Economics 245
Lecture 1: Differentiability

Cecilia Fieler

January 12, 2012

Leading Examples

- Find a firm's supply function:

$$\max_{y \geq 0} \; py - c(y).$$

- Find a firm's input demand functions:

$$\max_{x_1, x_2 \geq 0} \; pf(x_1, x_2) - w_1 x_1 - w_2 x_2.$$

Derivatives

The Usual Loose Definition. The “derivative” of a function f at a point a is the slope of the tangent line.

Definition (Better)

A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a iff it is "very well approximated" near a by a linear function of the form

$$g(x) = f(a) + (x - a)b.$$

The number b, the constant slope of this approximating function, is called the derivative of f at a. It is denoted as $f'(a)$ or $\frac{df(a)}{dx}$ or $Df(a)$.

What does very well approximated mean?

"Very Well Approximated"

- Two ways in which f may be approximated at a by $g(x) = f(a) + (x - a)b$:

  Way 1. $f(a) = g(a)$.
  Way 2. $\lim_{x \to a} [f(x) - g(x)] = 0$.

- These ways are not good enough. E.g., whether f is approximated by g in either way does not depend on the value of *b!*

The stronger notion of approximation we need is that as $x \to a$, the difference $f(x) - g(x)$ goes to zero so fast that

$$\text{Way 3.} \quad \lim_{x \to a} \frac{f(x) - g(x)}{x - a} = 0.$$

Since $g(x) = f(a) + (x - a)b$, this can be written as

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = b,$$

the usual beginner's definition of a derivative.

Definition (Proper)

A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a iff the limit,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

exists (i.e., yields the same number regardless of how x converges to a). When it exists, the limit is called the derivative of f at a.

Example

$f(x) = x^2$.
We of course know that $f'(a) = 2a$.
Here's the proof:

$$\begin{aligned}
\frac{f(x) - f(a)}{x - a} &= \frac{x^2 - a^2}{x - a} \\
&= \frac{(x-a)(x+a)}{x-a} \\
&= x + a \\
&\to 2a \text{ as } x \to a. \blacksquare
\end{aligned}$$

First-Order Expressions

- Given the differentiable f, the point a, and the approximating function

$$g(x) = f(a) + f'(a)(x - a),$$

we now change the variable from x to $h = x - a$, the "deviation" of x away from a.

- We also define the remainder function, $R(h;a)$, to be the difference between f and g at the point $x = a + h$:

$$\begin{aligned} R(h;a) &= f(a+h) - g(a+h) \\ &= f(a+h) - \left[f(a) + f'(a)h\right]. \end{aligned}$$

- Hence, $f(a+h)$ is the sum of the linear approximation and the remainder:

$$f(a+h) = f(a) + f'(a)h + R(h;a).$$

- Because f is differentiable, $R(h;a)/h \to 0$ as $h \to 0$.

- This is the meaning of the first-order expression,

$$f(a+h) \approx f(a) + f'(a)h.$$

In the texts you will see a first-order approximation written as

$$\Delta f \approx f'(a)\,\Delta x \quad (\text{or } \Delta y \approx f'(a)\,\Delta x),$$

which is the same as on the previous slide, with $\Delta f = f(a+h) - f(a)$ being the change in f and $\Delta x = h$ being the change in x.

You will also see this in differential terms:

$$df = f'(a)\,dx \quad (\text{or } dy = f'(a)\,dx).$$

This expression refers to movements along the tangent line, i.e., actually to changes in g rather than f.

Differentiable Monotonicity

Key Fact: If f is differentiable at a and the approximating linear function g is strictly increasing (decreasing), then f is strictly increasing (decreasing) in a neighborhood of a.

This is geometrically, "intuitively," obvious.

Formal statement, for the strictly increasing case:

Theorem

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a, with $f'(a) > 0$. Then $\delta > 0$ exists such that for any h satisfying $|h| < \delta$,

$$f(a+h) > f(a) \text{ if } h > 0,$$

$$f(a+h) < f(a) \text{ if } h < 0.$$

Proof.

- By definition of the remainder, for $h \neq 0$ we have

$$
\begin{aligned}
f(a+h) - f(a) &= f'(a)h + R(h;a) \\
&= \left[f'(a) + \frac{R(h;a)}{h} \right] h.
\end{aligned}
$$

- Since f is differentiable at a, $R(h;a)/h \to 0$ as $h \to 0$. Thus, since $f'(a) > 0$, there exists $\delta > 0$ such that

$$0 < |h| < \delta \quad \Longrightarrow \quad f'(a) + \frac{R(h;a)}{h} > 0.$$

- So, for any h satisfying $|h| < \delta$, $f(a+h) - f(a)$ and h have the same sign. This proves the theorem.

□

Consequence for Optimization

Corollary

If f is defined on an open interval, and it is maximized or minimized on this interval at a point a at which f is differentiable, then it satisfies the following necessary first-order condition:

$$\text{(FOC)} \quad f'(a) = 0.$$

So if we want to find the maximizer of a differentiable function f on an interval $[\underline{x}, \bar{x}]$, we can restrict attention to the endpoints, $\underline{x}$ and $\bar{x}$, and to the solutions of the equation $f'(x) = 0$, the critical points of f.

Examples

1. Maximize $f(x) = 6x - x^2$ on $[0, \infty)$.
   - Obviously 0 is not a maximizer. So any maximizer x^* is interior, and so satisfies

     $$f'(x^*) = 6 - 2x^* = 0.$$

     The only possible maximizer is therefore 3. If we knew from a separate argument that a maximizer exists, we would now know that it is 3.
   - The following is a direct proof that 3 is the maximizer:
     - $f'(x) > 0$ for any $x < 3$.
       $\Rightarrow f$ is strictly increasing on $[0, 3)$.
       $\Rightarrow f(3) > f(x)$ for all $x < 3$.
     - $f'(x) < 0$ for any $x > 3$.
       $\Rightarrow f$ is strictly decreasing on $(3, \infty)$.
       $\Rightarrow f(3) > f(x)$ for all $x > 3$.

Examples

2. Maximize $6x - x^2$ on $[0, 2]$.
   - Answer: $x^* = 2$, a corner. Note that $f'(x^*) = 2 \neq 0$.

3. Maximize $6x - x^2$ on $[0, 2)$.
   - Answer: no solution.

4. Minimize $6x - x^2$ on $[0, \infty)$.
   - Answer: No solution.

Examples

5. Maximize and minimize $f(x) = \sin x$ on $[0, 2\pi]$.

- Answer: maximizer is $\pi/2$, minimizer is $3\pi/2$.

  Proof. We know $f'(x) = \cos x$, the zeros of which are $\pi/2$ and $3\pi/2$. So the maximizer(s) lie in the set $\{0, \pi/2, 3\pi/2, 2\pi\}$. Checking each of these points yields $f(x) = 0, 1, -1$, and 0, respectively.

6. Maximize $f(x) = x^3$ on $\mathbb{R}$.

- Answer: no solution.

  Proof. There are no corners, and only one critical point:

  $$f'(x) = 3x^2 = 0 \iff x = 0.$$

  But 0 does not maximize x^3.

Multivariable Differentiability

A function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$ iff numbers $f_1(a), \dots, f_n(a)$ exist such that

$$\lim_{h \to 0} \frac{R(h;a)}{\|h\|} \to 0,$$

where $R(\cdot;a)$ is the remainder function defined at any $h \in \mathbb{R}^n$ by

$$f(a+h) = f(a) + \sum_{i=1}^{n} f_i(a) h_i + R(h;a).$$

So, f is differentiable at a iff it is approximated near a by the linear function

$$g(x) = f(a) + \sum_{i=1}^{n} f_i(a)\left(x_i - a_i\right).$$

- The numbers $f_i(a)$ are the partial derivatives of f at a, and are sometimes written as $\frac{\partial f}{\partial x_i}(a)$ or $D_i f(a)$.

- The gradient of f at a is the (column) vector of partial derivatives:

$$\nabla f(a) = \begin{pmatrix} f_1(a) \\ \vdots \\ f_n(a) \end{pmatrix}.$$

The gradient at a is sometimes denoted as $Df(a)$.

Unconstrained Multivariable Optimization

Theorem

If $x^* \in \mathbb{R}^n$ is a maximizer or a minimizer of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, then it satisfies the following necessary first-order condition:

$$\text{(FOC)} \quad \nabla f(x^*) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(We shall usually write the FOC as $\nabla f(x^*) = 0$, where it is understood that 0 is the n-vector of zeros.)

Proof. Same as for one variable. If $f_i(x^*) \neq 0$, consider deviations h such that $h_j = 0$ for all $j \neq i$.