Repeated Partnership with Limited Information Flows

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Abstract

The paper studies a model of random matching in which a matched pair plays the prisoners dilemma repeatedly until partners separate and rematch. Existing literature uses this type of model to explain complex behavior in long-term social and economic relationships. In particular, it is shown that long periods of gradual cooperation building and asymmetries in conduct based on payoff-irrelevant marks (such as skin color) Pareto dominate simpler symmetric strategies. By enriching the strategy space of the stage game from a discrete to a continuous action space, I prove that all payoffs in the set of symmetric subgame perfect equilibria can be attained with very simple symmetric strategy profiles employing at most three actions. I also show that some of these strategies are efficient, even when efficiency is defined with respect to all correlated equilibria. Thus, with a rich enough action space, social efficiency cannot explain complex behavior.

JEL Classification C73, C78

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1 Introduction

Cooperative behavior plays an essential role in several economic and social interactions. It is widely observed even in the absence of contracts and other legal enforcement, or with limited information flow. In order to explain this phenomenon in large populations, a recent body of literature refers to marriage models with a continuum of anonymous agents (Datta (1993), Eeckhout (2005), Ghosh and Ray (1996), Kranton (1996), Lindsey et al. (1999), Watson (1999)). Married couples typically play the prisoners dilemma repeatedly until one of the agents in the partnership decides to terminate it. Cooperative equilibria in these models are characterized by self-sustaining action paths. Precisely because of the limited flow of information, social efficiency requires such paths to have an
“incubation phase” with low payoffs preceding the phase of high cooperation. The threat of restarting a new relationship and having to re-enter the incubation period reinforces cooperative behavior in the later stage of the action path.

But an incubation phase is only one feature of such social and economic relationships. They also involve complex patterns of behavior. The incubation phase involves a variety of actions. There may also be asymmetries in conduct based on seemingly irrelevant heterogeneity across agents (e.g. skin color). Existing literature (e.g. Datta (1993), Eeckhout (2005)) attempts to explain these complexities in behavior through simple marriage models with identical agents. To the extent that efficiency provides a positive theory of behavior, they show that strategies involving “racial prejudice” or gradual cooperation Pareto dominate simpler symmetric strategies.

The objective of this paper is to show that under a simple assumption regarding the richness of the strategy space, the criterion of social efficiency cannot predict complex incubation strategies, nor can it be used to explain asymmetries of behavior that depend on racial markers. In particular, I establish the efficiency of very simple strategy profiles in which at most two periods of noncooperation are called for (perhaps at different intensities) before the fully cooperative action profile is reached and adhered to. To go further in explaining more complex behavior, one will need to assume information asymmetries, perhaps along the lines introduced by Ghosh and Ray (1996) or Watson (1996).

I study a marriage market model similar to those described above: there is a continuum of identical anonymous agents, and married couples repeatedly play a version of the prisoners dilemma. I prove that all payoffs in the set of symmetric subgame perfect equilibria can be attained with a strategy profile containing at most three actions (Proposition 1). The simplicity of these strategy profiles is a significant extension of Datta (1993) who, in a more restrictive setting, shows that the best equilibrium payoff can be attained with a finite (but indeterminate number) of actions.

I then apply this insight to a recent paper by Eeckhout (2005). This paper shows that
coordinated mixed strategies can Pareto-improve welfare with respect to uncoordinated ones, and uses this result to build a theory of racial segregation: “color” acts as a substitute for a public randomization device when the latter is not available. The model I present is identical to Eeckhout’s except for the version of the prisoners’ dilemma used as a stage game. The only difference is that I allow for continuous levels of cooperation (as in Ghosh and Ray (1996)), whereas Eeckhout uses a discrete action space.

Of course there is no general result that allows correlation to fall out of the richness of action sets. Yet – and in sharp contrast to the findings of Eeckhout – I am able to prove that some symmetric three-action strategy profile is constrained Pareto optimal and that no correlated equilibrium Pareto dominates it (Proposition 2).

Taken together, these results show that in rich action spaces, an incubation phase, while possible, is of a very limited nature, and that there is no role for asymmetric standards of behavior. In particular, one should be cautious in applying Eeckhout’s result to explain racial segregation.

Section 2 presents the model and the results. In subsection 2.1 I establish the limited incubation result, and in 2.2 I introduce correlation devices to address efficiency questions.

## 2 The Model

### 2.1 Basic model

Time is discrete and infinite; there is no initial nor final date \((t = \ldots, -1, 0, 1, 2, \ldots)\). At each time period, a unit mass of single agents is born. At the beginning of a period, agents are either single or married. All single agents are matched up in pairs, whereas the married ones remain with their previous partners. Pairs then play a stage game described below. After observing the outcome of the game, players choose to continue or discontinue the partnership. If both agents decide to continue, they repeat the stage game against each other in the subsequent period. If one of the agents discontinues, they both become
single at the beginning of the next period.

All agents are identical and anonymous. The payoff of any agent in the supergame is the sum of his payoffs in stage games discounted by a factor $\delta \in (0, 1)$ and normalized by multiplying by $(1 - \delta)$.

The stage game is symmetric, and the action space for each player is $A$. Let $\Pi : A^2 \to \mathbb{R}$ be the continuous payoff function, where $\Pi(a, a')$ is the payoff of an agent when he plays $a$ and his opponent plays $a'$. Assume there exists an element of $A$, denoted by 0, which is the unique strictly dominant strategy of the game. That is, $\Pi(0, a) > \Pi(a', a)$ for all $a$ and all $a' \neq 0$. Define $v(a) = \Pi(a, a)$, and $g(a) = [\Pi(0, a) - \Pi(a, a)]$. Normalize $\Pi(0, 0)$ to be zero. Define also the auxiliary function $W : A \to \mathbb{R}$ by

$$W(a) \equiv [v(a) - (\frac{1-\delta}{\delta}) g(a)].$$

I will make the following assumptions:

**Assumption 1** For any $a \in A$ and any $v' \in [0, v(a)]$, there exists an $a' \in A$ such that $v(a') = v'$ and $g(a') \leq g(a)$. In addition, $v$ is bounded.

**Assumption 2** $\max_{a \in A} \{W(a)\}$ exists.

**Assumption 3** For all $a, a' \in A$, $\Pi(a, a) + \Pi(a', a') \geq \Pi(a, a') + \Pi(a', a)$.

In A1, assuming $v$ is bounded is necessary to eliminate Ponzi schemes: otherwise any cooperation level $a$ can be sustained in equilibrium by an increasing and unbounded sequence of future cooperation levels. Thus, without it, equilibrium payoffs would also be unbounded. Assumption 2, which almost follows from 1, assures that the equilibrium set is closed. (Assumption 3 is discussed below.)

An **incubation strategy** is given by a sequence of cooperation levels $\{a_t\}_{t=1}^\infty$ where the prescribed action for the $t^{th}$ period of a partnership is always $a_t$. If the outcome of the stage game at time $t$ is the stipulated $(a_t, a_t)$ choose to continue the partnership. Otherwise, terminate it.

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1The assumption that $v$ is bounded and that $W$ attains a maximum can be dispensed with by letting $A$ be a compact subset of $\mathbb{R}$. If the set $A$ is a subset of $\mathbb{R}$, the first part of assumption 1 is satisfied if $v$ and $g$ are non-decreasing.
I use the standard definition of a subgame perfect equilibrium to define an equilibrium. An equilibrium is *symmetric* if equilibrium payoffs are the same for all agents within the same generation, and it is *generation-independent* if the payoff is the same across all generations.

**Proposition 1** Assume 1, and 2, and define $V \equiv \max_{a \in A} \{W(a)\}$. Then,

(i) for any $V \in [0, V]$, there exists an equilibrium with symmetric incubation strategies and payoffs $V$. Indeed, there always exists an equilibrium incubation strategy that uses no more than three distinct actions.

In addition, assume 3. Then,

(ii) the set of symmetric generation-independent equilibrium equals $[0, V]$.

**Proof.** For all $a$ such that $W(a) > 0$, there exists an integer $k \geq 1$, and an $\tilde{a} \in A$ such that $g(\tilde{a}) \leq g(a)$ and $(1 - \delta)\delta^k v(\tilde{a}) + \delta^{k+1} v(a) = W(a)$, where the left-hand-side equals the payoff of a newly formed partnership in which agents play 0 for $(k - 1)$ periods, $\tilde{a}$ in the $k^{th}$ period, and $a$ in all subsequent periods.\(^2\) Hence, by construction this incubation strategy has three actions and yields payoffs equal to $W(a)$. To check if it is a best response, for any action $a$, interpret $W(a)$ as the upper bound on the value of being single, $V$, that sustains the constant cooperation level $a$ in equilibrium: $(1 - \delta)[g(a) + v(a)] + \delta V \leq v(a) \iff V \leq W(a)$. Hence, agents will not deviate in the high cooperation phase, and in the period where $\tilde{a}$ is played, cooperation is guaranteed by the assumption that $g(\tilde{a}) \leq g(a)$ (see details in the appendix). Part (ii) is a corollary of proposition 2 proved below. ■

The feature that relationships are slowly built is a desideratum present in all the previously mentioned models of repeated partnership amongst a continuum of anonymous agents. While several cooperation levels may be required to attain the optimal symmetric equilibrium in the presence of asymmetric information regarding payoffs (Ghosh and Ray, 1996, and Watson, 1999), by proposition 1, in the absence of such asymmetries, at most

\(^2\)To see the existence of $k$ and $\tilde{a}$, just choose $k$ to satisfy $\delta^{k+1} v(a) < W(a) \leq \delta^k v(a)$, a possible choice since $v(a) \geq W(a)$. The existence of $\tilde{a}$ such that $\delta^k [(1 - \delta) v(\tilde{a}) + \delta v(a)] = W(a)$ follows from assumption 1.
three levels are needed to attain any symmetric equilibrium payoff. This result significantly extends Datta (1996), who, in a more restrictive setting, proves the existence of an equilibrium with maximum symmetric payoffs that attains the maximal trust level in finite time.

2.2 Correlation, Color, and Pareto Improvements

To motivate the analysis, consider the following stage game taken from Eeckhout (2005):

<table>
<thead>
<tr>
<th></th>
<th>$C$</th>
<th>$D$</th>
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<tbody>
<tr>
<td>$C$</td>
<td>$1,1$</td>
<td>$-l,1+g$</td>
</tr>
<tr>
<td>$D$</td>
<td>$1+g,-l$</td>
<td>$0,0$</td>
</tr>
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</table>

where $g > 0$, $l > 0$, and $(g - l) \leq 0$. Notice that the action space is discrete; for this reason (and this reason alone) the example is not a special case of our model. As observed by Eeckhout (and explained below), for general value of the parameters, no uncorrelated equilibrium is constrained efficient. My subsequent analysis shows, however, that this inefficiency result does not perdure if the action space is continuous.

Consider the problem of sustaining $\{C, C\}$ forever, starting from some period in the match. More specifically, suppose as of period $T$ individuals always play $C$, and discontinue the partnership only if the outcome of the stage game is not $\{C, C\}$. For this cooperative strategy to be a best response to itself, the value of being single can be no greater than $W(C) = 1 - g \left( \frac{1 - \delta}{\delta} \right) = \nabla$, the constrained optimal payoff.

To verify whether $\nabla$ can be attained, we check exhaustively all possible behavior strategies for period $T - 1$. A strategy for period $T - 1$ is described by $\sigma$, the probability of cooperating, and the probability of terminating a partnership conditional on each one of the outcomes of the stage game. Notice, however, that the payoff from continuing the partnership ($\delta$) is always greater than the value of terminating it, which is bounded above by $\delta\nabla < \delta$. Therefore, if a player knows that his opponent will decide to continue the partnership with any positive probability, he will decide to continue it with probability...
1. Thus, the set of relevant behavior strategies is reduced to those of the form \( s^* = \{ \sigma, (P_1, P_2, P_3) \} \) where \( P_1, P_2, P_3 \) are the probabilities of continuing the partnership when the outcome of the stage game is \( \{ C, C \} \), \( \{ C, D \} \) (or \( \{ D, C \} \)), and \( \{ D, D \} \), respectively, and \( (P_1, P_2, P_3) \in \{0,1\}^3 \). For several values of the parameters, however, all of these strategies yield, from the perspective of period \( T - 1 \) (before initiating the stage game), payoffs smaller than \( \bar{V} \). Consequently, if the value of being single was indeed \( \bar{V} \), then no cooperation would be possible in periods preceding \( T - 1 \), making it unviable to attain \( \bar{V} \).

To take a concrete example, let \( \delta = 0.8 \), \( g = 0.5 \), and \( l = 0.5 \) so that \( \bar{V} = 0.875 \). The behavior strategy \( s^* = \{0, (0, 0, 1)\} \) yields suboptimal payoffs \( v_{T-1} = 0 + \delta = 0.8 \). If \( \sigma \in (0,1) \), players in equilibrium must be indifferent between playing \( C \) or \( D \). Supposing that the value of being single is \( \bar{V} \), this indifference condition is

\[
\sigma [1 - \delta + \delta(P_1 + (1 - P_1)\bar{V})] + (1 - \sigma) [-l(1 - \delta) + \delta(P_2 + (1 - P_2)\bar{V})] \\
= \sigma [(1 + g)(1 - \delta) + \delta(P_2 + (1 - P_2)\bar{V})] + (1 - \sigma)\delta(P_3 + (1 - P_3)\bar{V}) \] (1)

which uniquely determines \( \sigma \), and consequently \( v_{T-1} \), for each tuple \( (P_1, P_2, P_3) \). It is easy to check, however, that for all \( (P_1, P_2, P_3) \in \{0,1\}^3 \), the solution to (1) does not satisfy \( \sigma \in (0,1) \), confirming our conclusion that \( \bar{V} = 0.875 \) cannot be attained in any uncorrelated equilibrium.

Now suppose, before each stage game, players could observe the realization of a public randomization device \( \omega \) drawn independently in each period from a uniform distribution on \([0,1]\). Let \( T = 2 \), and in the first period of a partnership, let agents play \( C \) if \( \omega < 0.375 \), and \( D \) otherwise, and discontinue the partnership only if the prescribed actions were not taken. The payoffs from following this strategy, easily verified to be an equilibrium, are \( [(1 - \delta)0.375 + \delta] = 0.875 \), precisely the constrained optimal which Pareto dominates all
uncorrelated equilibria.

The result that, even though mixed strategies convexify the strategy space, they do not guarantee constrained efficiency is due to Eeckhout (2005), who proceeds to show how, if a public randomization device is not available, color can substitute it in Pareto improving uncorrelated equilibria. If, for instance, individuals only cooperated in the first period of a partnership if matched with someone their color, then the value of being single would be a weighted average between the full cooperation (1 in our example) and the one period incubation strategy payoffs (0.8). By judiciously choosing population sizes, Eeckhout proves that a color equilibrium always Pareto dominates all inefficient uncorrelated equilibria. Proposition 2 below shows that the discreteness of the action space of the stage game is essential for this finding.

In the supergame described in section 2.1, suppose players observed the realization of a public randomization device before each stage game. To add color, let each individual be endowed with one of \( n \) payoff-irrelevant characteristics, which partition the population into \( n \) groups with a continuum of individuals each. Assume that the fraction of the population in each group is constant across generations and that these characteristics are freely observed. Players can condition their strategies on both characteristics and on the public randomization device. As before, because there is a continuum of individuals in each group, players cannot infer whether an individual is single in a certain time period because he has previously defaulted or because he is a newborn. I refer to an equilibrium of this new game as a correlated equilibrium.

**Proposition 2** Under assumptions 1 through 3, no correlated generation-independent equilibrium Pareto-dominates the symmetric generation-independent equilibrium with payoff \( V \).

I provide the intuition for the proof, which is in the appendix. Suppose there is such an equilibrium where the payoffs of individuals \( i \) and \( j \) of colors \( I \) and \( J \) exceed \( V \) by some \( \Delta > 0 \). If one admits asymmetric or mixed strategies as we do here, it is
difficult to guess which agent will deviate from such a strategy profile: an agent who is more cooperative than his opponent may be tempted to abandon a partnership because he receives a lower payoff from it, but his deviation will be less profitable since $g(a)$ is increasing in cooperation levels (A1). We can, nevertheless, say something about the sum of their temptations. To understand it, suppose once the partnership is established, players $i$ and $j$ follow static strategies described by cumulative distribution functions $F_i$ and $F_j$. By assumption 3, perfect coordination is better than alternatingly beating on each other, so that the sum of payoffs from the partnership would be no greater than $\int A v(a) dF_i(a) + \int A v(a) dF_j(a)$. But even if payoffs were this high, best responses require the value of being single to be bounded by the discounted weighted average of the gains from defaulting on the opponent and starting a new partnership the next period. The net sum of these gains equals $\int A W(a) dF_j(a) + \int A W(a) dF_i(a)$ which by definition is smaller than $2V$, which in turn is smaller than the sum of $I$ and $J$’s expected values of being single. So, to deter deviations, it must be the case that strategies are not static, and that future payoffs exceed the current period’s by at least $\Delta$, thus violating the assumption that $v$ is bounded.

Although the game proposed by Eeckhout (2005) satisfies assumption 3, it is still interesting to question its necessity. Its violation implies that a coordinated action profile may dominate the use of symmetric strategies. Color is particularly relevant in this context as it facilitates the use of asymmetric strategies. Still, I will argue heuristically that long-term partners can in general coordinate their actions, even without color. Consider a situation where blacks and whites are both strictly better off than in a color-blind equilibrium by means of a certain (possibly mixed and asymmetric) strategy profile. For simplicity, suppose also that the expected value of a partnership as of period $t = 2$ is slightly higher for whites than it is for blacks. Then, they can use the incubation period, where deviations are not binding, to play a first price auction to bid for the value of being white. That is, in the first period of a partnership, they play a symmetric mixed strategy.
profile (with no mass points just like in a first price auction) and the player who chooses the highest cooperation level - thus the lowest current period payoff - plays the white’s strategy as of the subsequent period, while the other agent plays the black’s. Such a scheme precludes the use of color as a coordination device.\footnote{To fully construct a color-blind equilibrium, it would probably be necessary also to change the action profile in order to make the equilibrium value of being single equal to the minimum between the blacks’ and the whites’ value of being single. If the payoffs of a established partnership are equal for blacks and whites a “trembling” of the equilibrium strategies will generate the asymmetry needed for the argument above.}

Eeckhout (2005) proves that in a generic partnership economy the best color-blind equilibrium is constrained inefficient and that there exist distributions of colors that Pareto-dominate the best uncorrelated equilibrium. These two results motivate the rest of his paper: to the extent that efficiency of equilibrium provides a positive theory of behavior in society, they deliver a possible explanation for racial segregation. Proposition 2 together with the argument above shows that these results rely on the discreteness of the actions in the stage game assumed by Eeckhout. More specifically, by Propositions 1 and 2, when a continuum of cooperation levels is viable, it is possible to attain the constrained optimum symmetric payoff with simple incubation strategies containing only three actions. Hence, applications of Eeckhout’s theory should be restricted to contexts where payoffs are perceived to be discrete. And the explanation for gradual relationship building should be sought in more complex environments.
3 Appendix

Proof of Proposition 1. To complete the proof presented in the main body, it only remains to prove that action $\tilde{a}$ is a best response in period $k$:

$$(1 - \delta) [g(\tilde{a}) + v(\tilde{a})] + \delta W(a) \leq (1 - \delta)v(\tilde{a}) + \delta v(a)$$

\[\Leftrightarrow (1 - \delta)g(\tilde{a}) \leq \delta v(a) - \delta W(a) \]

\[= (1 - \delta)g(a) \]

which follows from assumption 1. ■

Proof of Proposition 2. Denote by $V_I$ and $V_J$ the payoffs of individuals $i$ and $j$ of colors $I$ and $J$, respectively. Let $F_t(a_i, a_j)$ be the joint cumulative distribution over $A^2$ of $i$ and $j$’s equilibrium actions on the stage game they play against each other at time $t$, before the realization of the random variable, following some history (omitted for notational convenience). Let $F_{it}$ and $F_{jt}$ be the marginal distributions of the first and the second argument of $F_t$, respectively. Denote by $U_{i,t+1}$ and $U_{j,t+1}$ players $i$ and $j$’s expected continuation payoffs when they both follow their equilibrium strategies. Notice that, at any period, $i$ has the option of playing $a = 0$ in the stage game irrespective of the realization of the random variable, discontinuing the partnership and collecting $V_I$ in the next period. Hence, for his equilibrium strategy to be a best response for $j$’s, it must satisfy
\[
(1 - \delta) \int_A \int_A \Pi(a, a') dF_t(a, a') + \delta U_{i,t+1} \\
\geq (1 - \delta) \int_A \Pi(0, a') dF_{jt}(a') + \delta V_I \\
= (1 - \delta) \int_A [v(a') + g(a')] dF_{jt}(a') + \delta \int_A W(a') dF_{jt}(a') + \delta (V_I - \bar{V}) \\
\geq (1 - \delta) \int_A [v(a') + g(a')] dF_{jt}(a') + \delta \int_A [v(a') - (1 - \delta) g(a')] dF_{jt}(a') + \delta (V_I - \bar{V}) \\
= \int_A v(a') dF_{jt}(a') + \delta (V_I - \bar{V})
\]

where line (2) follows from the fact that \( \bar{V} \geq W(a) \) for all \( a \), and the others from definitions or rearrangements of the terms. Summing the equation above with the equivalent one for \( j \), we get

\[
(1 - \delta) \int_A \int_A \left[ \Pi(a, a') + \Pi(a', a) \right] dF_t(a, a') + \delta \left[ U_{i,t+1} + U_{j,t+1} \right] \\
\geq \int_A v(a') dF_{jt}(a') + \int_A v(a) dF_{it}(a) + \delta (V_I + V_J - 2\bar{V}) \\
= \int_{a' \in A} v(a') \int_{a \in A} dF_t(a, a') + \int_a v(a) \int_{a' \in A} dF_t(a, a') + \delta (V_I + V_J - 2\bar{V}) \\
= \int_A \int_A \left[ v(a') + v(a) \right] dF_t(a, a') + \delta (V_I + V_J - 2\bar{V}) \\
\geq \int_A \int_A \left[ \Pi(a, a') + \Pi(a', a) \right] dF_t(a, a') + \delta (V_I + V_J - 2\bar{V})
\]

The assumption that \( v \) is bounded and continuous was used to invert the order of the integrals in line (3), and line (4) comes from assumption 3. Rearranging and dividing by \( \delta \), we have

\[
[U_{i,t+1} + U_{j,t+1}] - (V_I + V_J - 2\bar{V}) \geq \int_A \int_A \left[ \Pi(a, a') + \Pi(a', a) \right] dF_t(a, a')
\]
Note that the RHS is the sum of payoffs of $i$ and $j$ in period $t$. If this equilibrium Pareto dominates the symmetric one, $(V_I + V_J - 2\bar{V}) > 0$ for some colors $I$ and $J$. Since it must hold for all $t$, equation (5) then implies that there is an unbounded sequence of one-period payoffs. This contradicts the assumption that $v$ is bounded combined with assumption 3. ■

**Note on generation-dependent equilibria.** The symmetric generation-independent equilibrium with payoffs $\bar{V}$ may be Pareto dominated by some color-blind generation-dependent ones. However, this occurs only in zero measure events and only when the minimum and maximum of the set $\bar{A} = \{v(a) : W(a) = \bar{V}\}$ are different. Letting $\upsilon$ and $\underline{\upsilon}$ be the maximum and minimum of $\bar{A}$, one can show that the payoffs $\{V_t\}_{t=-\infty}^{\infty}$ of all these equilibria must satisfy $\{\sum_{t=-\infty}^{\infty} [V_t - \bar{V}] \leq \upsilon - \underline{\upsilon}\}$. These are clearly very particular cases.

**References**


