

Economics 245

Lecture 2: Constrained Optimization

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Example: Input
Demand Functions

Consumer Problem

When is Budget
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Substitution
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Readings

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Example: Input Demand Functions

$$\max_{x_1, x_2 \geq 0} p(\sqrt{x_1} + \sqrt{x_2}) - w_1 x_1 - w_2 x_2.$$

The FOCs are

$$\frac{p}{2\sqrt{x_i}} - w_i = 0 \text{ for } i = 1, 2.$$

These are two equations in two unknowns, x_1 and x_2 .
Solving them yields the demand functions,

$$x_i^* = \left(\frac{p}{2w_i} \right)^2 \text{ for } i = 1, 2,$$

which express the firm's optimal inputs as functions of the three prices, (p, w_1, w_2) .

Consumer Problem

For positive prices p_1, \dots, p_n and income I , the *consumer problem* is

$$(CP) \quad \max_{x \in \mathbb{R}_+^n} u(x) \text{ such that}$$

$$(B) \quad \sum_{i=1}^n p_i x_i \leq I.$$

This is a constrained maximization problem, and we only know how to solve unconstrained problems.

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Consumer Problem: Example

Example $u(x_1, x_2) = x_1x_2$. For positive prices p_1, p_2 and income I , the *consumer problem* is

$$(CP) \quad \max_{x \in \mathbb{R}_+^n} x_1x_2 \text{ such that}$$

$$(B) \quad p_1x_1 + p_2x_2 \leq I.$$

The utility function is monotone. Therefore, the budget constraint is binding.

Budget Constraint is Binding: Proof

The utility function is monotone. Therefore, the budget constraint is binding.

Proof. Assume the opposite is true. That is, let x_1^* and x_2^* be a solution and

$$p_1x_1^* + p_2x_2^* < I.$$

Then, there exists $\Delta > 0$ such that

$$p_1(x_1^* + \Delta) + p_2(x_2^* + \Delta) \leq I.$$

and

$$(x_1^* + \Delta)(x_2^* + \Delta) > x_1^*x_2^*.$$

It follows that x_1^* and x_2^* cannot be a solution. A contradiction. *QED*

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Budget Constraint is Binding: Proposition

Definition

For any $A \subseteq \mathbb{R}^n$ and function $f : A \rightarrow \mathbb{R}$, f is *strictly increasing on A* iff $\forall x, y \in A$,

$$x \neq y \text{ and } x_i \leq y_i \forall i \Rightarrow f(x) < f(y).$$

Theorem

If u is strictly increasing on \mathbb{R}_+^n , then any solution of the consumer problem satisfies the budget constraint with equality (without slack).

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Consumer Problem, example

So the substitution method can be used to solve the consumer problem if u is strictly increasing on \mathbb{R}_+^n .

Back to the example. For positive prices p_1, p_2 and income I , the *consumer problem* is

$$(CP) \quad \max_{x \in \mathbb{R}_+^n} x_1 x_2 \text{ such that}$$

$$(B) \quad p_1 x_1 + p_2 x_2 \leq I.$$

We can express x_1 from the budget constraint:

$$x_1 = \frac{I - p_2 x_2}{p_1}$$

Consumer Problem, example

Substitution in the objective function gives:

$$\begin{aligned} \text{(CP)} \quad & \max_{x \in \mathbb{R}_+^n} \frac{I - p_2 x_2}{p_1} x_2 \text{ such that} \\ \text{(B)} \quad & x_1 = \frac{I - p_2 x_2}{p_1}. \end{aligned}$$

The first order condition is

$$\begin{aligned} \frac{I}{p_1} - \frac{2p_2}{p_1} x_2 &= 0 \\ x_2 &= \frac{I}{2p_2} \end{aligned}$$

From which

$$x_1 = \frac{I}{2p_1}$$

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Substitution Method

Consider a constrained maximization problem of n variables with k constraints.

1. Identify constraints that are binding at the solution
2. Express some variable from these constraints
3. Substitute these expressions into the objective of the problem
4. Maximize the objective function subject to the remaining constraints

Remark If all constraints are binding at the solution, the substitution method transfers the problem into unconstrained maximization problem of $n - k$ variables.

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Substitution Method: Disadvantages

1. It is not always feasible. For example, consider a firm's cost minimization problem:

$$\min_{x \in \mathbb{R}_+^n} \sum_{i=1}^n w_i x_i \quad \text{such that}$$
$$F(x) \geq Q$$

where F is the firm's production function.

- ▶ The functional form of F is not known but we would like to deduct properties of the solution given properties of F , such as concavity, constant, decreasing or increasing returns to scale, etc. This point is relevant even for the utility max problem.
 - ▶ One cannot solve for x_i explicitly. Example:
 $F(x) = x_1 + (x_1)^{.5} + x_2 + (x_2)^{.5}$
2. It is impractical in many circumstances, in particular in problems with symmetry.
 3. It cannot be extended to a continuum of goods.

Perturbation Method: Idea

The idea is to study a perturbation of solution that satisfies the constraint(s).

A necessary condition for solution is that a perturbation cannot increase the value of an objective function.

The method is akin to the tangency method of previous courses.

Consider again the consumer problem: For positive prices p_1, p_2 and income I , and a strictly increasing u ,

$$(CP) \quad \max_{x \in \mathbb{R}_+^n} u(x_1, x_2) \text{ such that}$$

$$(B) \quad p_1 x_1 + p_2 x_2 \leq I.$$

We have already established that, as u is strictly increasing, the budget constraint is binding at the solution.

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Perturbation Method: Perturbations

Therefore, without loss of generality, we can rewrite the consumer problem as: For positive prices p_1, p_2 and income I , and a strictly increasing u ,

$$(CP) \quad \max_{x \in \mathbb{R}_+^n} u(x_1, x_2) \text{ such that}$$

$$(B) \quad p_1 x_1 + p_2 x_2 = I.$$

Let now x_1^* and x_2^* be a solution. And consider a perturbation

$$x'_1 = x_1^* + \Delta_1,$$

$$x'_2 = x_2^* + \Delta_2,$$

where $\Delta_1, \Delta_2 \in \mathbb{R}$.

Perturbation Method: Feasible Perturbations

What perturbations should we consider?

A perturbation is feasible if it satisfies a budget constraint.

We should consider only feasible perturbations.

Substituting the expression for the perturbation into the budget constraint, we get

$$p_1 x_1^* + p_1 \Delta_1 + p_2 x_2^* + p_2 \Delta_2 = I,$$

or, equivalently,

$$p_1 \Delta_1 + p_2 \Delta_2 = 0.$$

Hence, a feasible perturbation satisfies

$$\Delta_2 = -\frac{p_1}{p_2} \Delta_1.$$

Perturbation Method: Feasible Perturbations

If x_1^* and x_2^* are both positive then

Δ_1 can be both positive and negative.

Otherwise,

Δ_1 must be positive if $x_1^* = 0$, while

Δ_1 must be negative if $x_2^* = 0$.

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Perturbation Method: First Order Approximation

To make use of perturbations, we should express the objective function in terms of Δ_1 .

We do it by using first order derivatives and the remainder:

$$\begin{aligned}u(x'_1, x'_2) &= u(x_1^*, x_2^*) + u'_{x_1}(x_1^*, x_2^*)\Delta_1 + u'_{x_2}(x_1^*, x_2^*)\Delta_2 \\ &+ R(\Delta_1, \Delta_2; x_1^*, x_2^*).\end{aligned}$$

or the first order approximation

$$\begin{aligned}u(x'_1, x'_2) - u(x_1^*, x_2^*) &\approx u'_{x_1}(x_1^*, x_2^*)\Delta_1 + u'_{x_2}(x_1^*, x_2^*)\Delta_2 \\ &= (p_2 u'_{x_1}(x_1^*, x_2^*) - p_1 u'_{x_2}(x_1^*, x_2^*)) \frac{\Delta_1}{p_2}.\end{aligned}$$

where in the second line we have the fact that the perturbation should be feasible

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Perturbation Method: Necessary Conditions

$$u(x'_1, x'_2) - u(x_1^*, x_2^*) \approx (p_2 u'_{x_1}(x_1^*, x_2^*) - p_1 u'_{x_2}(x_1^*, x_2^*)) \frac{\Delta_1}{p_2}.$$

Thus, a necessary condition for solution is that

If x_1^* and x_2^* are positive, then Δ_1 could take either sign and it must be that

$$\frac{u'_{x_1}(x_1^*, x_2^*)}{p_1} = \frac{u'_{x_2}(x_1^*, x_2^*)}{p_2}$$

If $x_1^* = 0$, then Δ_1 cannot be negative and it must be that

$$\frac{u'_{x_1}(x_1^*, x_2^*)}{p_1} \leq \frac{u'_{x_2}(x_1^*, x_2^*)}{p_2}$$

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Perturbation Method: Necessary Conditions

Finally, if $x_2^* = 0$, then Δ_1 cannot be positive and it must be that

$$\frac{u'_{x_1}(x_1^*, x_2^*)}{p_1} \geq \frac{u'_{x_2}(x_1^*, x_2^*)}{p_2}$$

Why are these conditions necessary?

We can use methods of last class to give a formal proof.

Example: Cobb-Douglas preferences

$$(CP) \quad \max_{x \in \mathbb{R}_+^n} x_1^\alpha x_2^{1-\alpha} \text{ such that}$$

$$(B) \quad p_1 x_1 + p_2 x_2 = I,$$

where $p_1 > 0$, $p_2 > 0$, $I > 0$ and $\alpha \in (0, 1)$.

- ▶ Find the solution using the substitution and the arbitrage method.
- ▶ Empirical applications of Cobb-Douglas preferences.