+1}(s_1,a)\right),$$

for all $s_1 > s_2$. By Envelope theorem, then:

$$v_a^{n+1}(s_2,a) \leq v_a^{n+1}(s_1,a).$$

It is straightforward to show that v^n converges to the value function V point-wise. Therefore,

$$V_a(s_2,a) \leq V_a(s_1,a),$$

for all $s_2 > s_1$.

Now, note that, by envelope theorem:

$$V_a(s,a) = u'(c(s,a)).$$

As s increases, $V_a(s,a)$ decreases. This implies $c(s,a)$ must increase.

Lemma 2 *There exists some $\hat{a} \in \mathbb{R}_+$, such that $\forall a \in [0,\hat{a}]$, $a'(a, s_{\min}) = 0$.*

It is easy to see that, for $a = 0$, $a'(a, s_{\min}) = 0$. First of all, note the first order condition:

$$u_c(c(s,a)) \geq \frac{\beta}{q}\sum_{s'}\Gamma_{ss'}u_c(c(s', a'(s,a))),$$

with equality when $a'(s,a) > 0$. Under the assumption that $\beta/q < 1$, we have:

$$\begin{aligned} u_c(c(s_{\min},0)) =& \frac{\beta}{q}\sum_{s'}\Gamma_{s_{\min}s'}u_c(c(s', a'(s_{\min},0))) \\ <& \sum_{s'}\Gamma_{ss'}u_c(c(s', a'(s_{\min},0))). \end{aligned}$$

By Lemma 1, if $a' = a'(0, s_{\min}) > a = 0$, then $c(s', a') > c(s_{\min}, 0)$ for all $s' \in S$, which leads to a contradiction.

Lemma 3 *$a'(s_{\min}, a) < a$, for all $a > 0$.*

Suppose not; as we showed in Lemma 1, then $a'(s_{\min}, a) > a > 0$:

$$u_c(c(s_{\min},a)) < \sum_{s'}\Gamma_{ss'}u_c(c(s', a'(s_{\min},a))).$$

Contradiction, since $a'(s_{\min}, a) > a$, and $s' \geq s_{\min}$, and the policy function in monotone.

Lemma 4 *There exits an upper bound for the agent's asset holding.*

Suppose not; we have already shown that $a'(s_{\min}, a)$ lies below the 45 degree line. Suppose this is not true for $a'(s_{\max}, a)$; i.e. for all $a \geq 0$, $a'(s_{\max}, a) > a$. Consider two cases.

In the first case, suppose the policy functions for $a'(s_{\max}, a)$ and $a'(s_{\min}, a)$ diverge as $a \to \infty$, so that, for all $A \in \mathbb{R}_+$, there exist some $a \in \mathbb{R}_+$, such that:

$$|a'(s_{\max}, a) - a'(s_{\min}, a)| \geq A.$$

Since S is finite, this implies, for all $C \in \mathbb{R}_+$, there exist some $a \in \mathbb{R}_+$, so that

$$c(s_{\min}, a) - c(s_{\max}, a) \geq C,$$

which is a contradiction, since c is monotone in s.

Next, assume $a'(s_{\max}, a)$ and $a'(s_{\min}, a)$ do not diverge as $a \to \infty$. We claim that, as $a \to \infty$, c must grow without bound. This is quite easy to see; note that, by envelope condition:

$$V_a(s, a) = u'(c(s, a)).$$

The fact that V is bounded, then, implies that V_a must converge to zero as $a \to \infty$, implying that $c(s, a)$ must diverge to infinity for all values of s, as $a \to \infty$. But, this implies, if $a'(s_{\max}, a) > a$,

$$u_c(c(s_{\max}, a'(s_{\max}, a))) \to \sum_{s'} \Gamma_{s_{\max}s'} u_c(c(s', a'(s_{\max}, a))).$$

As a result, for large enough values of a, we may write:

$$\begin{aligned}
u_c(c(s_{\max}, a)) =& \frac{\beta}{q} \sum_{s'} \Gamma_{s_{\max}s'} u_c(c(s', a'(s_{\max}, a))) \\
<& \sum_{s'} \Gamma_{s_{\max}s'} u_c(c(s', a'(s_{\max}, a))) \\
\approx& u_c(c(s_{\max}, a'(s_{\max}, a))).
\end{aligned}$$

But, this implies:

$$c(s_{\max}, a) > c(s_{\max}, a'(s_{\max}, a)),$$

which, by monotonicity of policy function, means $a > a'(s_{\max}, a)$, and this is a contradiction.

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