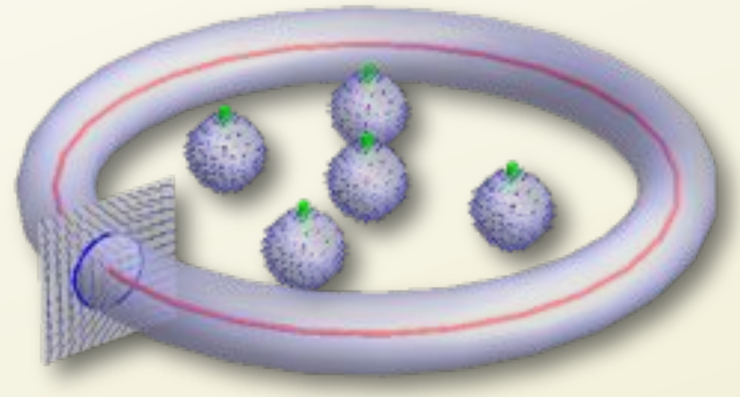
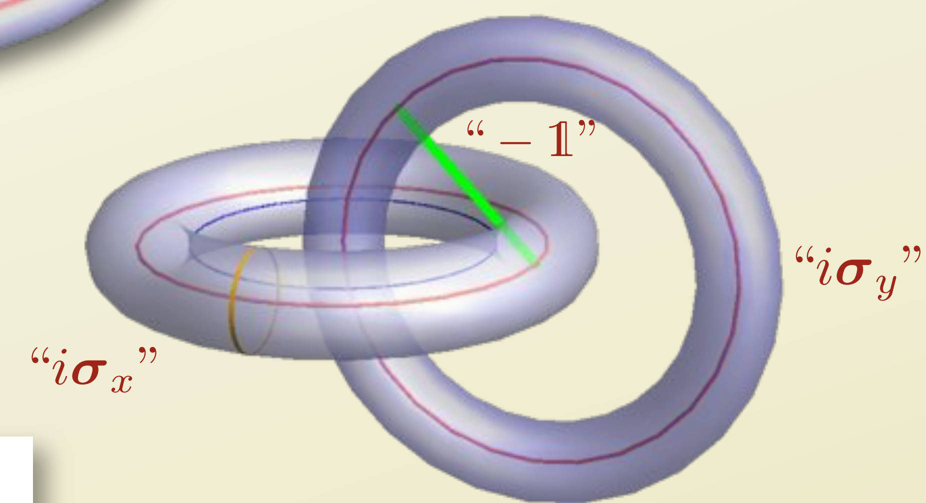
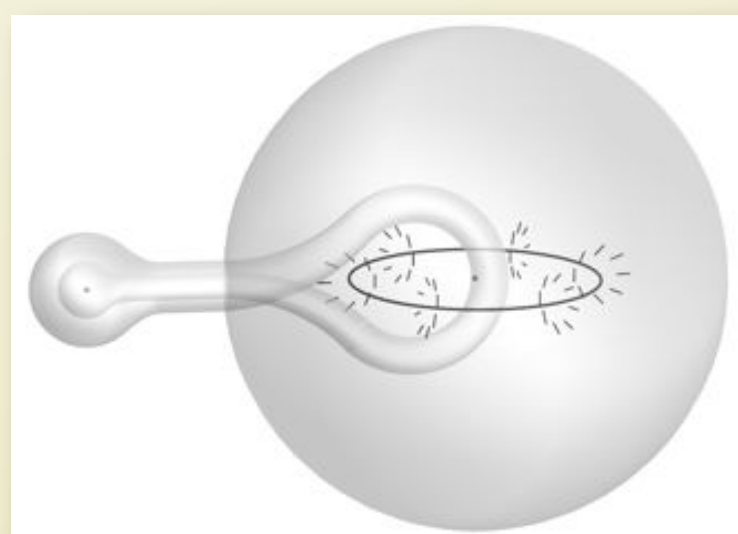
Entangled Defects





Gareth Alexander

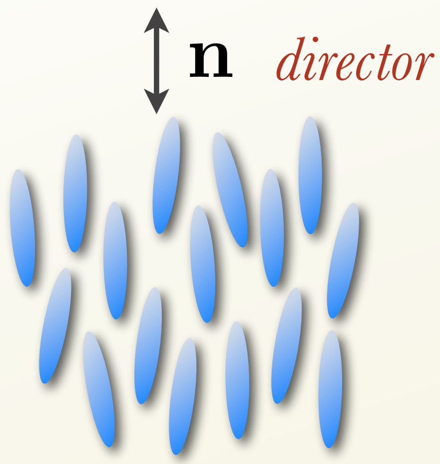
Penn, Warwick



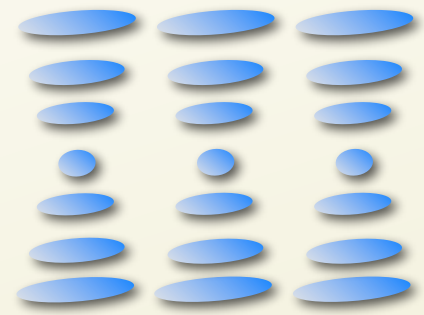
Topological Methods in Complex Systems
25th July-12th August 2011

Penn

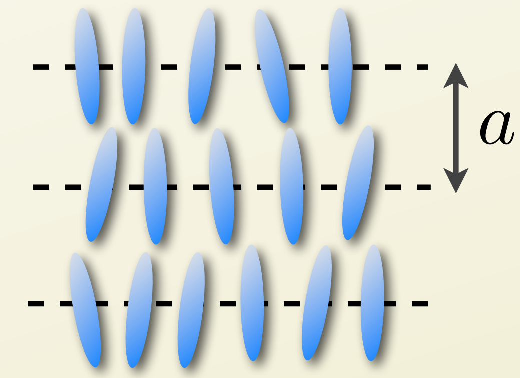
TEXTURES IN LIQUID CRYSTALS



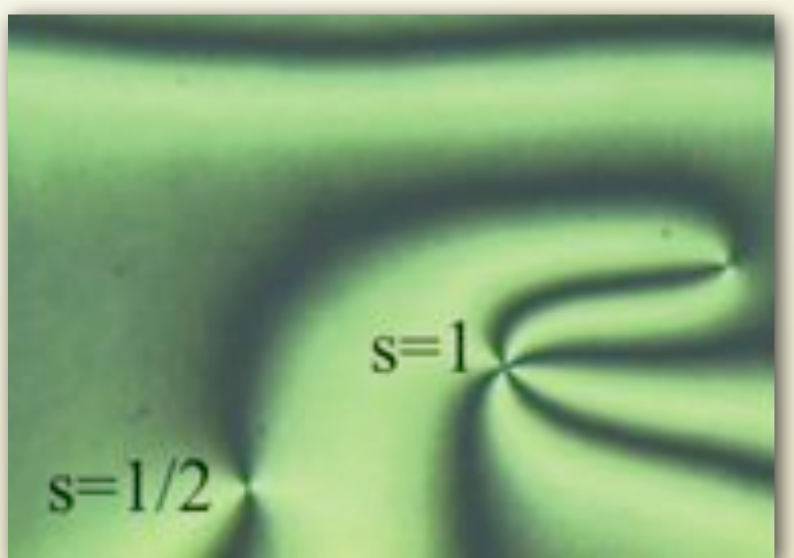
Nematic



Cholesteric



Smectic-A



courtesy of Ingo Dierking

Photo by Michi Nakata

courtesy of Noel Clark

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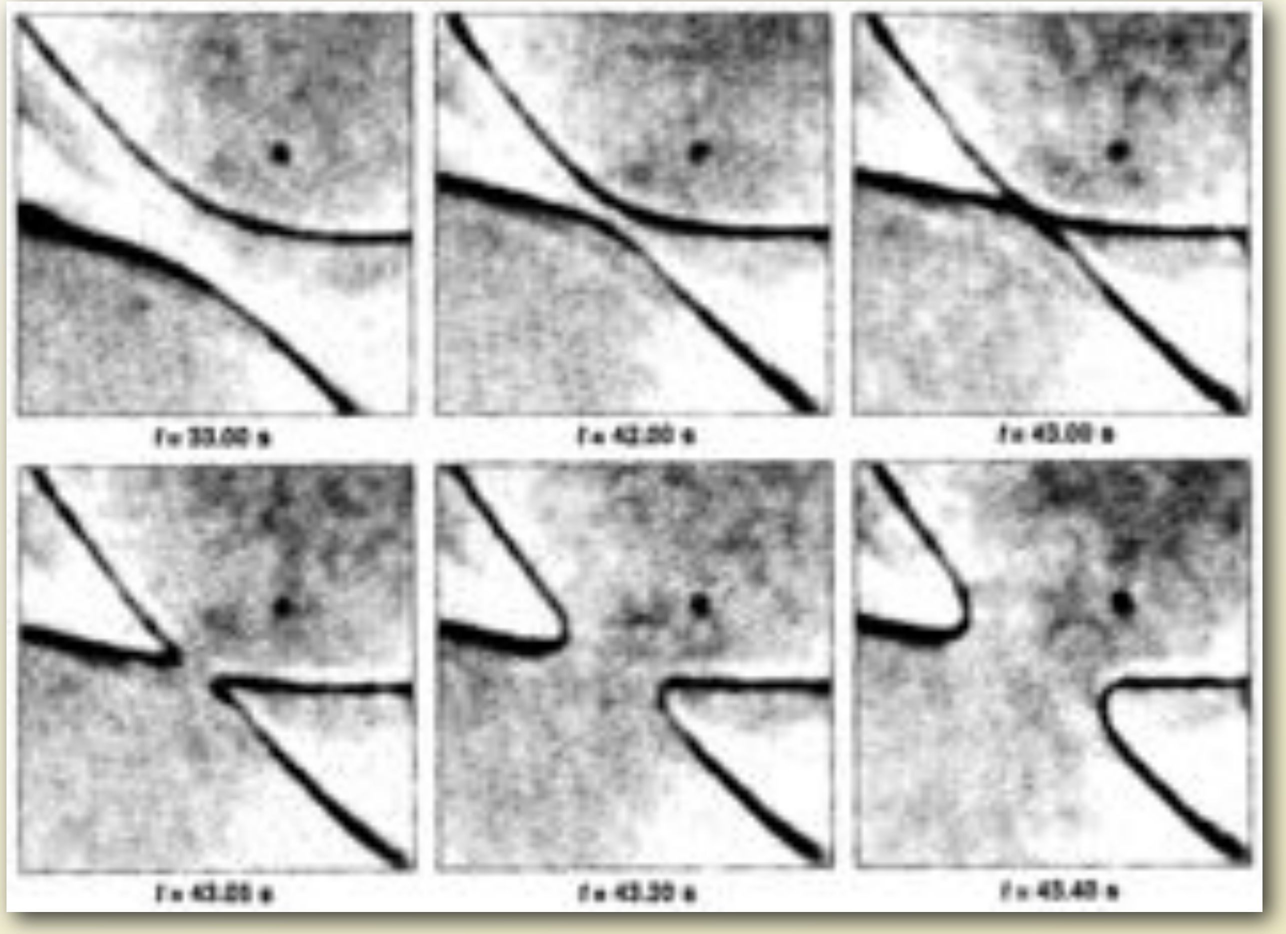
INTERACTION OF DEFECTS

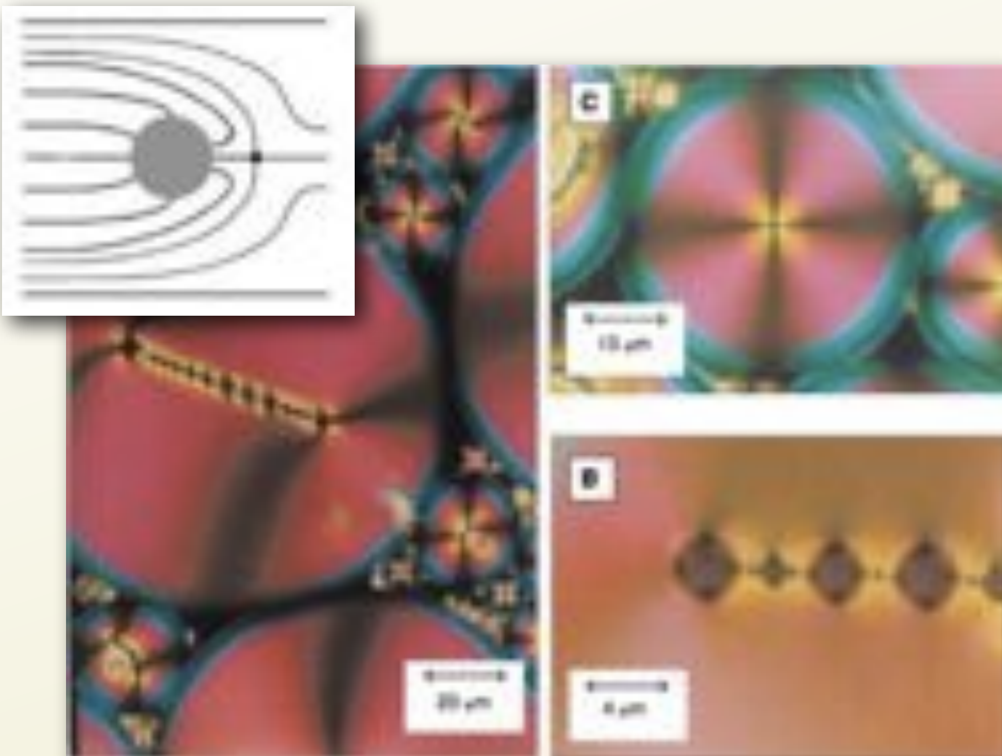
Coarsening and crossing



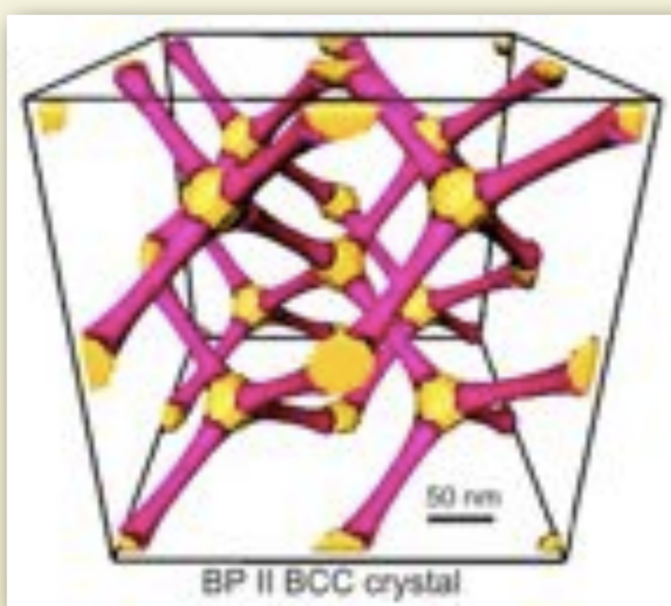
CHUANG, DURRER, TUROK & YURKE *Science* **251**, 1336–1342 (1991)

Interaction of defects

Colloids: self-assembly

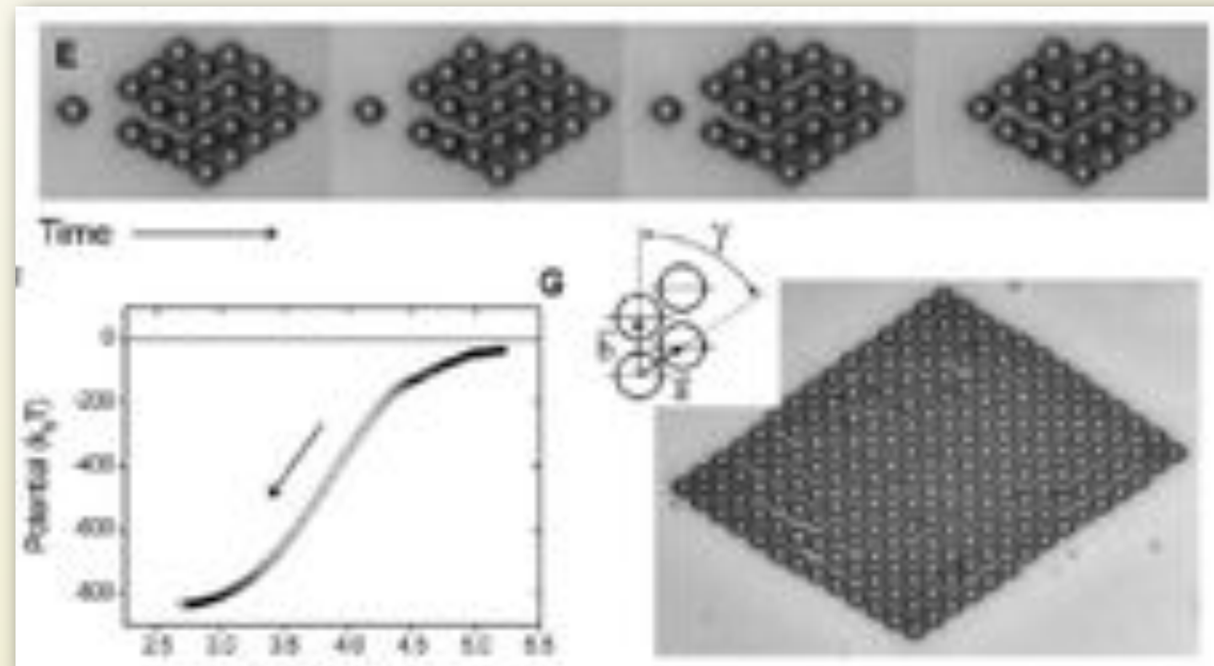


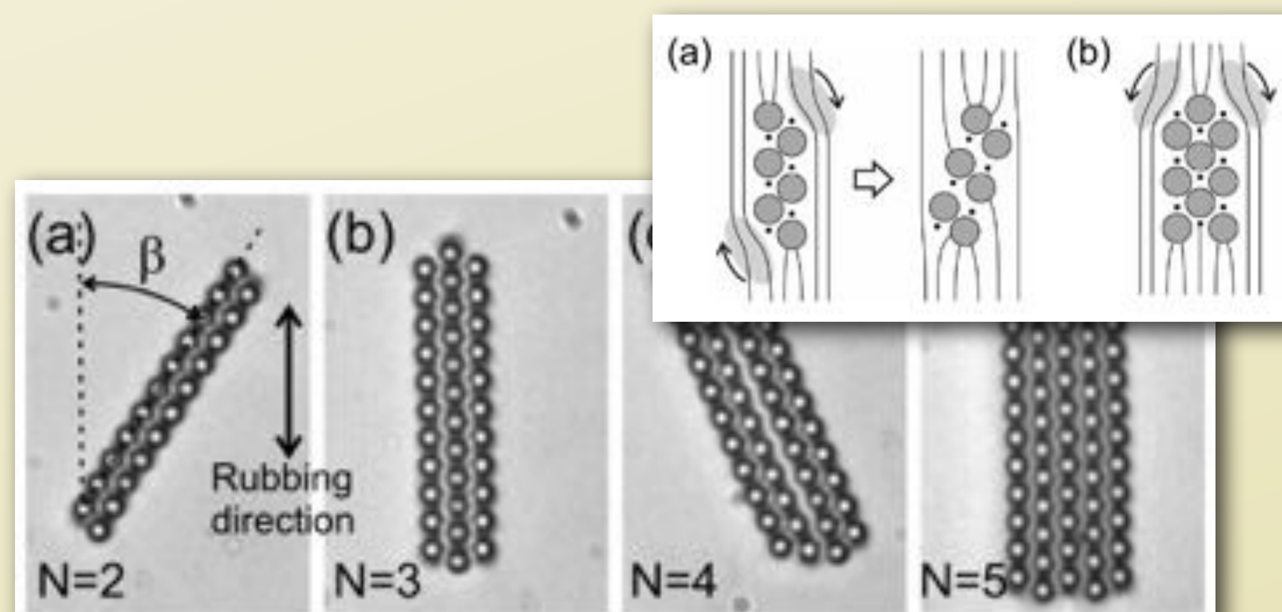
Poulin et al *Science* **275**, 1770–1773 (1997)



Ravnik, GPA, Yeomans & Žumer
PNAS **108**, 5188–5192 (2011)

Muševič et al *Science* **313**, 954–958 (2006)

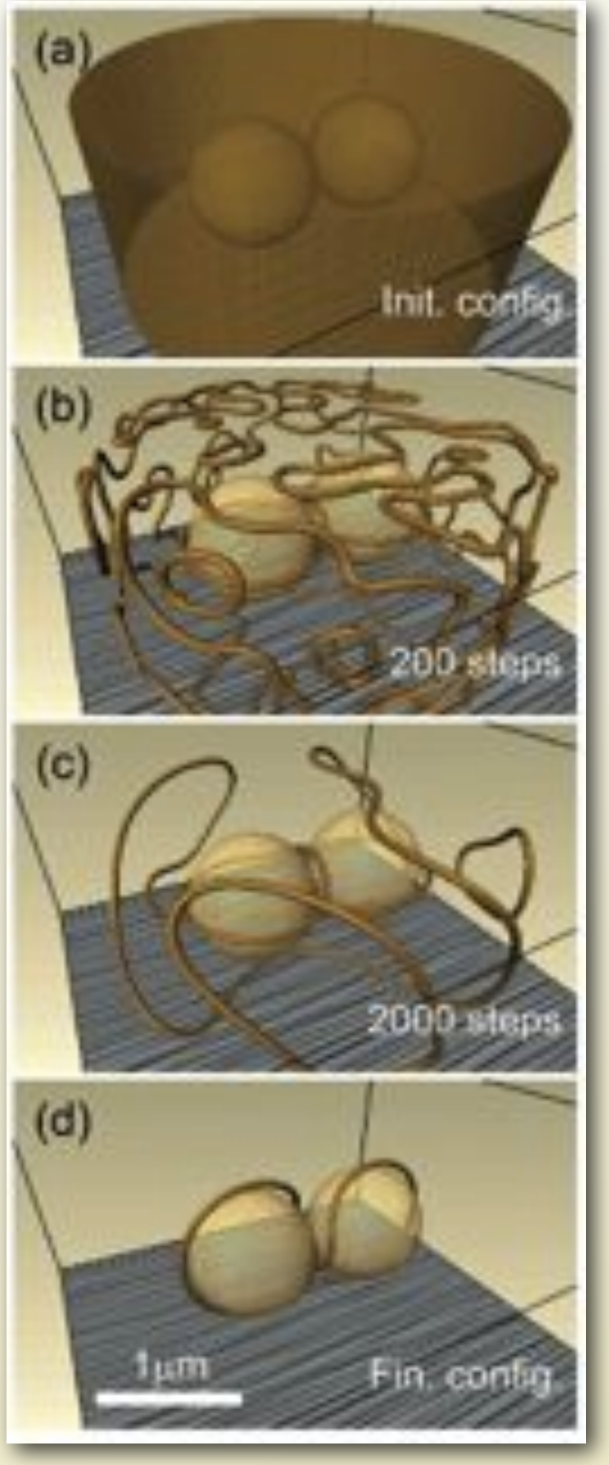


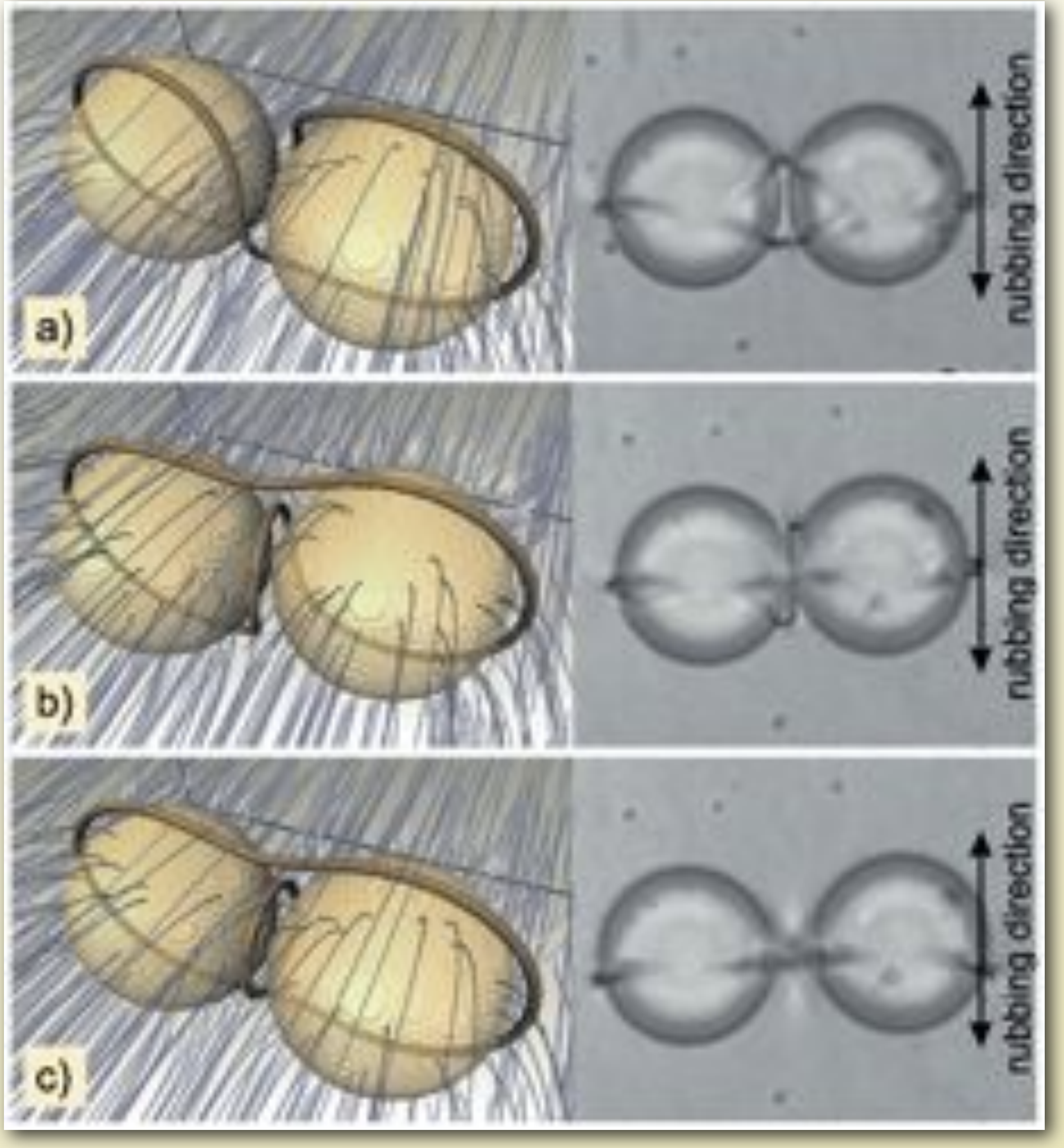


Škarabot et al *Phys. Rev. E* **76**, 051406 (2007)

Interaction of defects

Colloids: entangled defects





Ravnik & Žumer *Soft Matter* **5**, 269–274 (2009)

Biaxial nematics

uniaxial

symmetry of a right circular cylinder

symmetry group D_∞

ground state manifold

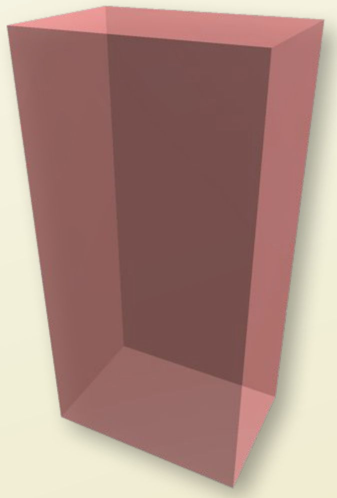
$$SO(3)/D_\infty$$

distinct orientations of a cylinder

biaxial

symmetry of a rectangular cuboid

symmetry group D_2



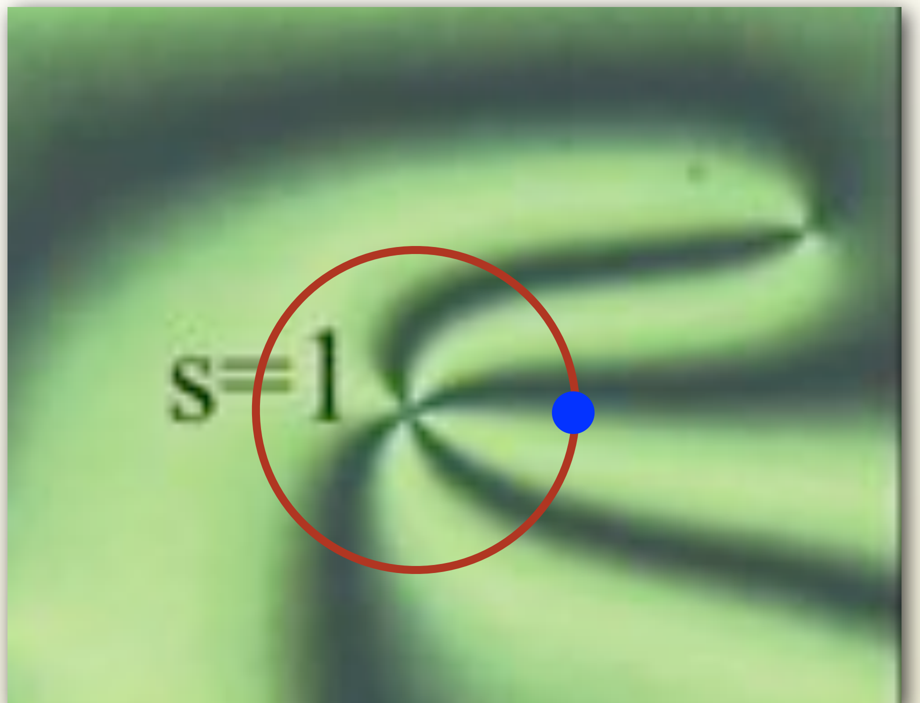
ground state manifold

$$SO(3)/D_2$$

distinct orientations of a brick

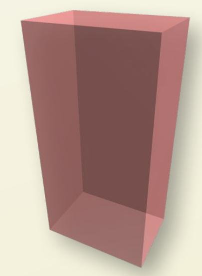
Penn

LINE DEFECTS: DISCLINATIONS



biaxial

ground state manifold $SO(3)/D_2$



distinct orientations of a brick

$$\pi_1(SO(3)/D_2) = Q_8$$

$$\{\pm\mathbb{1}, \pm i\boldsymbol{\sigma}_x, \pm i\boldsymbol{\sigma}_y, \pm i\boldsymbol{\sigma}_z\}$$

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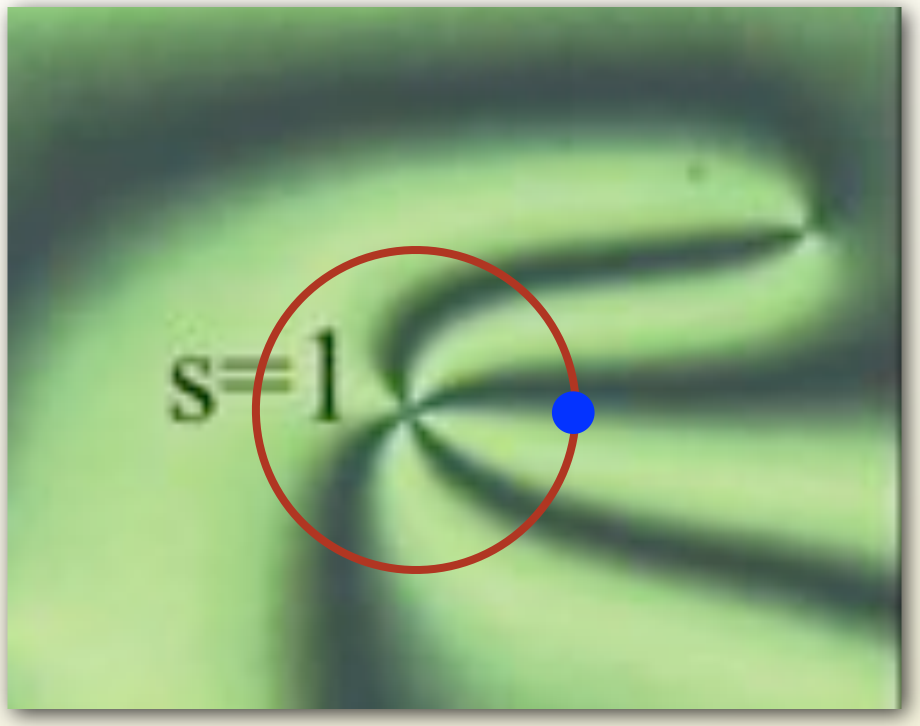
LINE DEFECTS: DISCLINATIONS



biaxial

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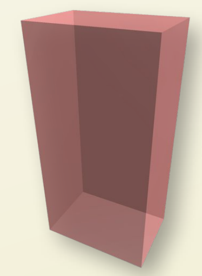

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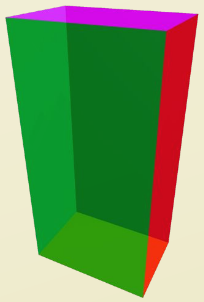
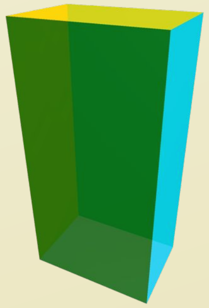
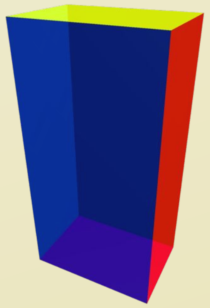


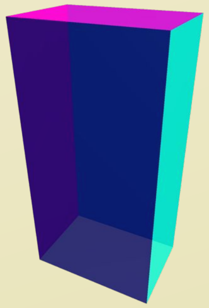
“$\mathbb{1}$”



“$i\boldsymbol{\sigma}_x$”



“$i\boldsymbol{\sigma}_y$”



“$i\boldsymbol{\sigma}_z$”

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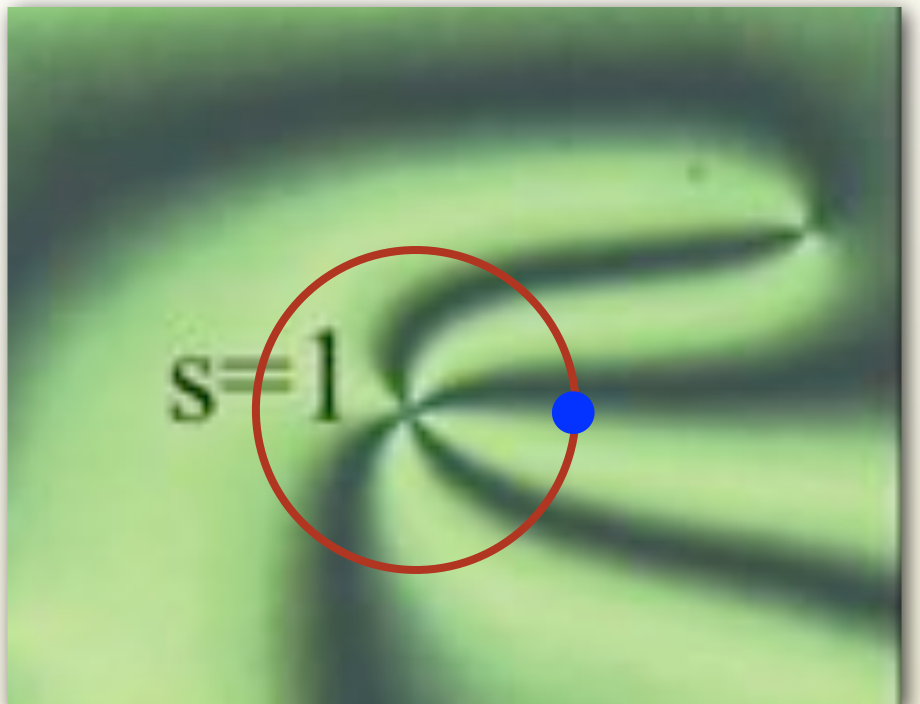
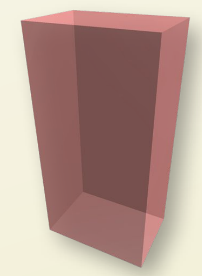
Line defects: disclinations

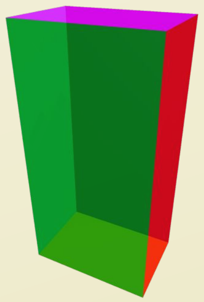


biaxial

ground state manifold $SO(3)/D_2$



distinct orientations of a brick

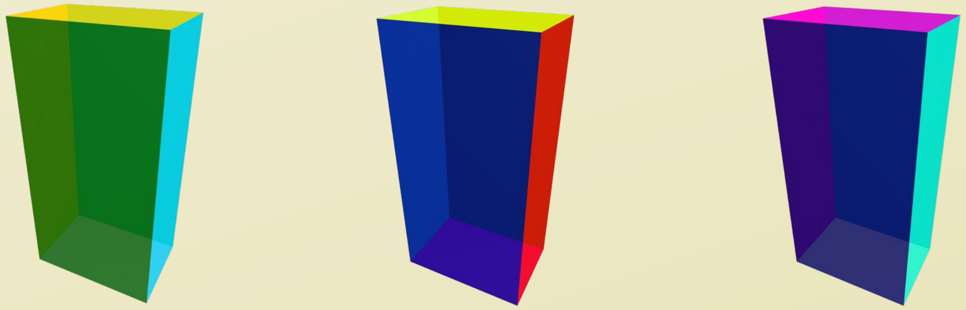


"$\mathbb{1}$"

"$-\mathbb{1}$"

$$\pi_1(SO(3)/D_2) = Q_8$$

$$\{\pm\mathbb{1}, \pm i\boldsymbol{\sigma}_x, \pm i\boldsymbol{\sigma}_y, \pm i\boldsymbol{\sigma}_z\}$$



"$\pm i\boldsymbol{\sigma}_x$" "$\pm i\boldsymbol{\sigma}_y$" "$\pm i\boldsymbol{\sigma}_z$"

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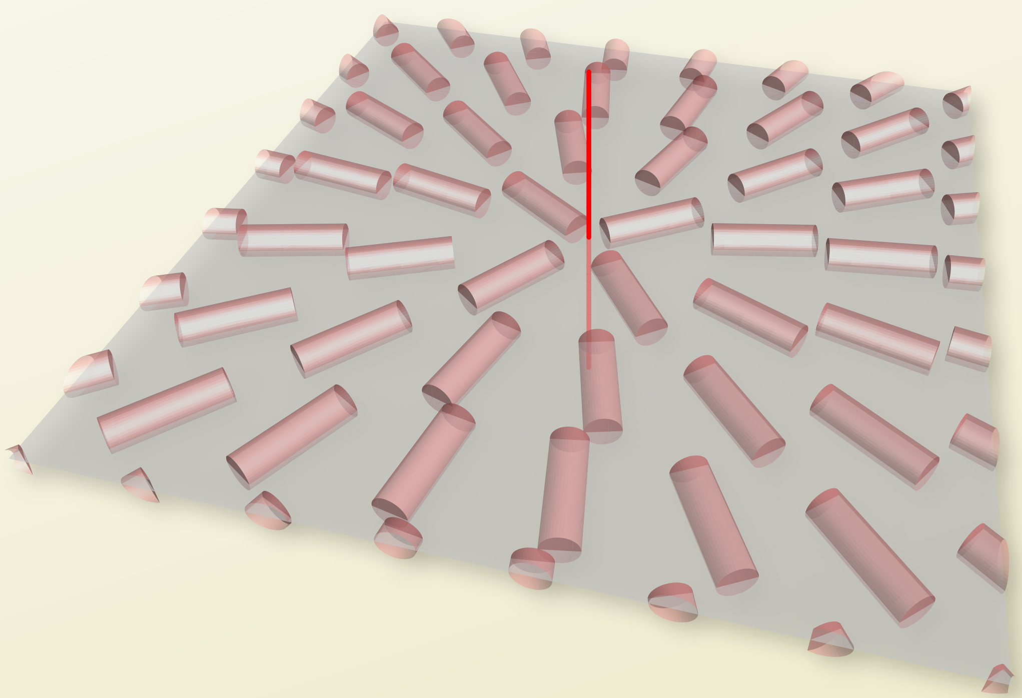
No escape for biaxials

- 2π disclinations are removable in uniaxial nematics

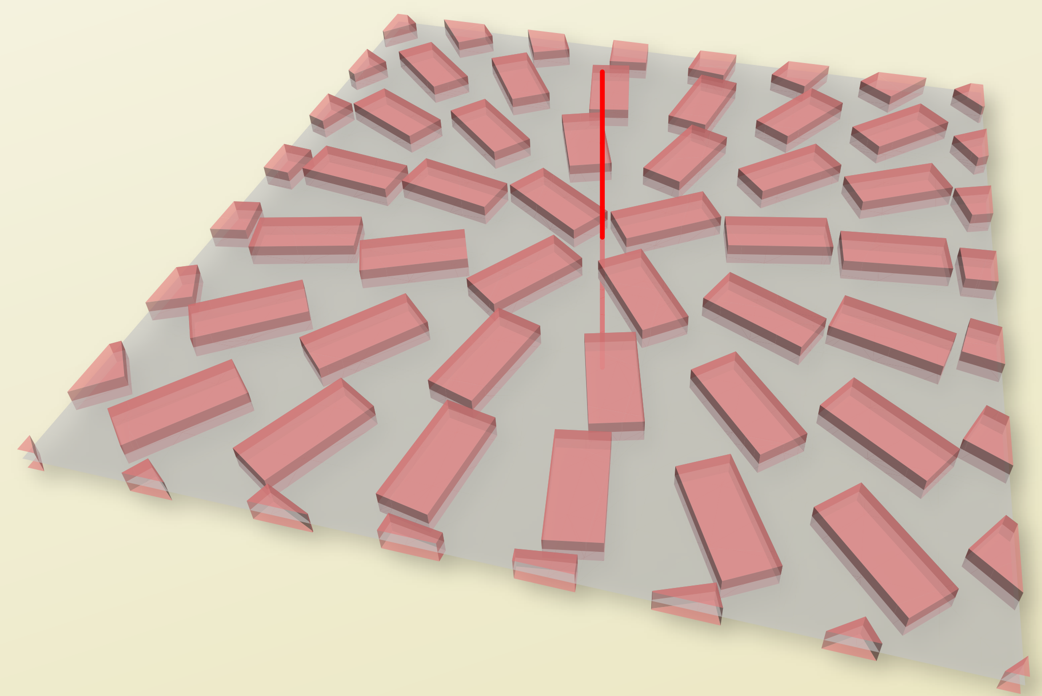
  "escape in the third dimension"

- but not in biaxials (Mermin-Ho)

uniaxial



biaxial



"– 1"

Penn

Meyer *Phil. Mag.* **27**, 405–424 (1973)
Mermin & Ho *Phys. Rev. Lett.* **36**, 594–597 (1976)

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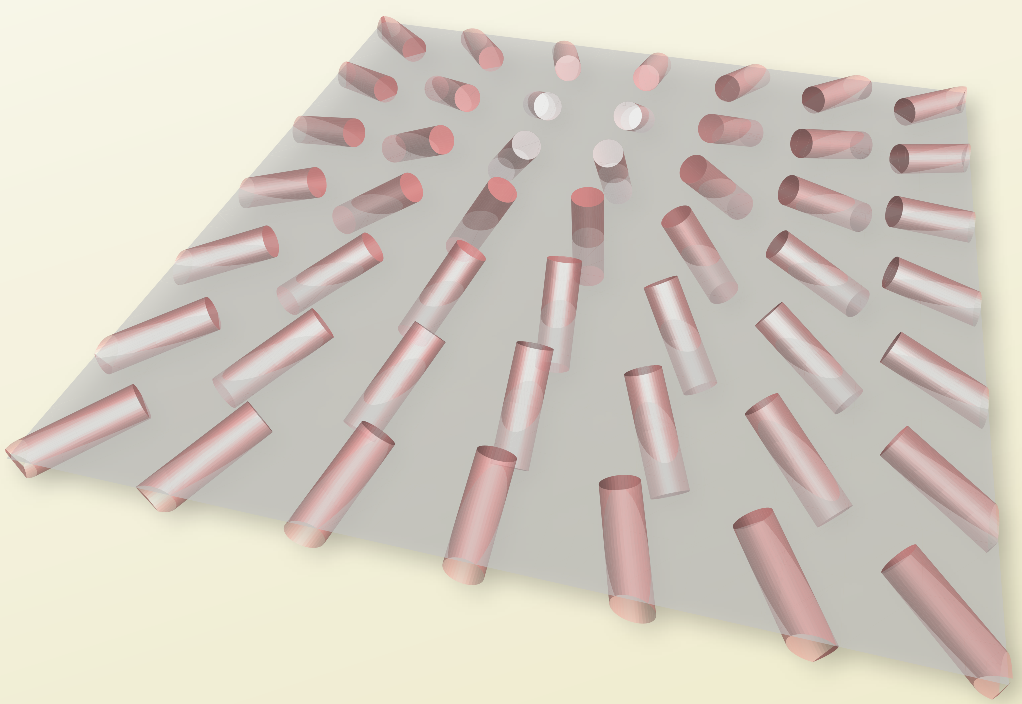
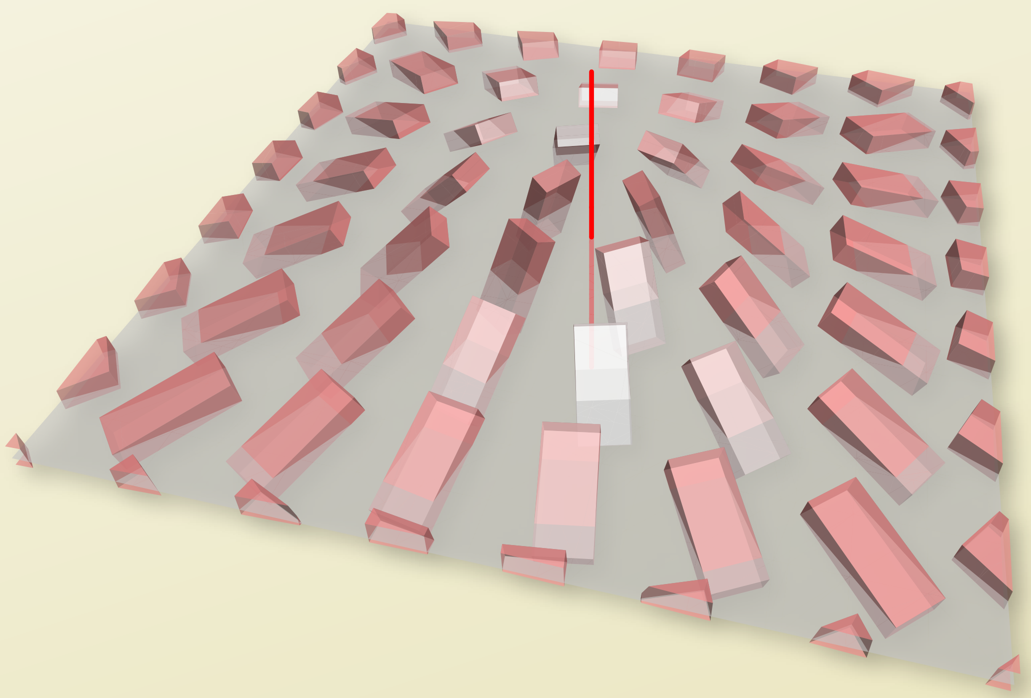


biaxial



" – 1"

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MEYER *Phil. Mag.* **27**, 405–424 (1973)
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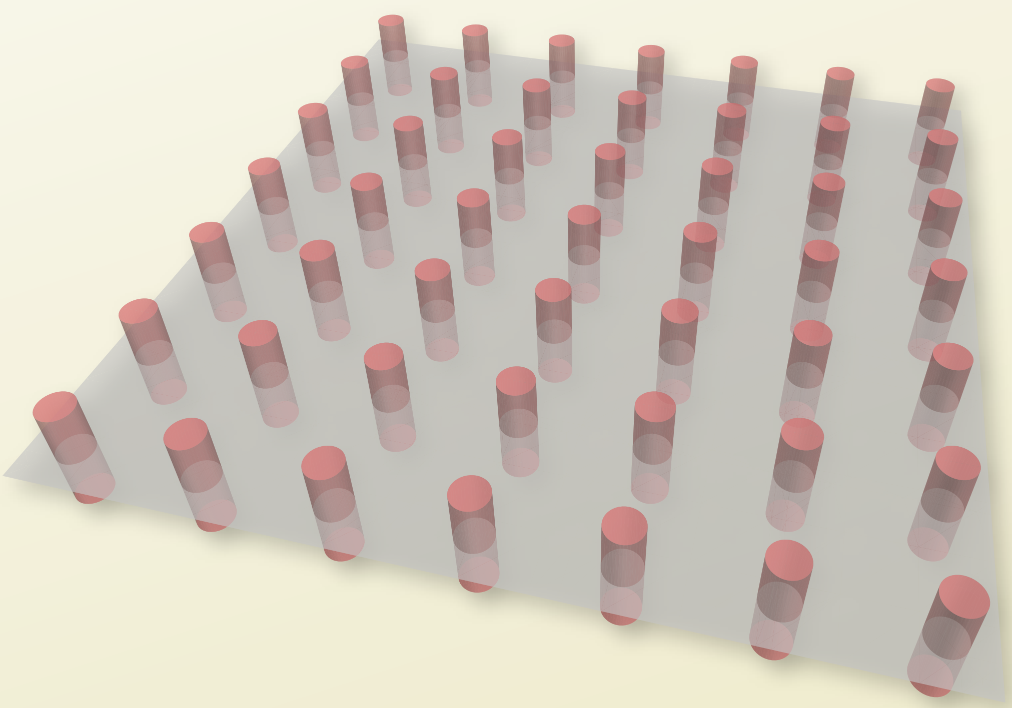
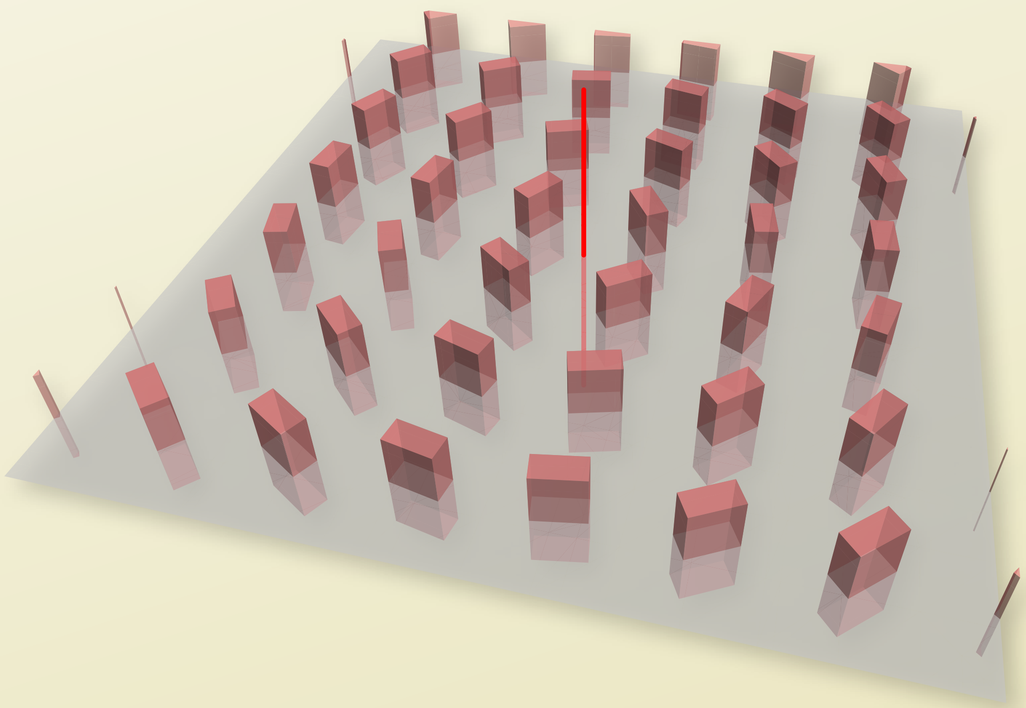


biaxial

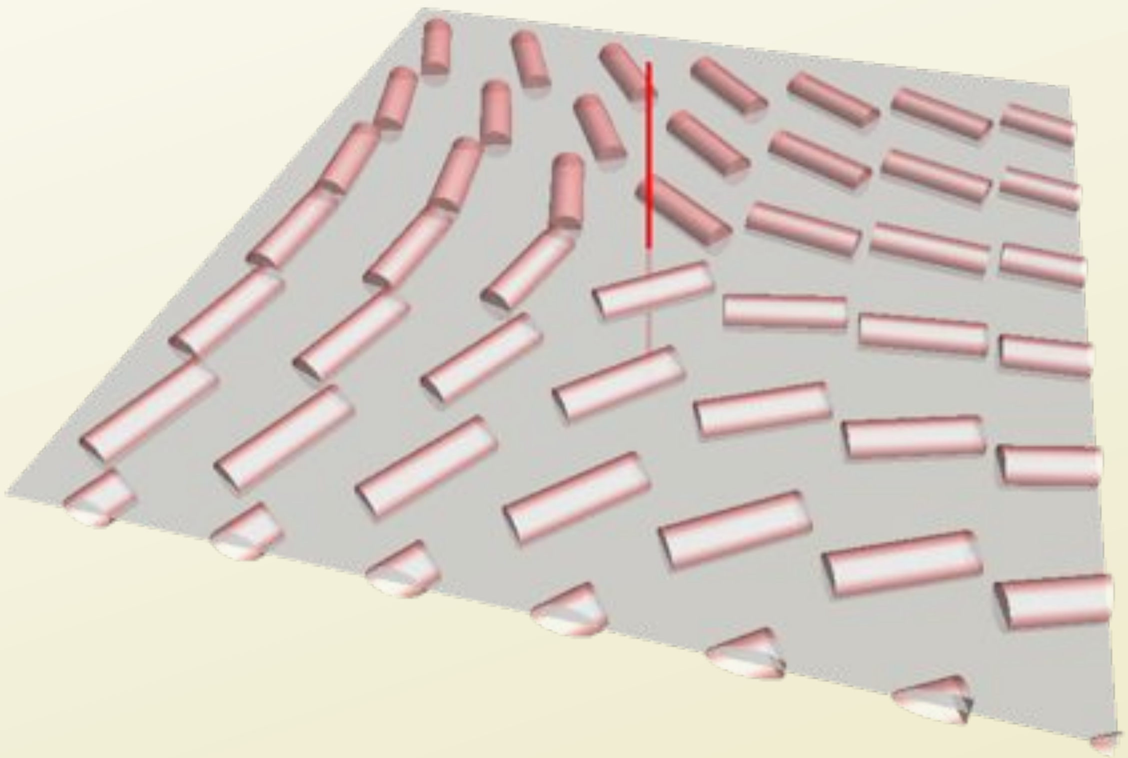


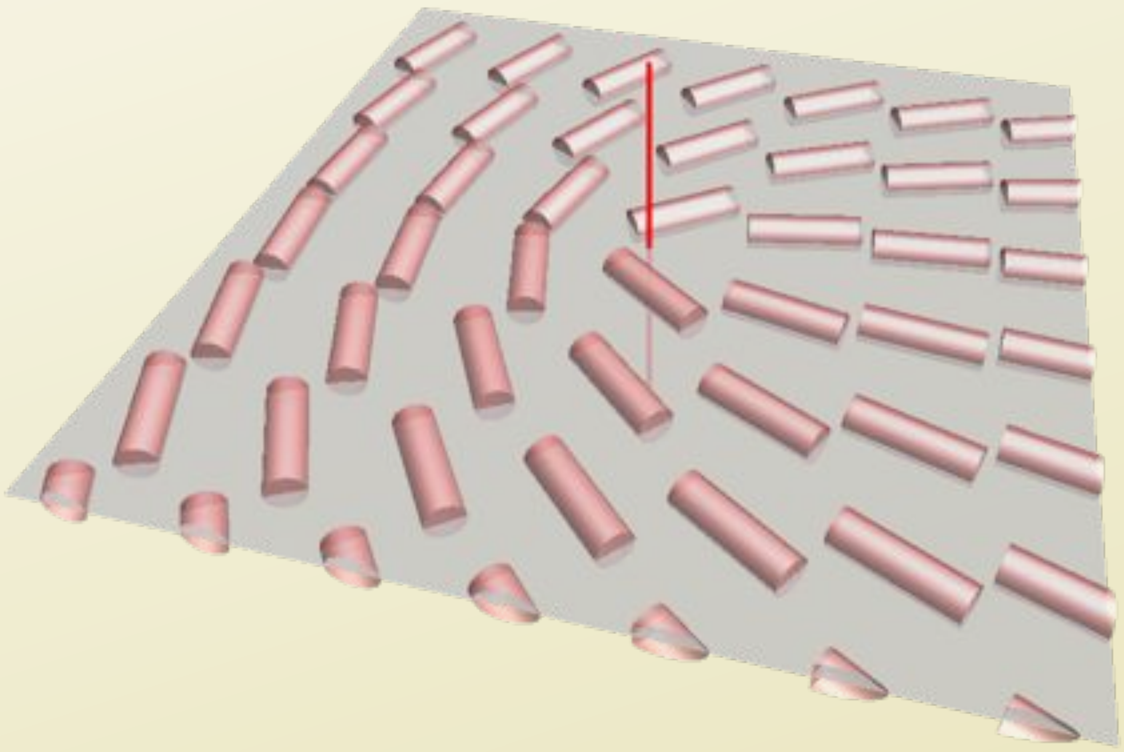
"-1"

Meyer *Phil. Mag.* **27**, 405–424 (1973)
Mermin & Ho *Phys. Rev. Lett.* **36**, 594–597 (1976)

Based and free: 1/2 disclinations

- $\pm\frac{1}{2}$ are homotopic in uniaxial nematics
- but not in biaxials





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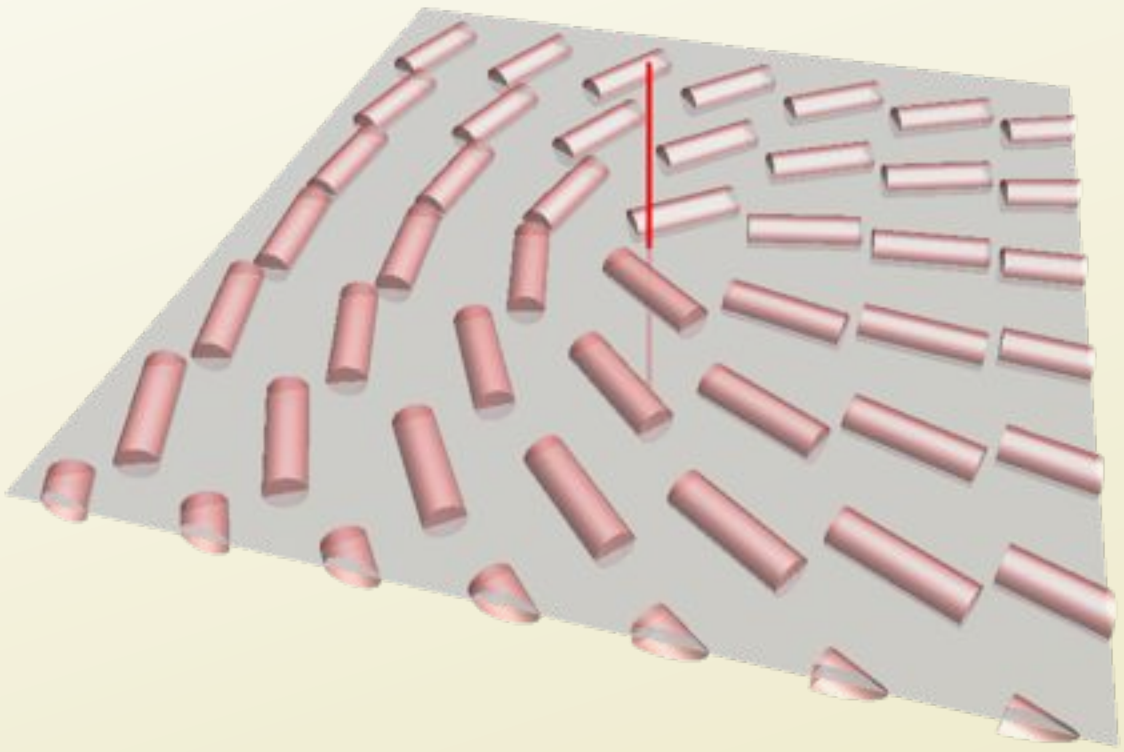

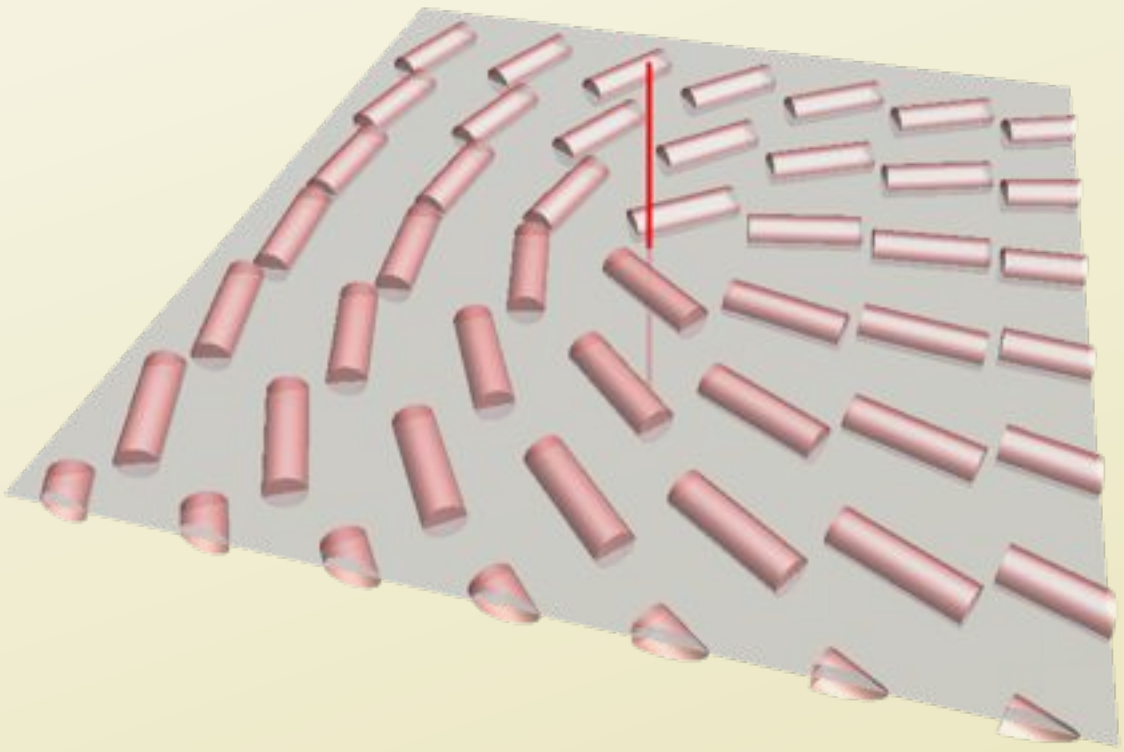

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BASED AND FREE: 1/2 DISCLINATIONS

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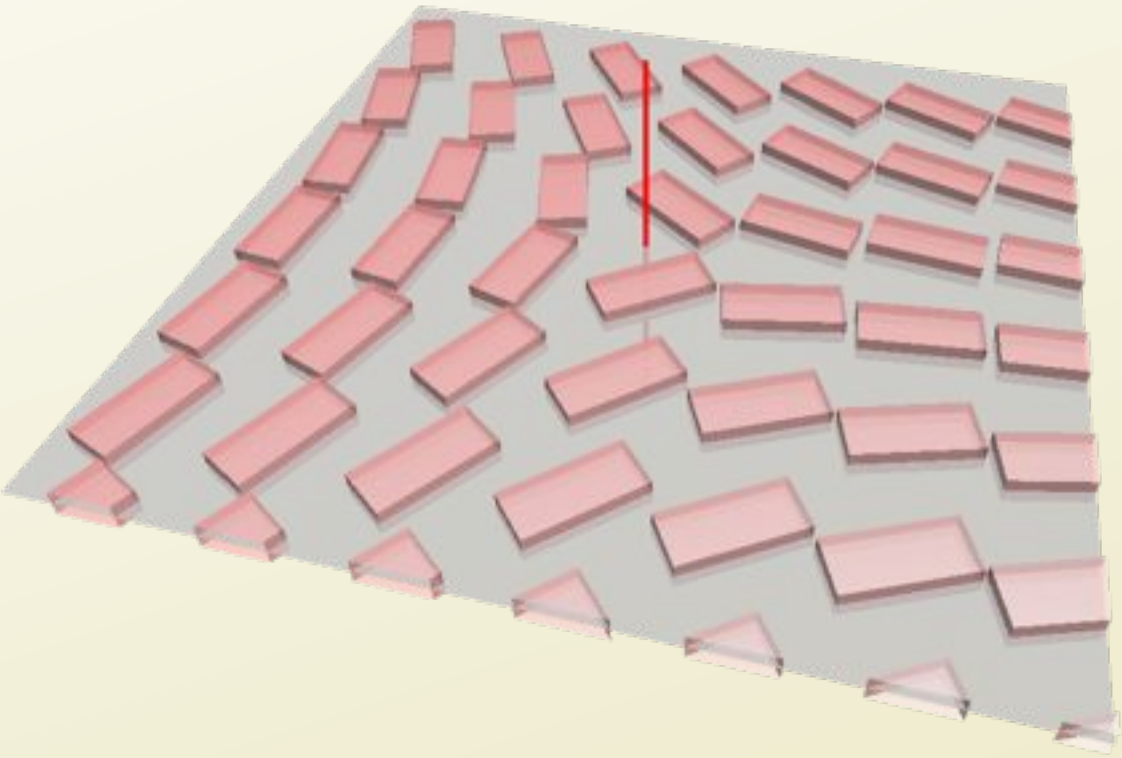

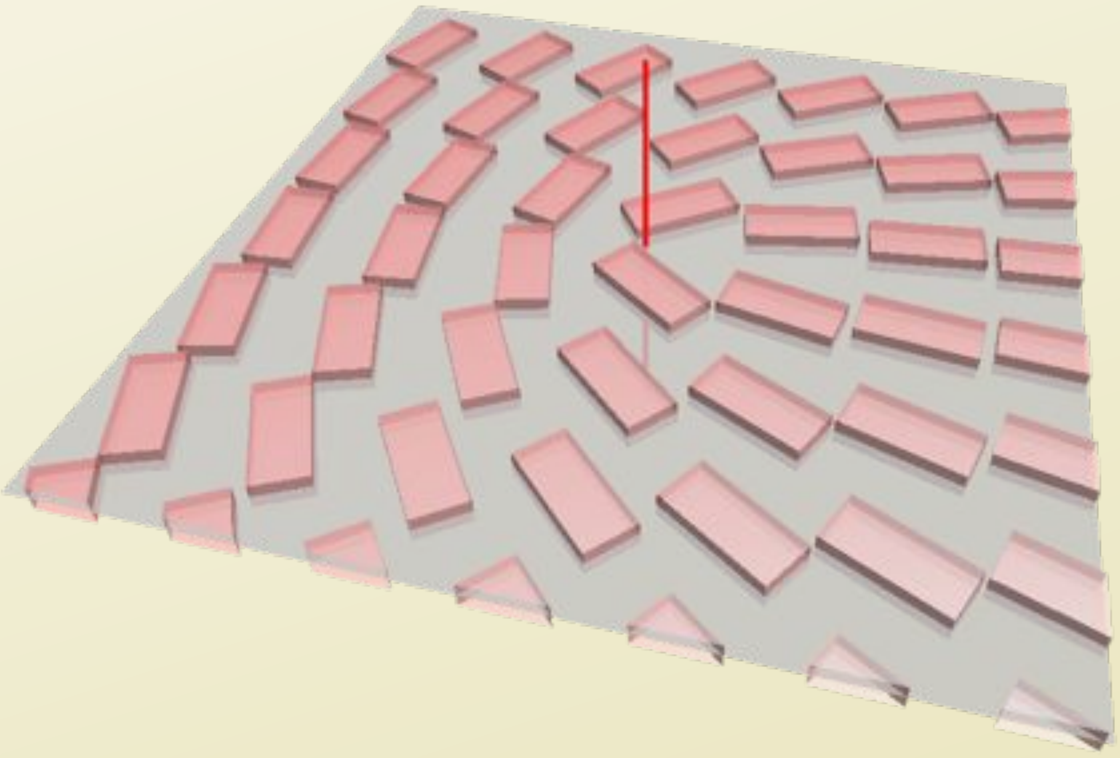

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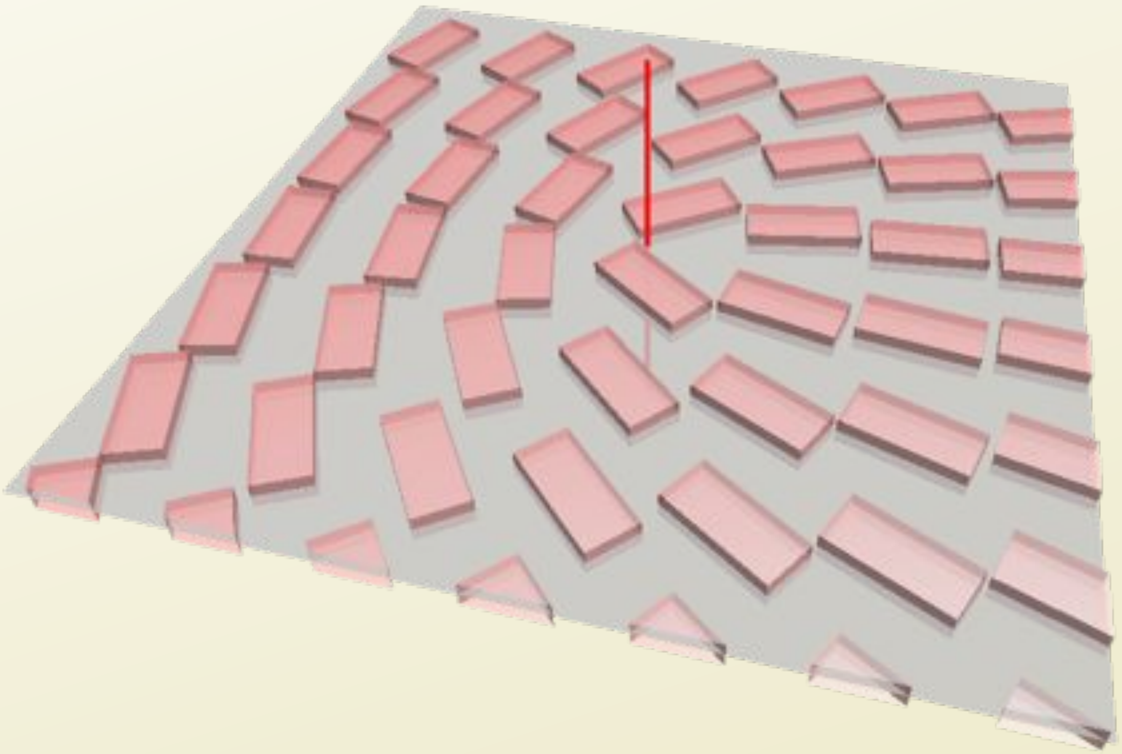

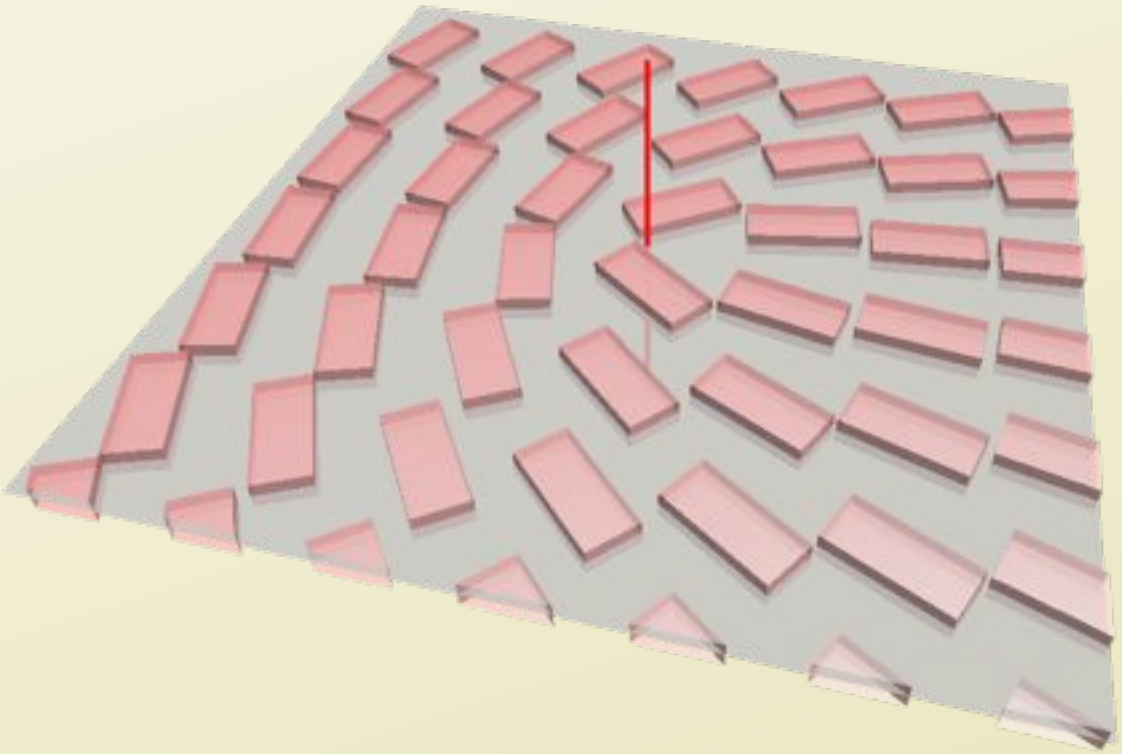

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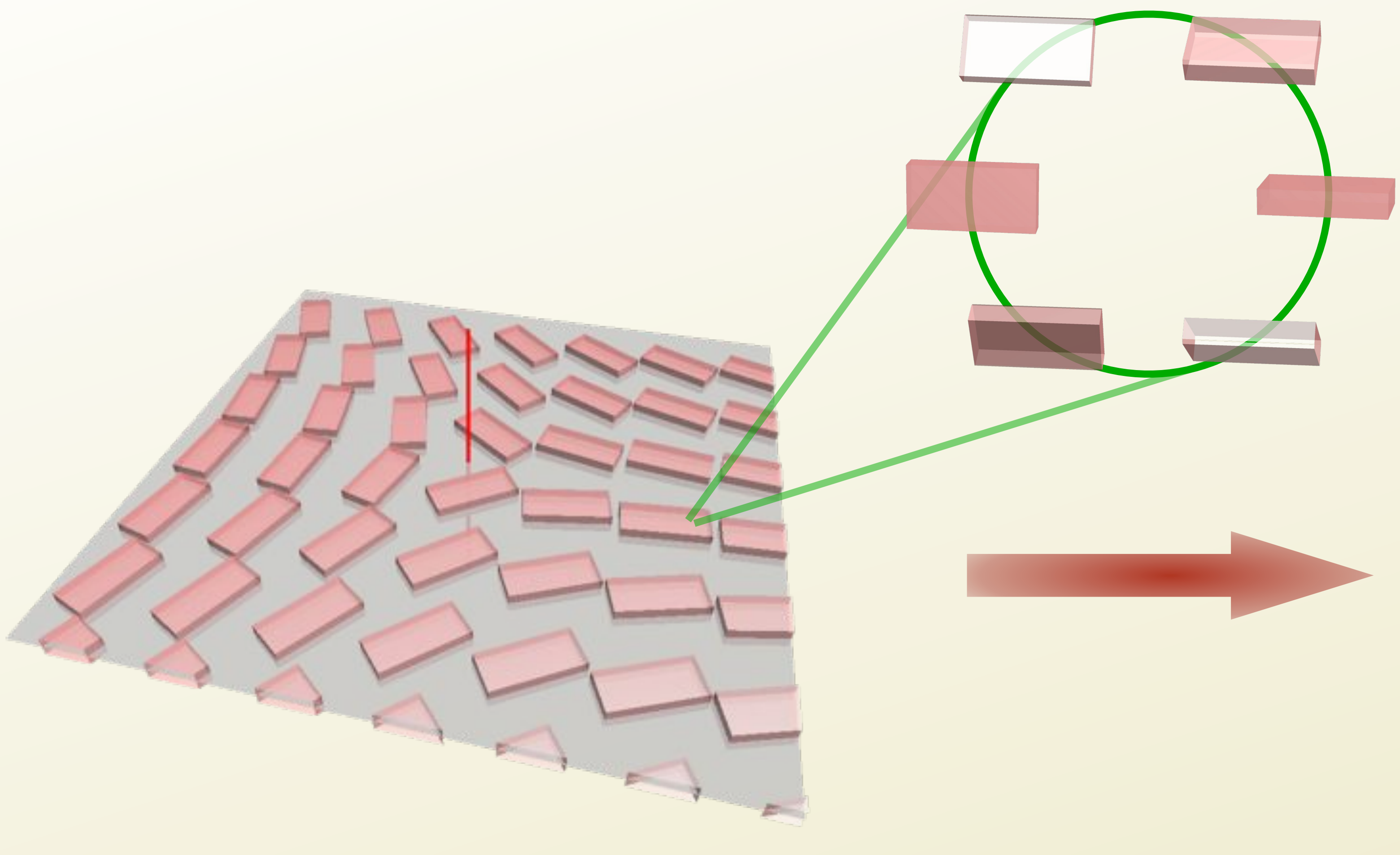
ACTION OF π_1 ON ITSELF

base point traverses a loop in the GSM $\beta \in \pi_1(X, x_0)$



initial defect in class

$\alpha \in \pi_1(X, x_0)$

$-i\boldsymbol{\sigma}_y$

final defect in class

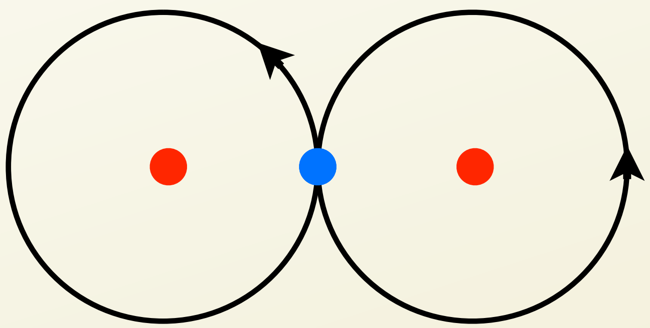
$\alpha^\beta \in \pi_1(X, x_0)$

$(i\boldsymbol{\sigma}_x)(-i\boldsymbol{\sigma}_y)(i\boldsymbol{\sigma}_x)^{-1} = (i\boldsymbol{\sigma}_y)$

where is the defect β *?*

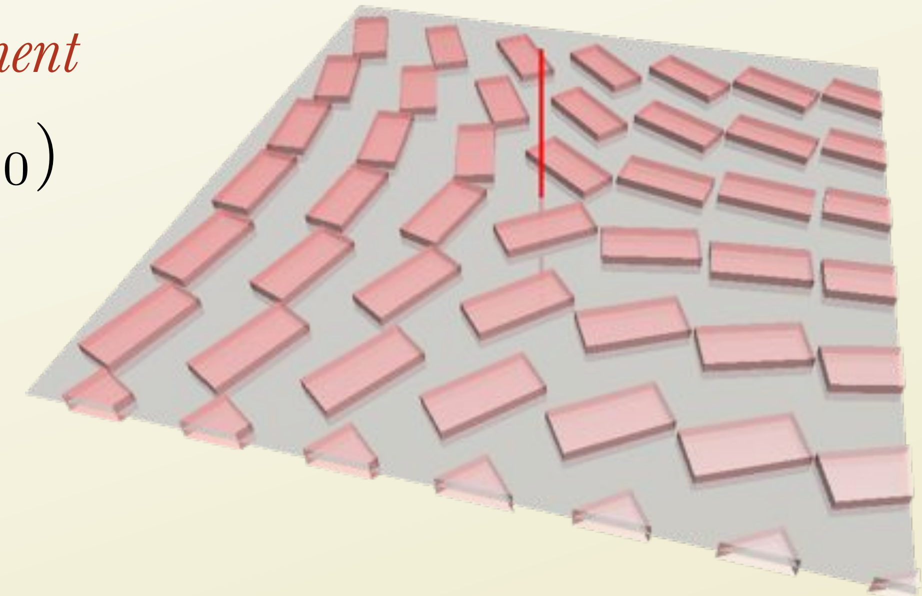
Interaction between defects

$\beta \in \pi_1(X, x_0)$



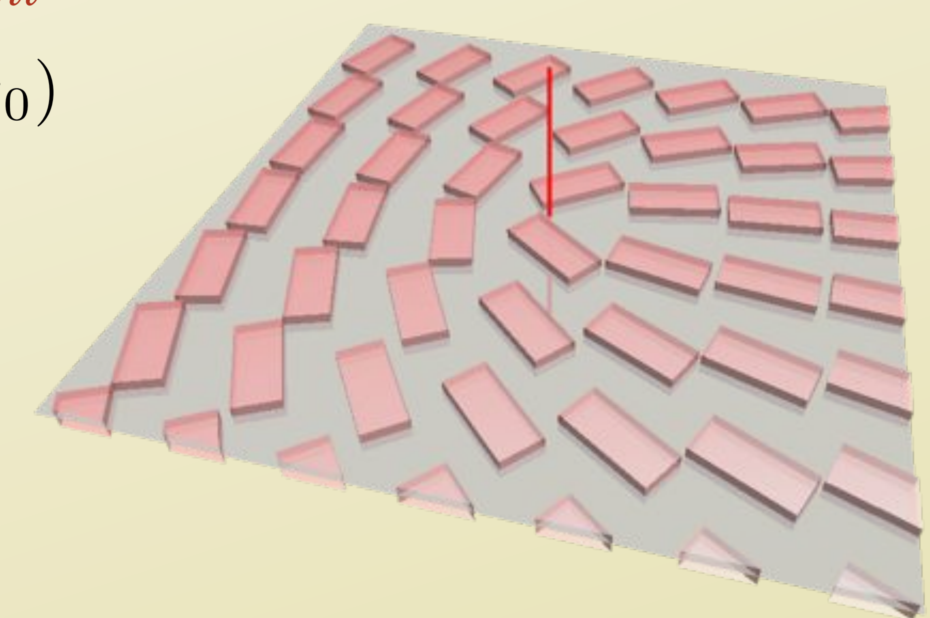
initial measurement

$\alpha \in \pi_1(X, x_0)$



new measurement

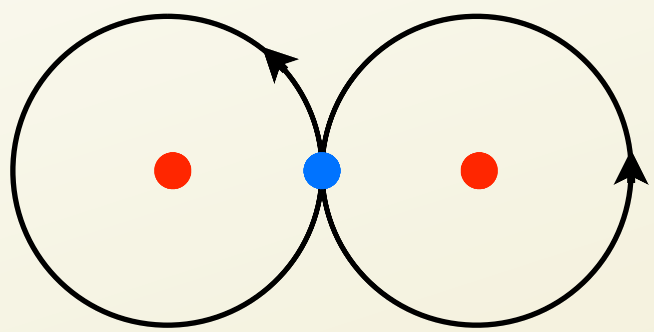
$\alpha^\beta \in \pi_1(X, x_0)$



drag one defect around another

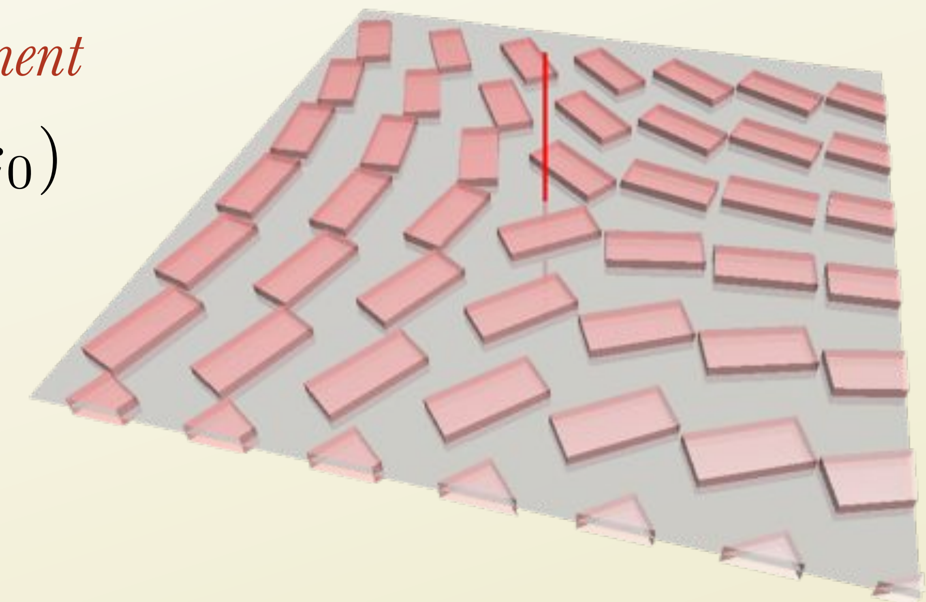
Interaction between defects

$\beta \in \pi_1(X, x_0)$



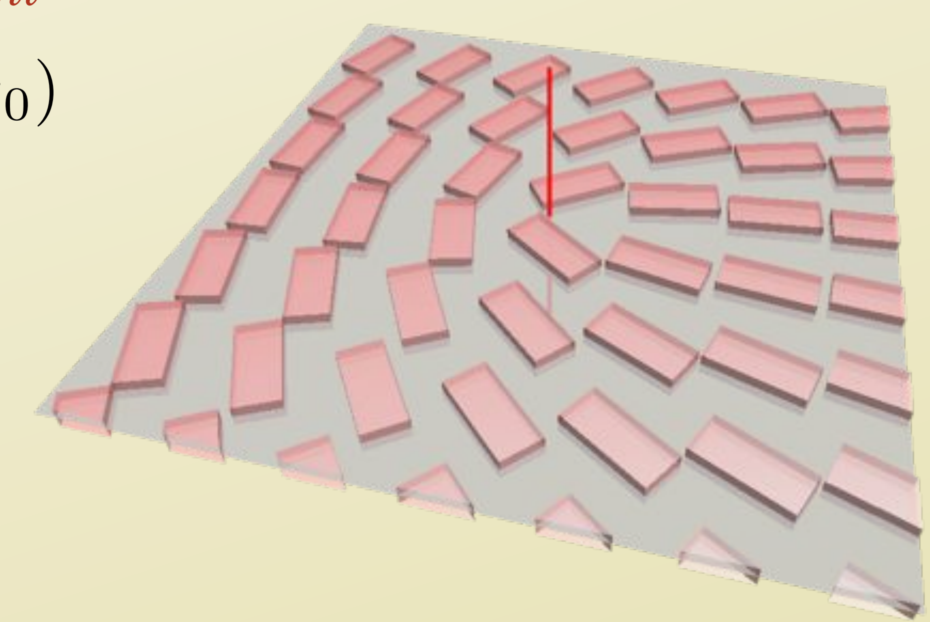
initial measurement

$\alpha \in \pi_1(X, x_0)$



new measurement

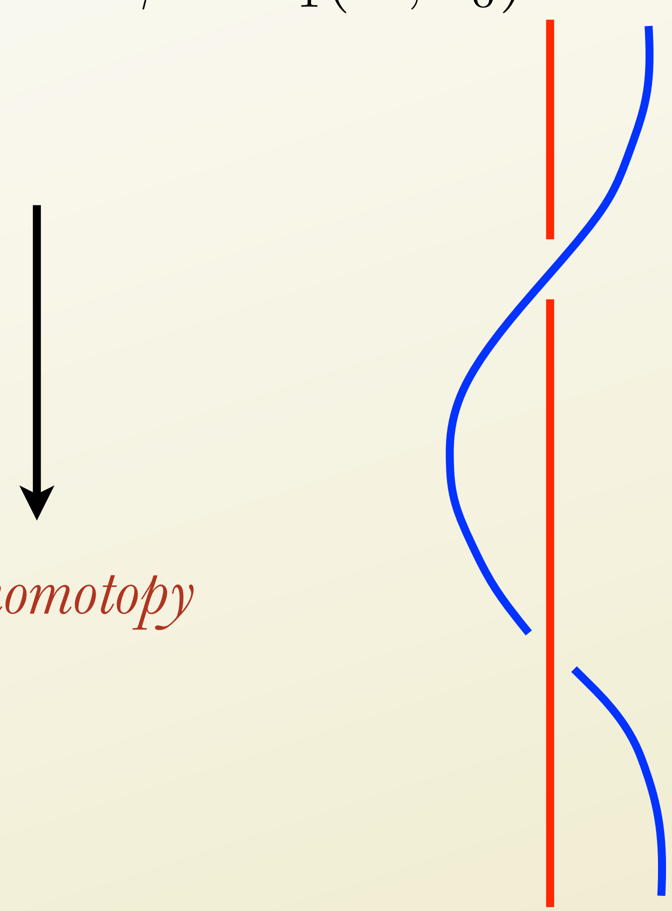
$\alpha^\beta \in \pi_1(X, x_0)$



drag one defect around another

Interaction between defects

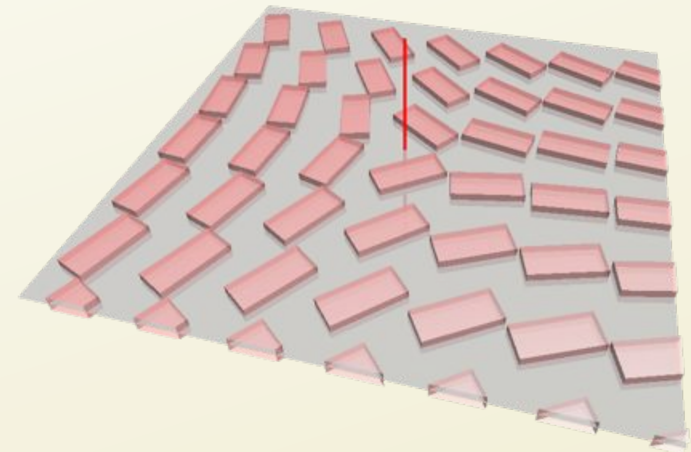
$\beta \in \pi_1(X, x_0)$ $\alpha \in \pi_1(X, x_0)$

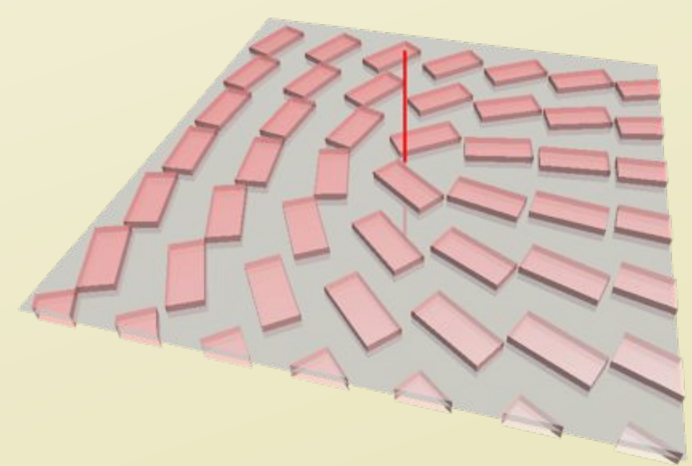


free homotopy

$\alpha^\beta \in \pi_1(X, x_0)$

“drag one defect around another”





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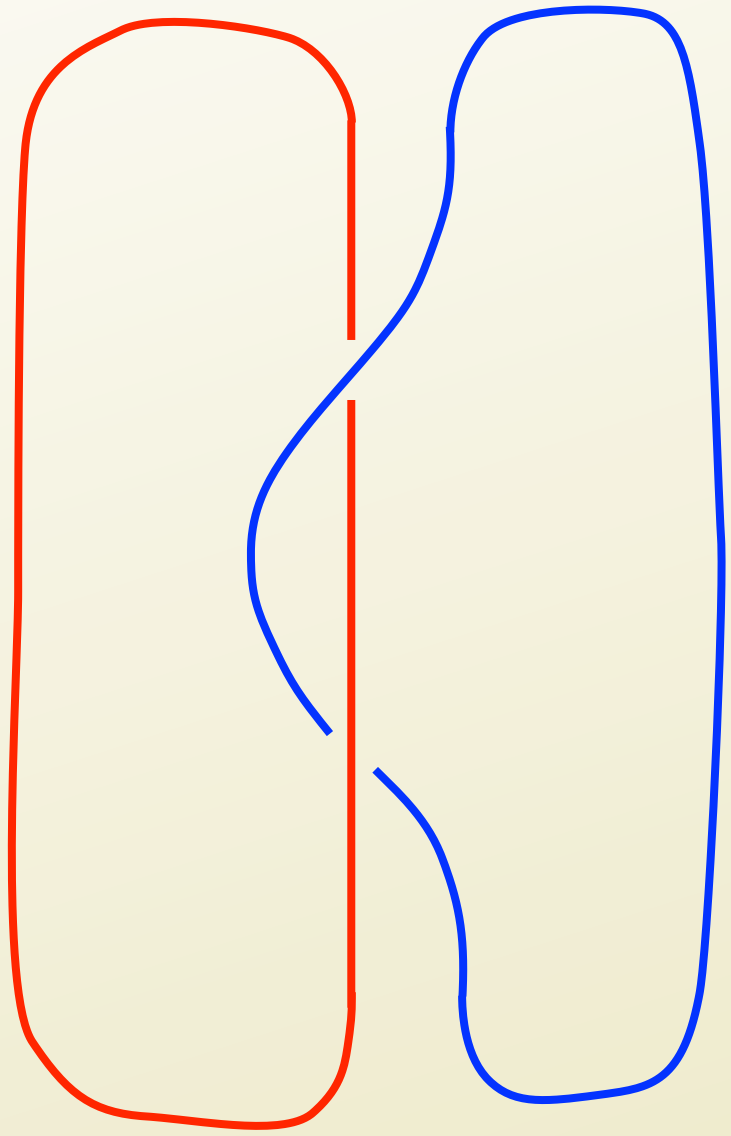
Interaction between defects

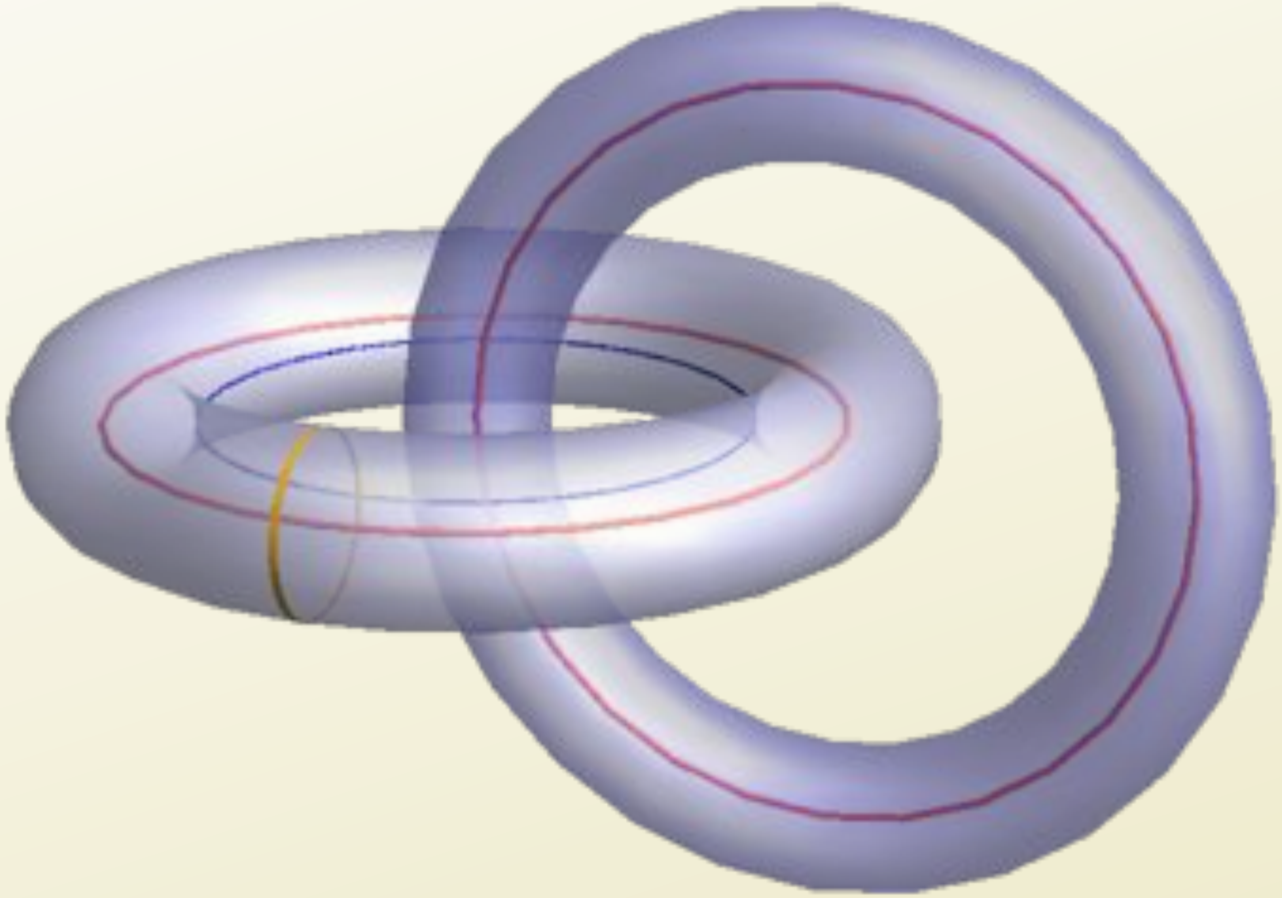


linked loops

but Randy said:
"our defects don't end"

Penn

Linked loops

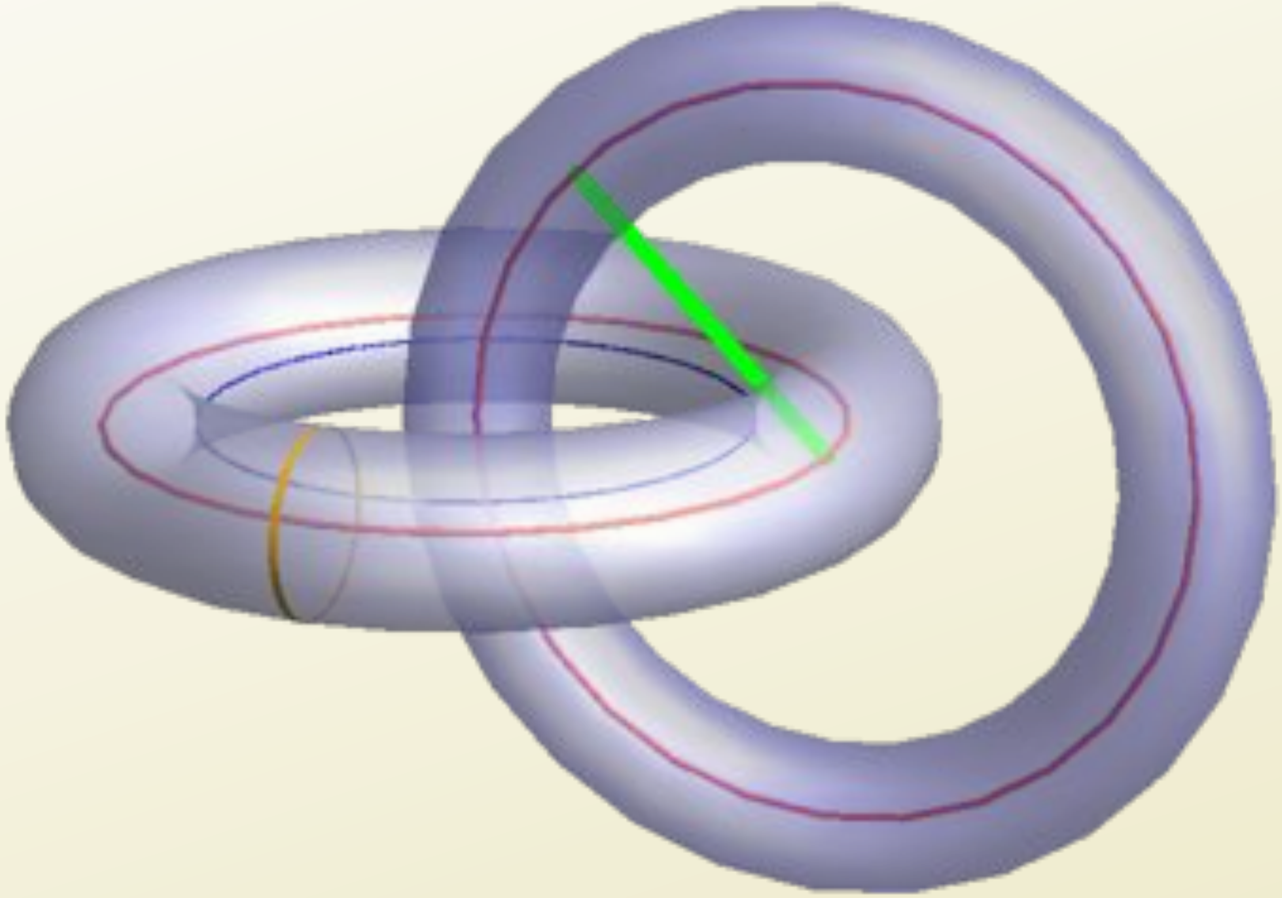


orange circle measures $\alpha \in \pi_1(X, x_0)$

blue circle measures $\beta \in \pi_1(X, x_0)$

Penn

LINKED LOOPS



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two defects collectively define a third

Penn

Linked loops

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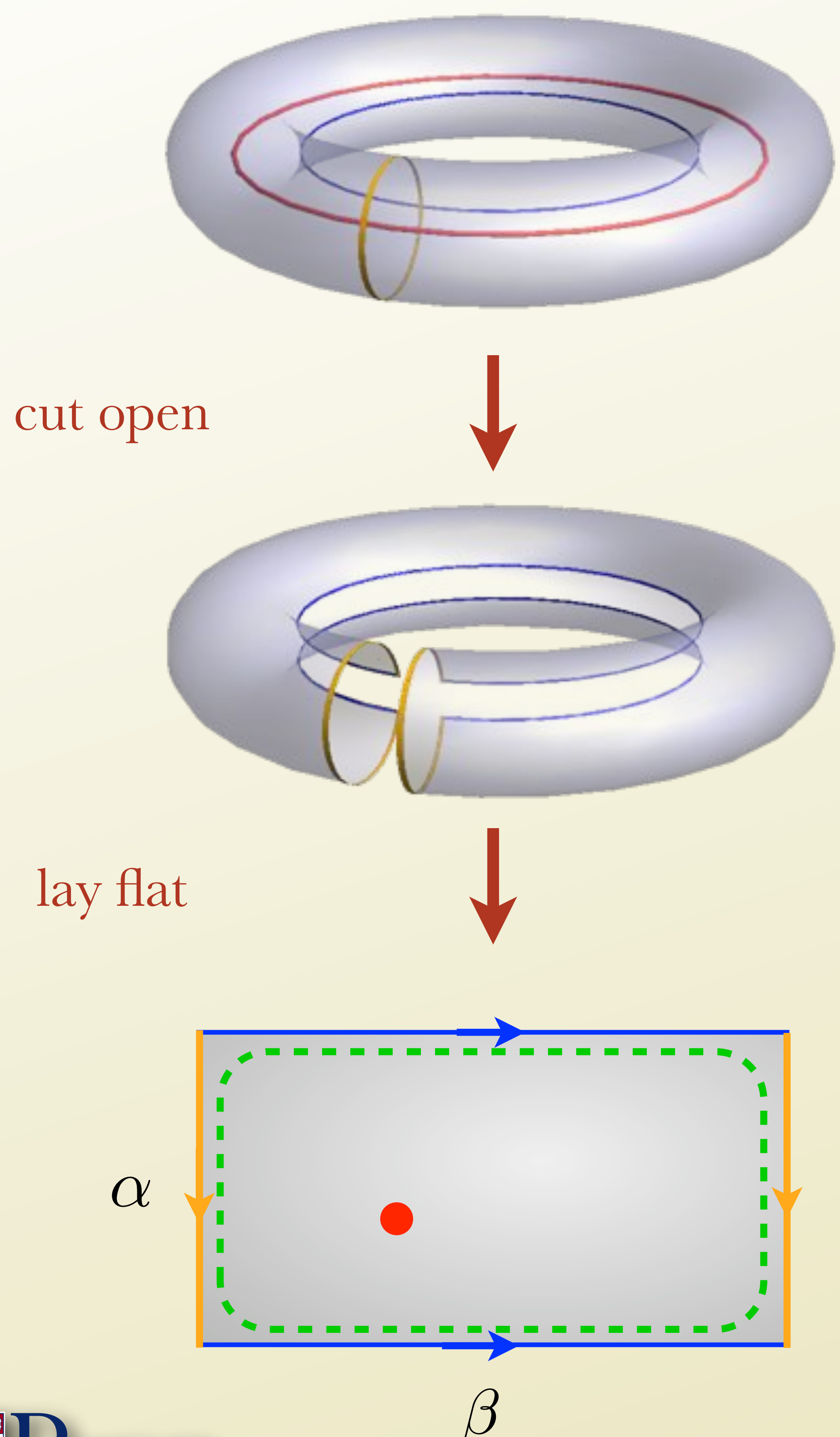
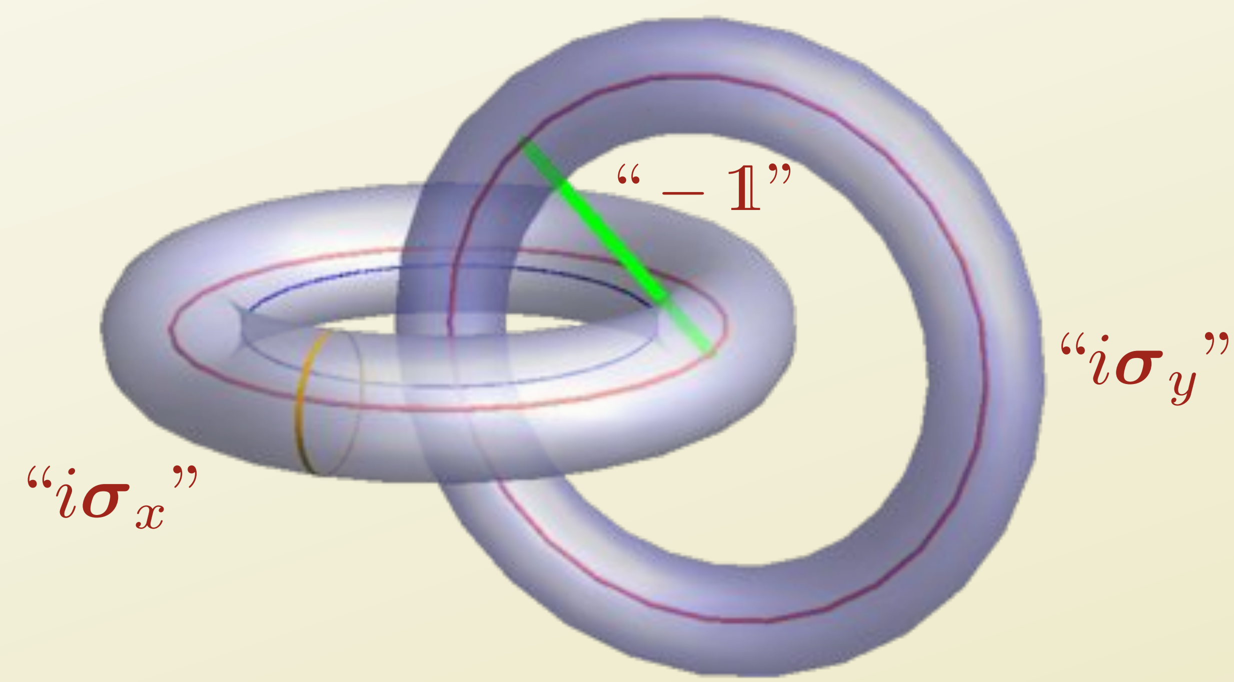


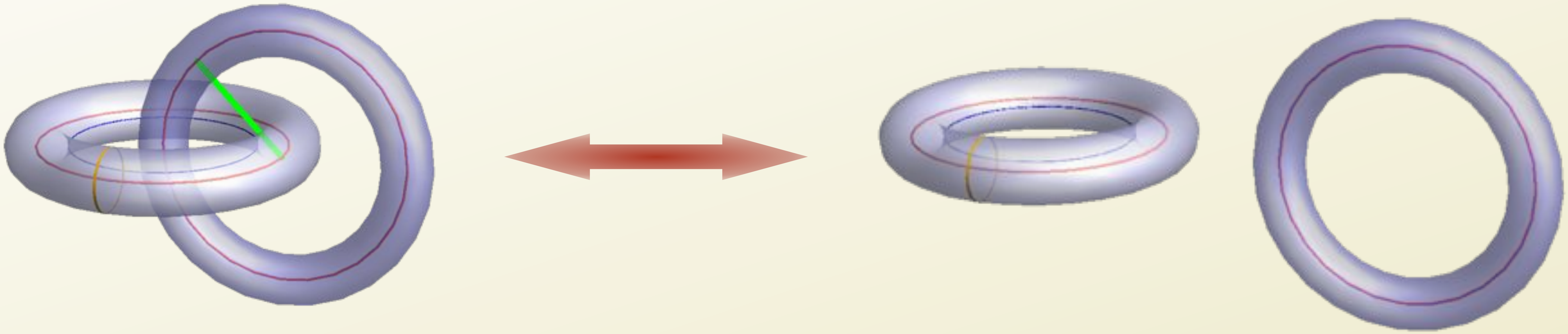


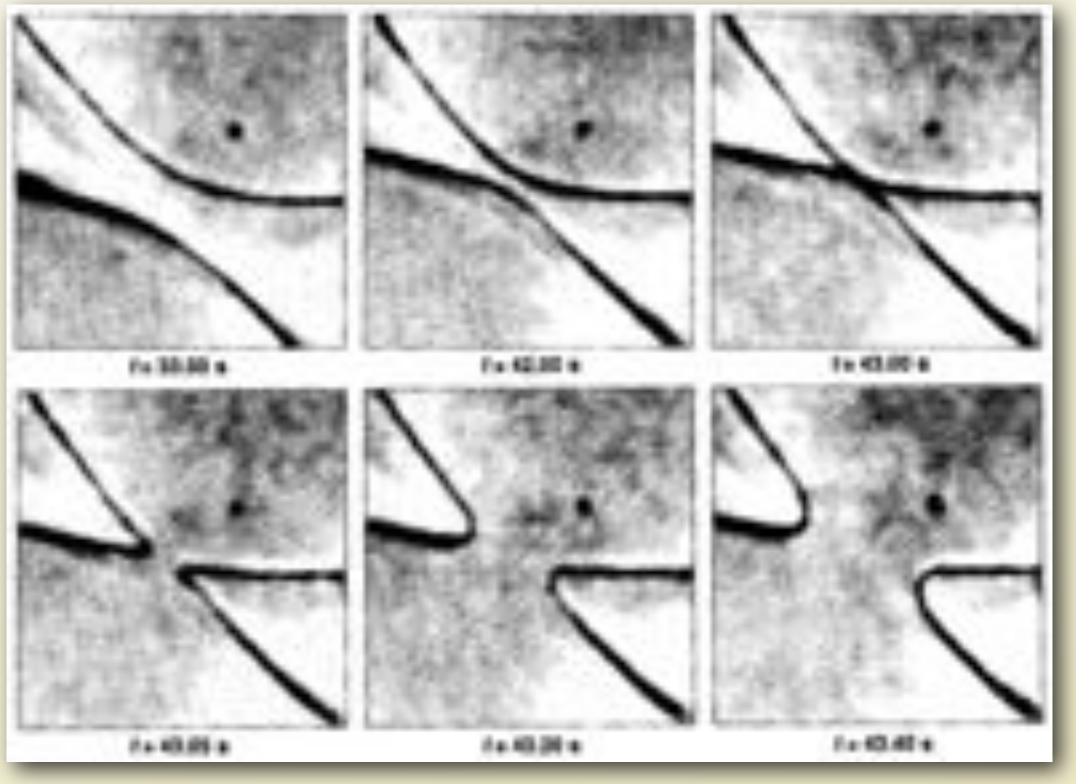
$$[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$$

Kléman *J. Phys. France Lett.* **38**, 199–202 (1977)
Poénaru & Toulouse *J. Phys. France* **38**, 887–895 (1977)

Defect crossing

when defects cross,
sometimes there's a tether





Chuang, Durrer, Turok & Yurke *Science* **251**, 1336–1342 (1991)

no experiments yet …

Poénaru & Toulouse *J. Phys. France* **38**, 887–895 (1977)

LINKED LOOPS

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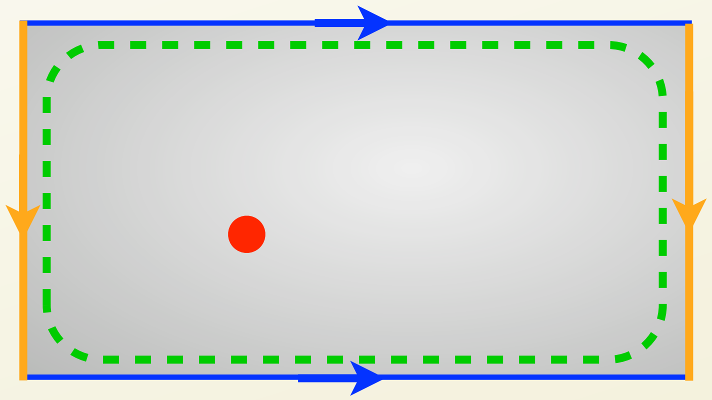
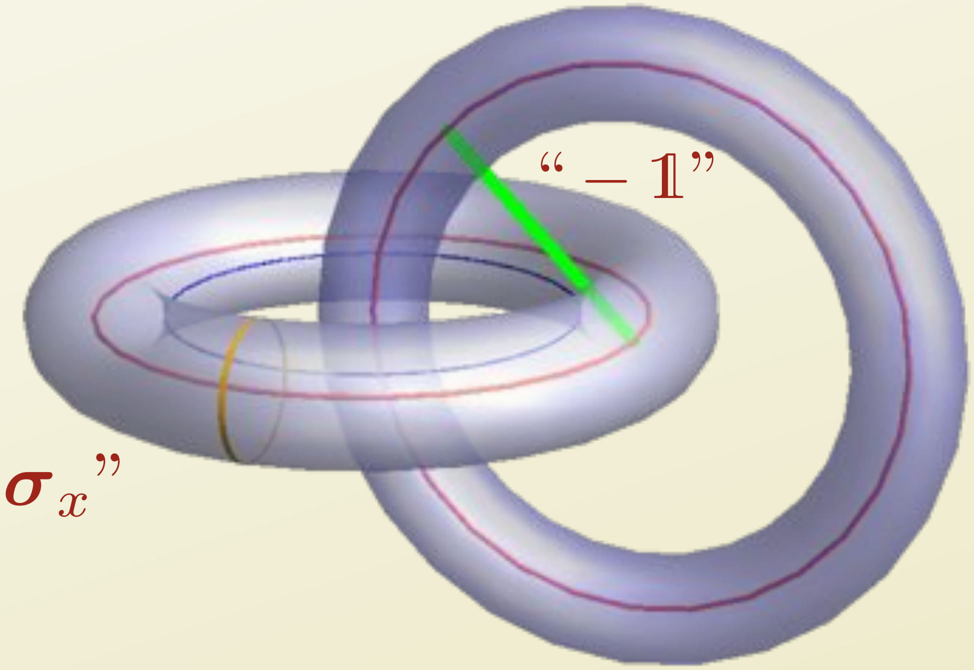
orange circle measures $\alpha \in \pi_1(X, x_0)$

blue circle measures $\beta \in \pi_1(X, x_0)$





Given two defects α, β in the form of linked loops they collectively define a third

$$\alpha,\, \beta \quad \rightarrow \quad [\alpha, \beta]$$

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \quad \rightarrow \quad \pi_1(X, x_0)$$

Whitehead product

WHITEHEAD *Ann. of Math.* **42**, 409–428 (1941)
KLÉMAN *J. Phys. France Lett.* **38**, 199–202 (1977)
POÉNARU & TOULOUSE *J. Phys. France* **38**, 887–895 (1977)

WHITEHEAD PRODUCTS

two defects collectively define a third

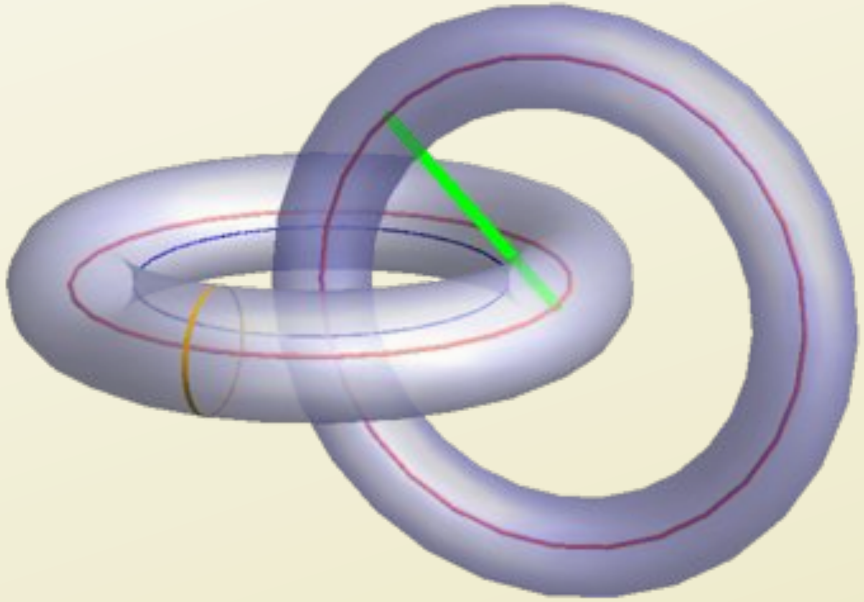
$$\pi_p(X, x_0) \times \pi_q(X, x_0) \quad \rightarrow \quad \pi_{p+q-1}(X, x_0)$$

think of a "p-defect" linking a "q-defect" in $\mathbb{R}^{p+q+1}$

surround the "p-defect" with a $S^p \times S^q$

cut this open along a $S^p \vee S^q$ to give a D^{p+q}

the map on the boundary $\partial D^{p+q} = S^{p+q-1}$
is the Whitehead product



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WHITEHEAD *Ann. of Math.* **42**, 409–428 (1941)

WHITEHEAD PRODUCTS

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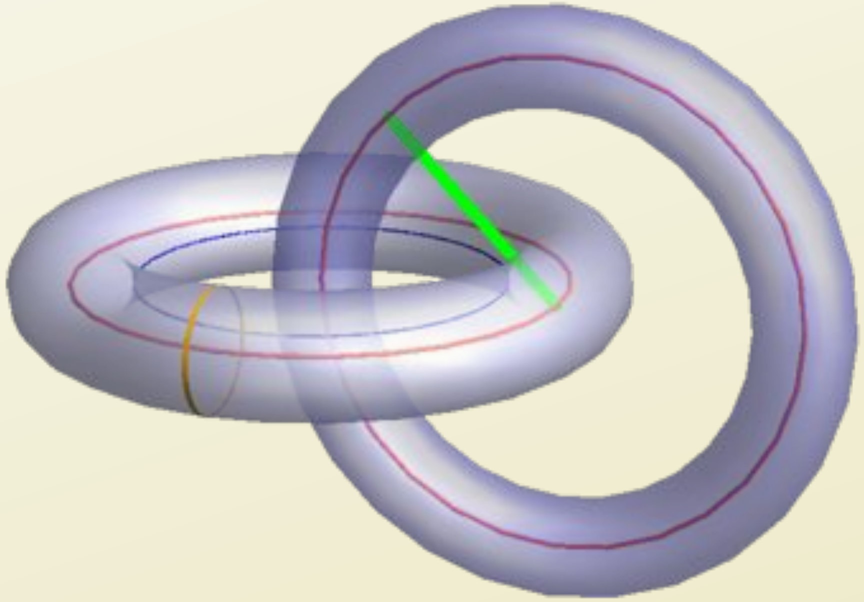
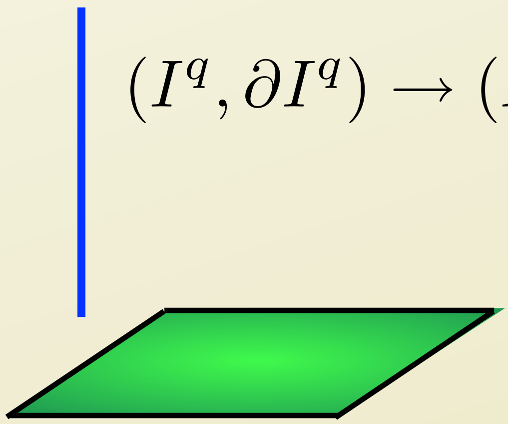
cut this open along a $S^p \vee S^q$ to give a D^{p+q}

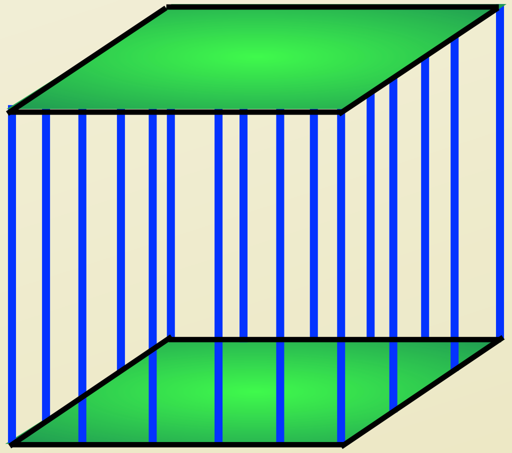
the map on the boundary $\partial D^{p+q} = S^{p+q-1}$
is the Whitehead product





$(I^q, \partial I^q) \to (X, x_0)$

$(I^p, \partial I^p) \to (X, x_0)$



$\partial I^{p+q} \to (X, x_0)$

simplest case $\quad p = q = 1$

next simplest $\quad p = 2,\ q = 1$

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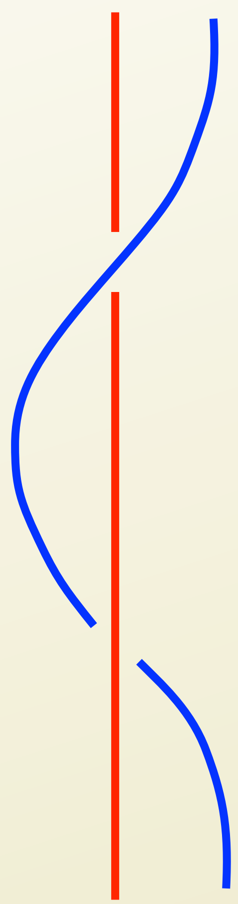
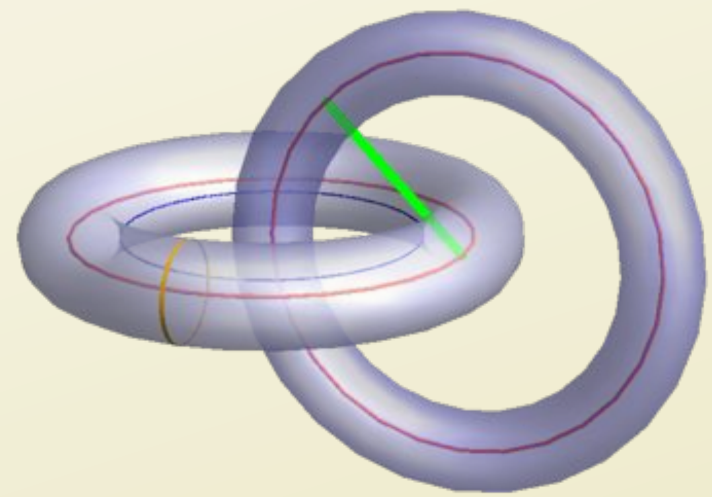
WHITEHEAD *Ann. of Math.* **42**, 409–428 (1941)

Remember how we got here

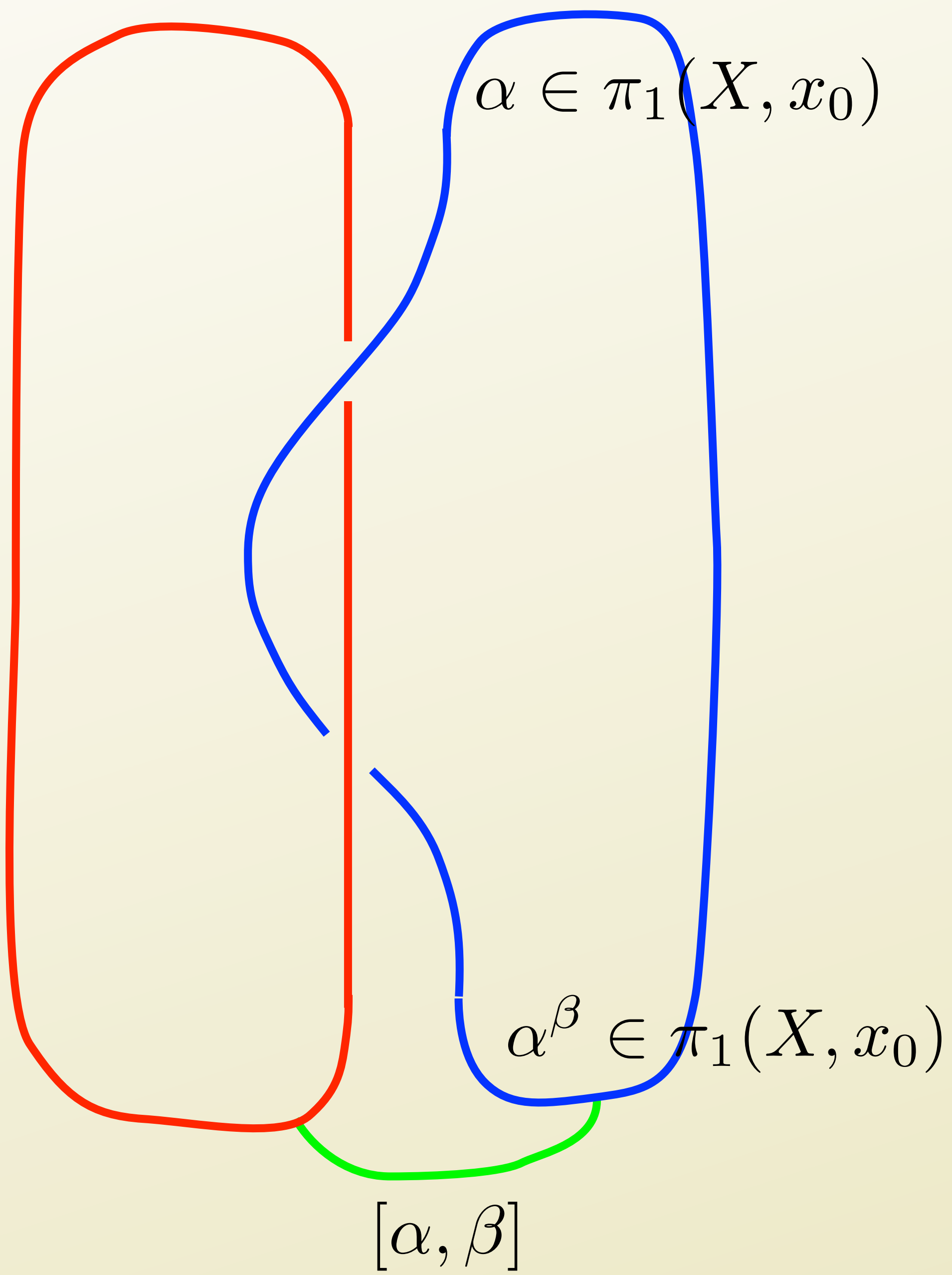
$\alpha \in \pi_1(X, x_0)$

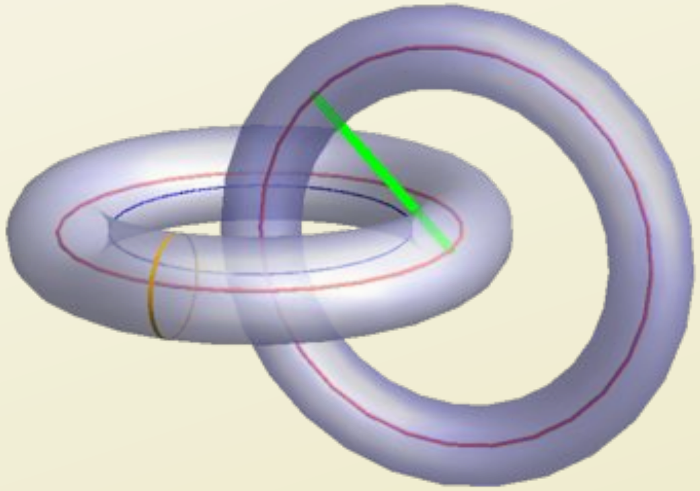


$\alpha^\beta \in \pi_1(X, x_0)$



REMEMBER HOW WE GOT HERE





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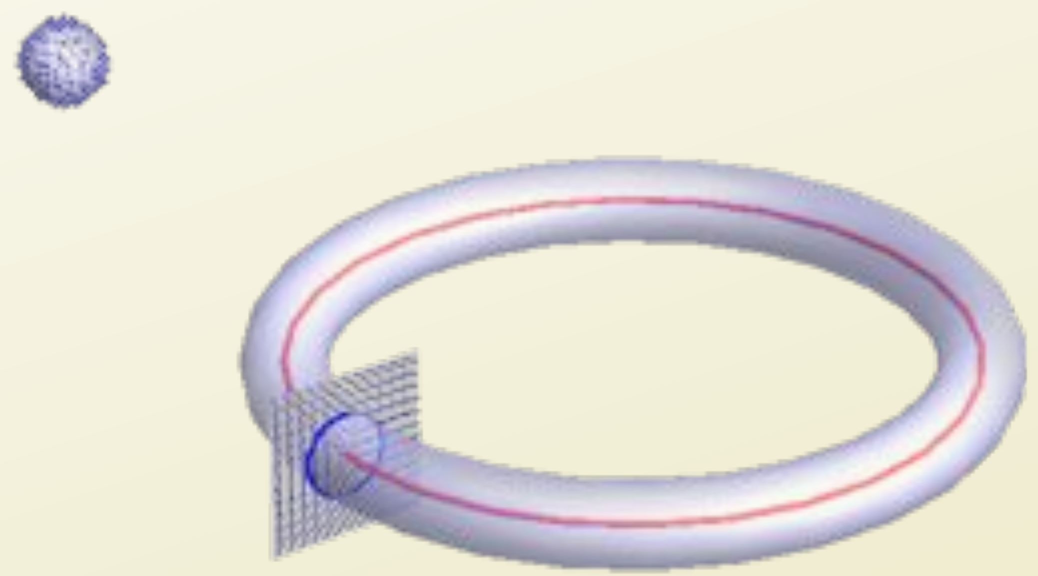
Moving a hedgehog around a disclination

hedgehog

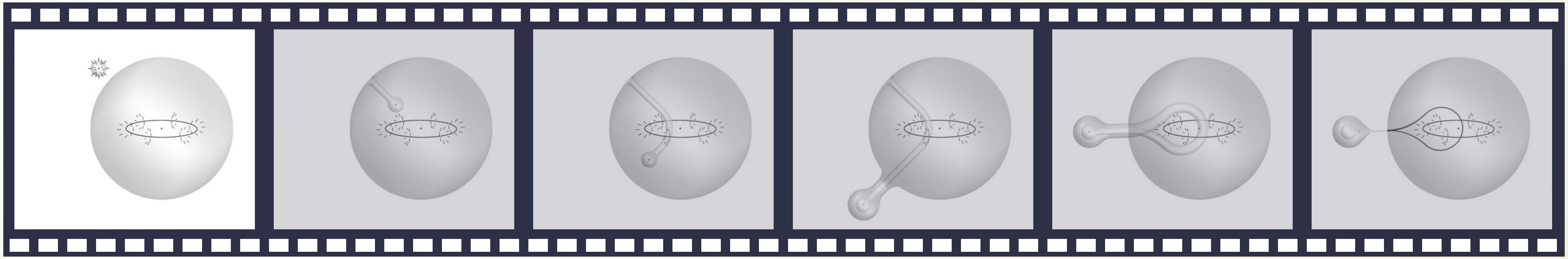
$p \in \pi_2(\mathbb{RP}^2, x_0)$

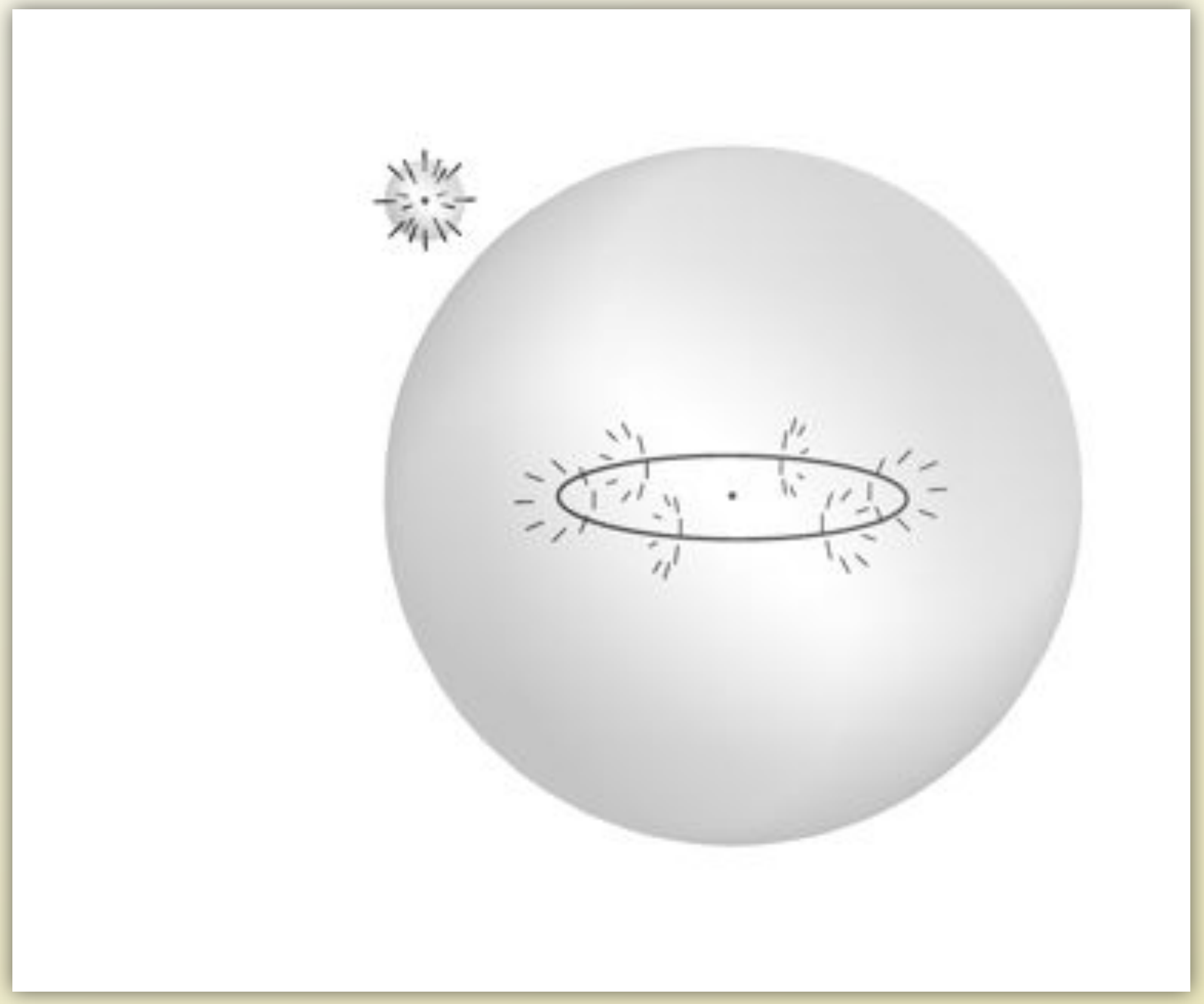


$-1 \in \pi_1(\mathbb{RP}^2, x_0)$

disclination loop

Moving a hedgehog around a disclination

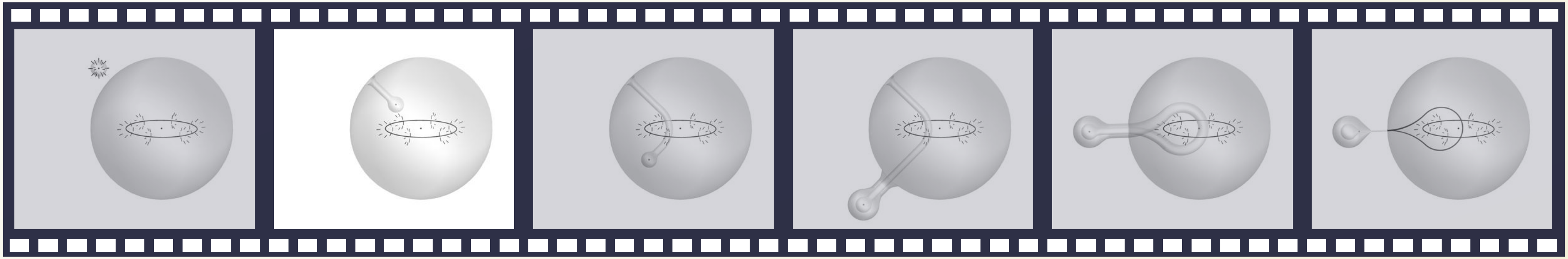


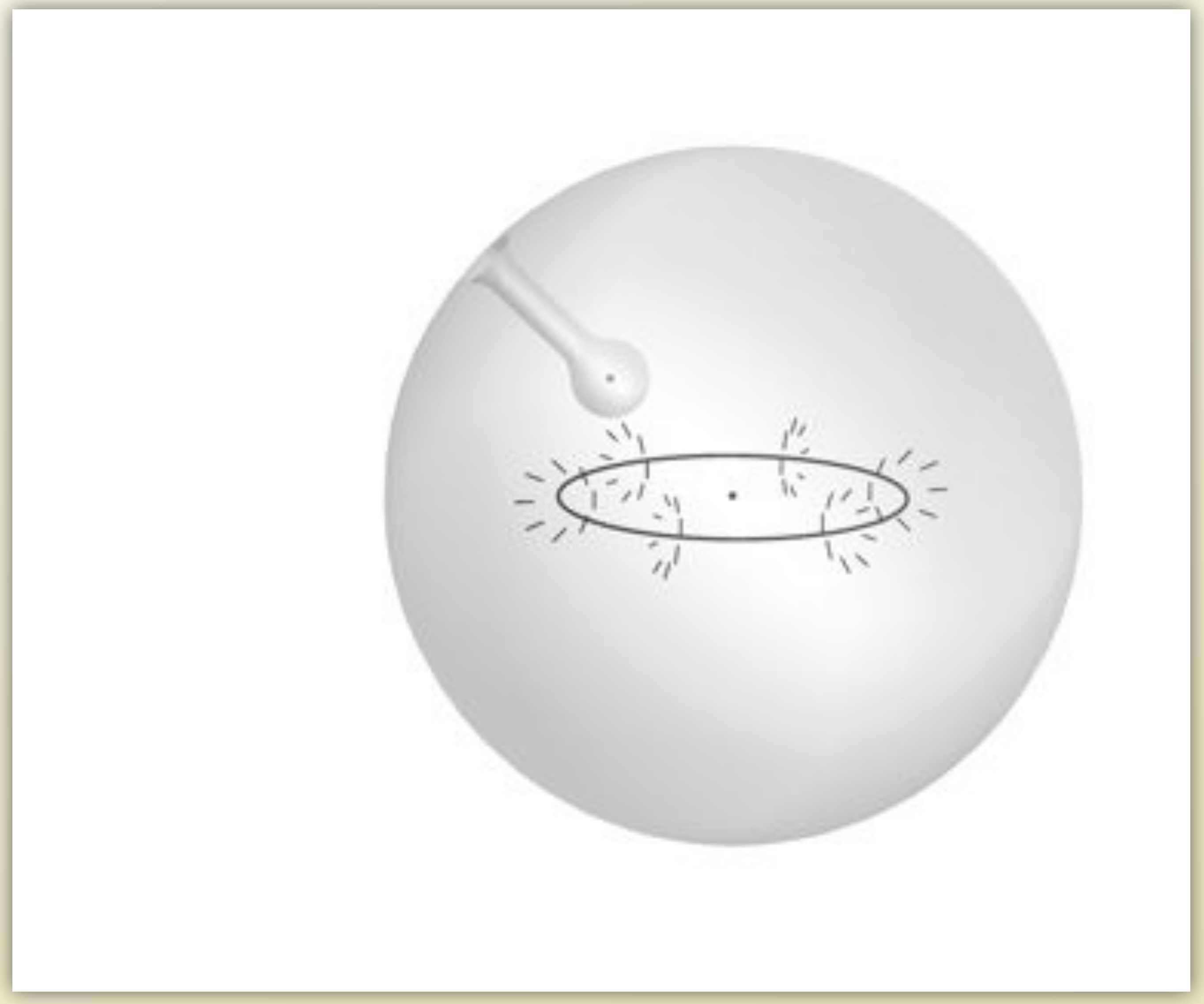


Penn

images: Sabetta Matsumoto

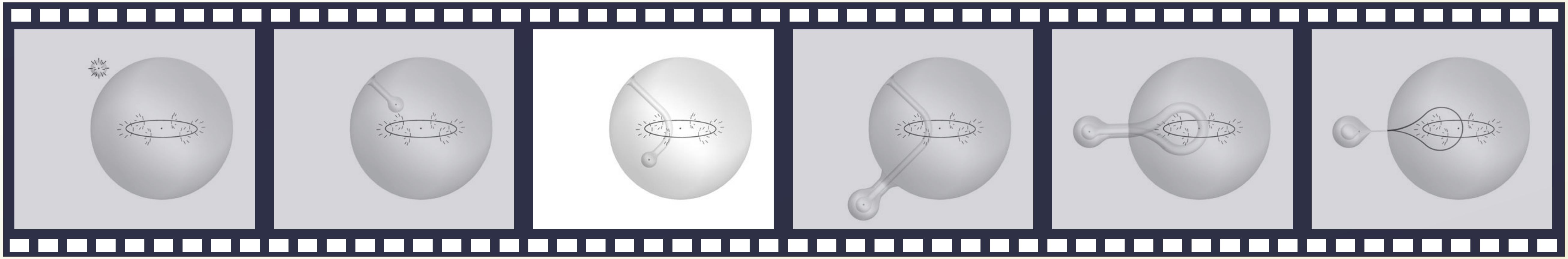
Moving a hedgehog around a disclination

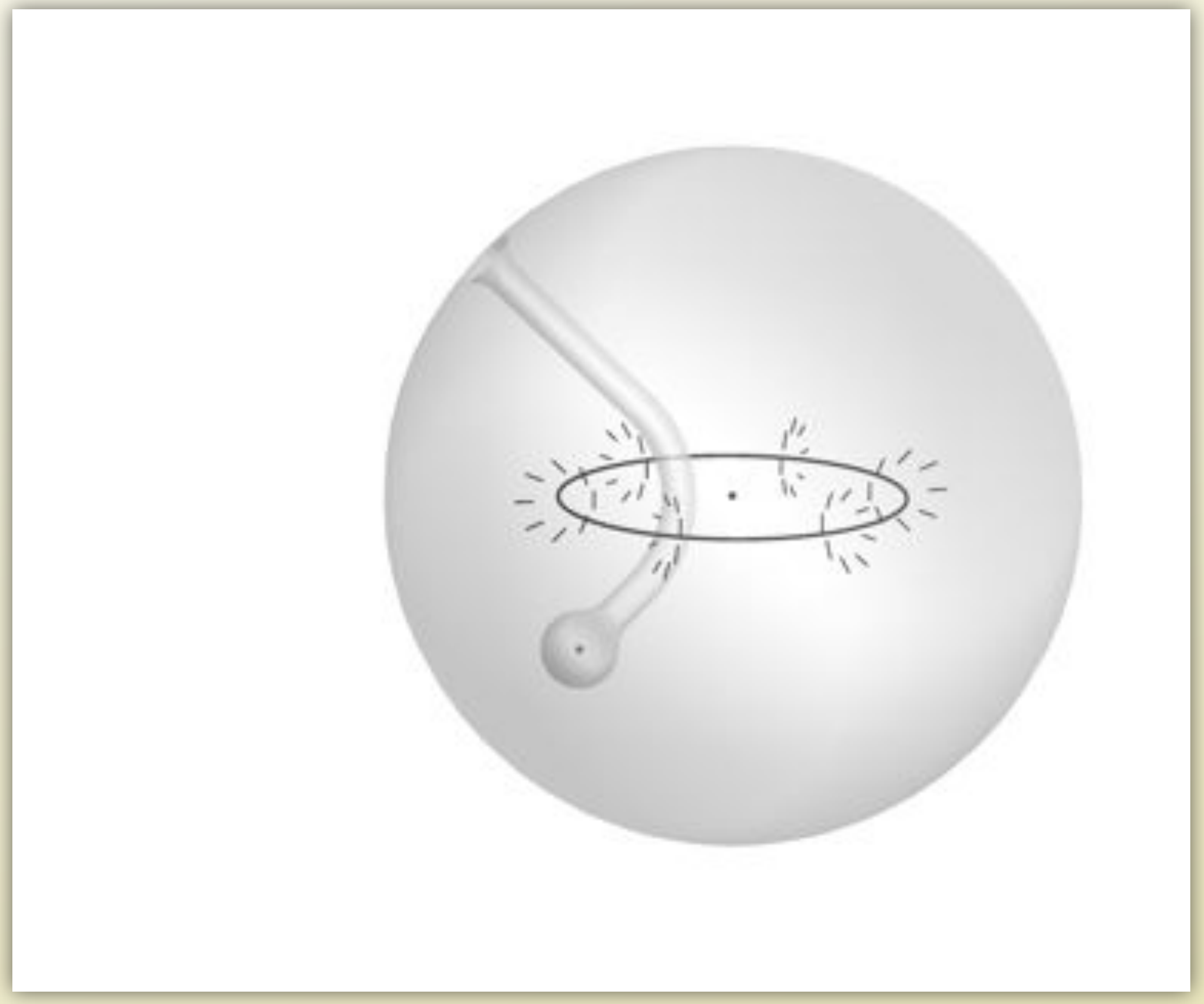




images: Sabetta Matsumoto

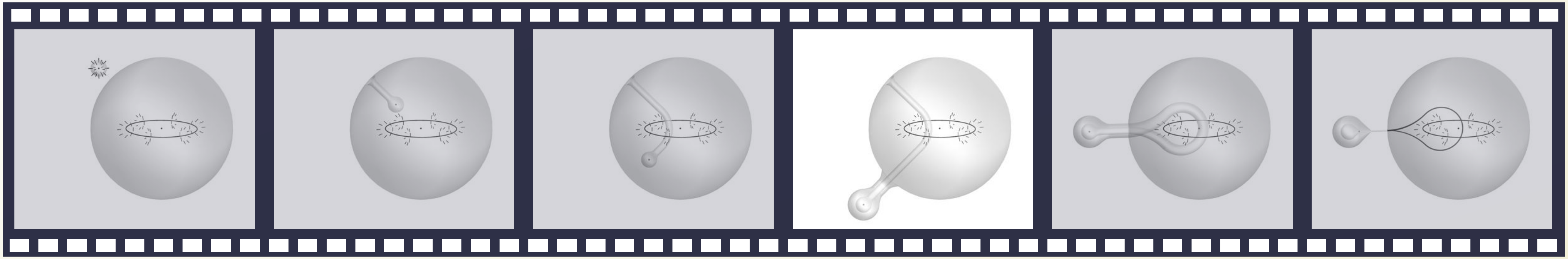
Moving a hedgehog around a disclination

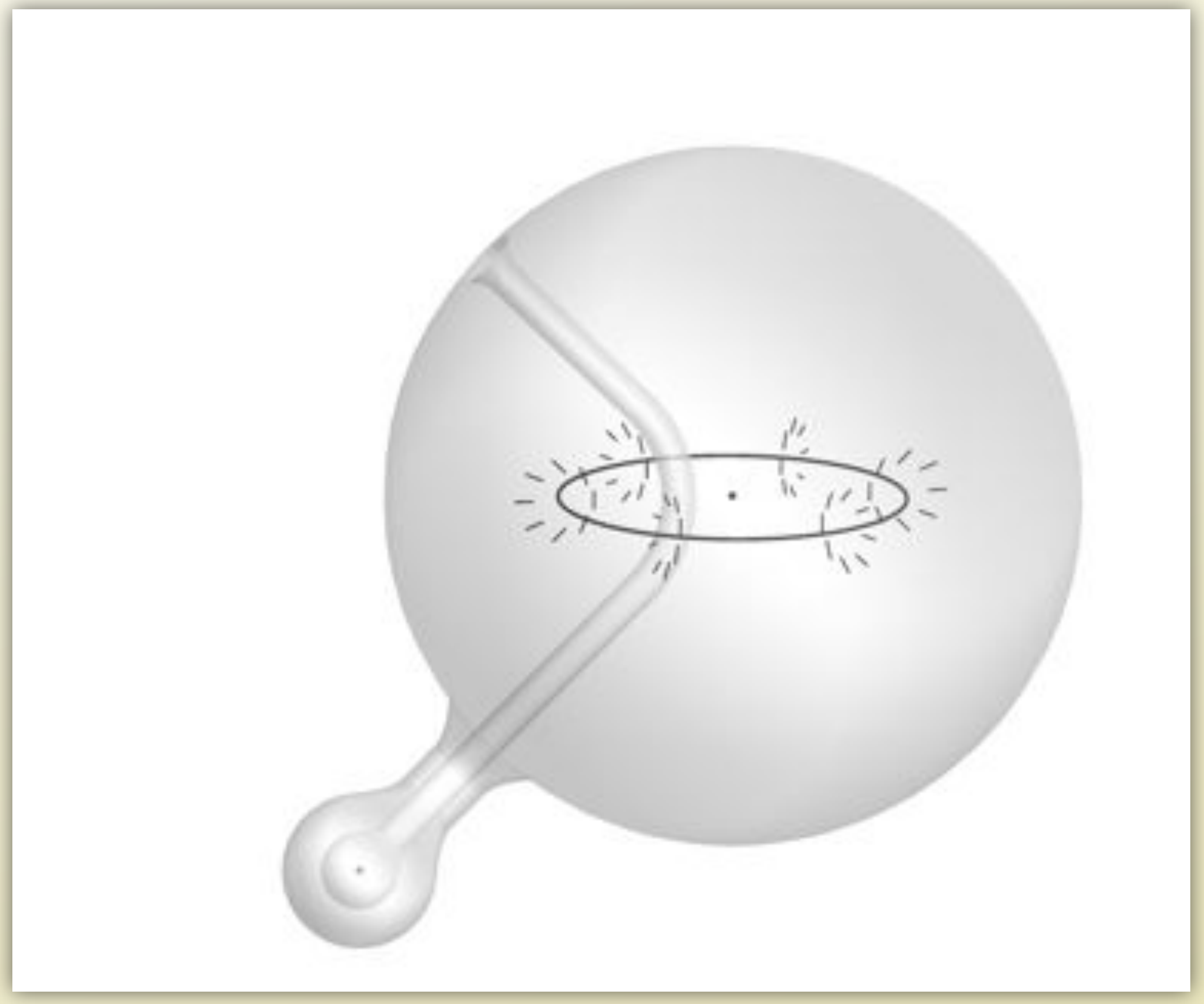




images: Sabetta Matsumoto

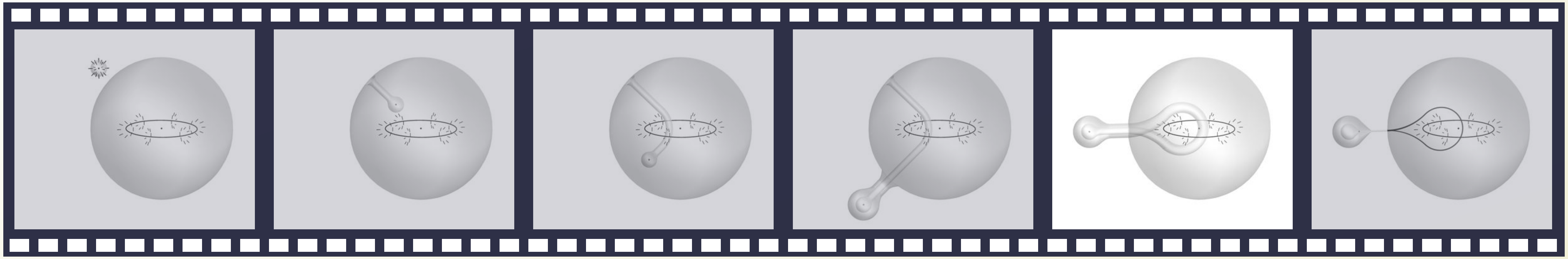
Moving a hedgehog around a disclination

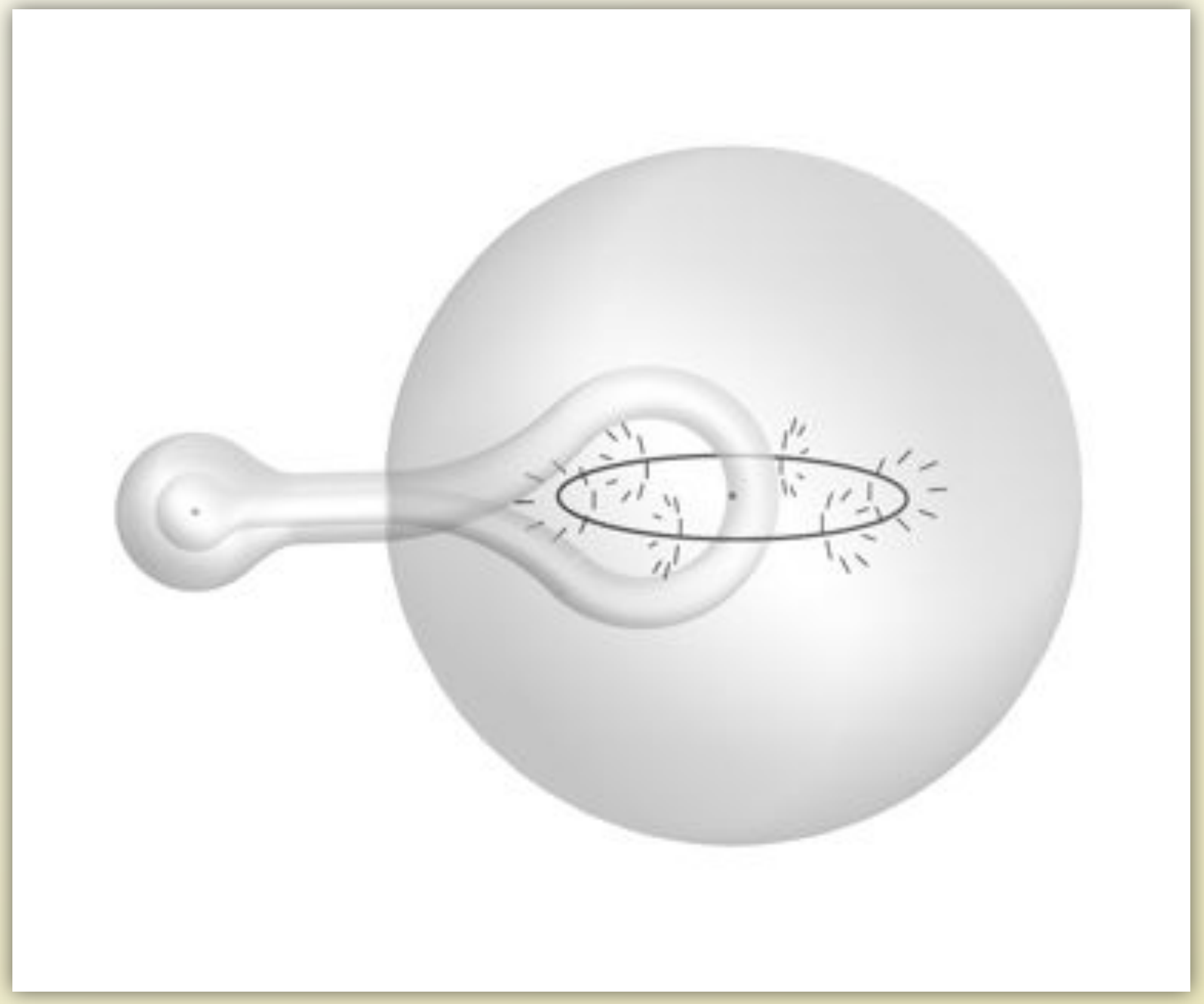




images: Sabetta Matsumoto

Moving a hedgehog around a disclination





images: Sabetta Matsumoto

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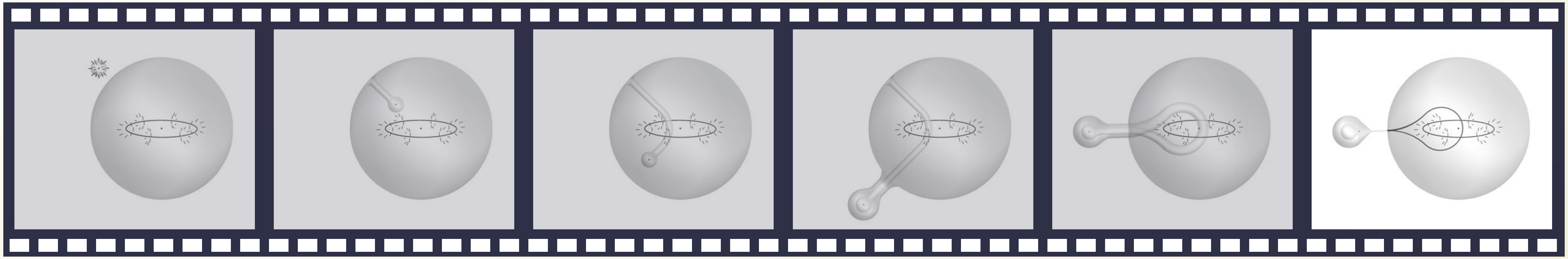

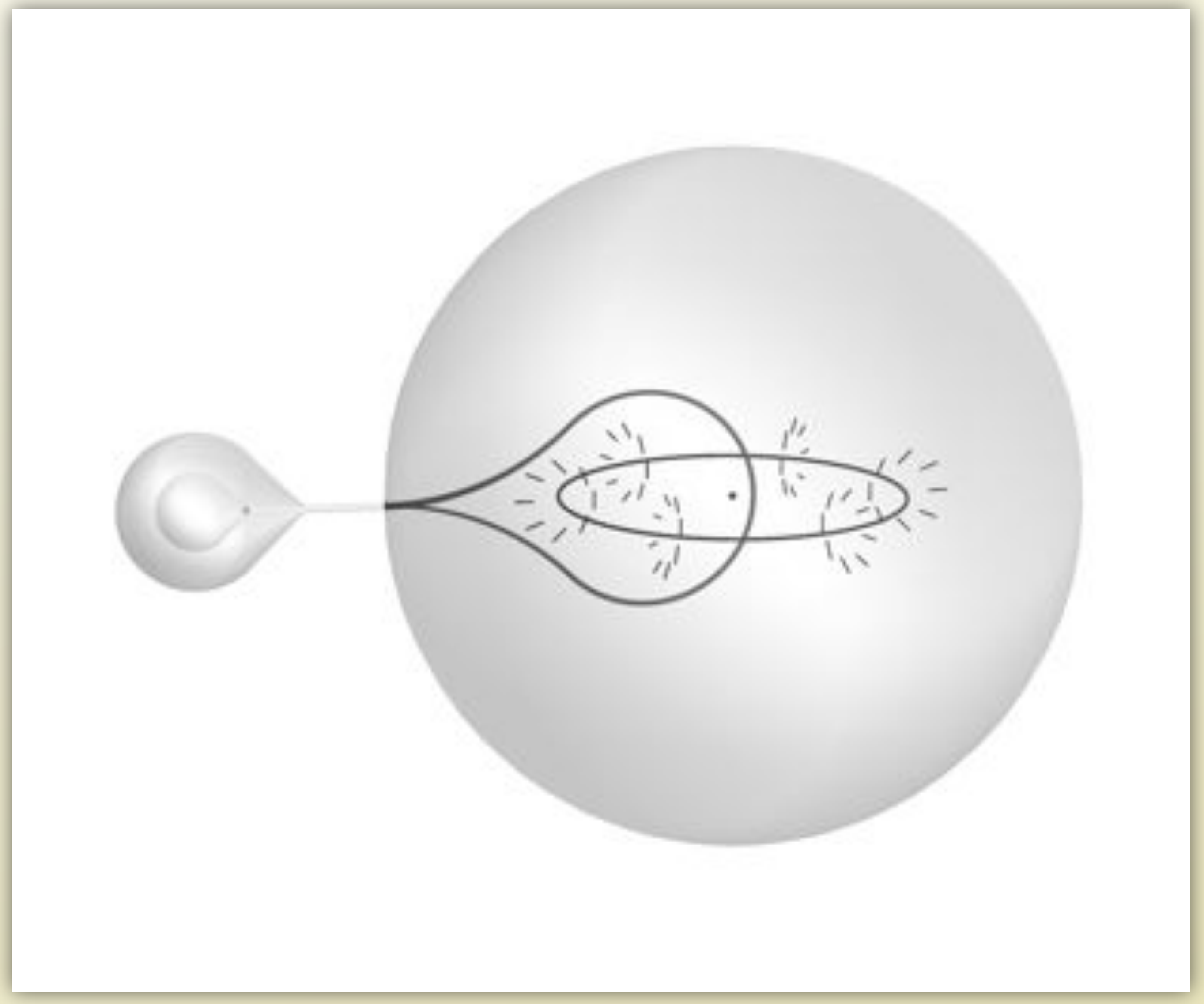

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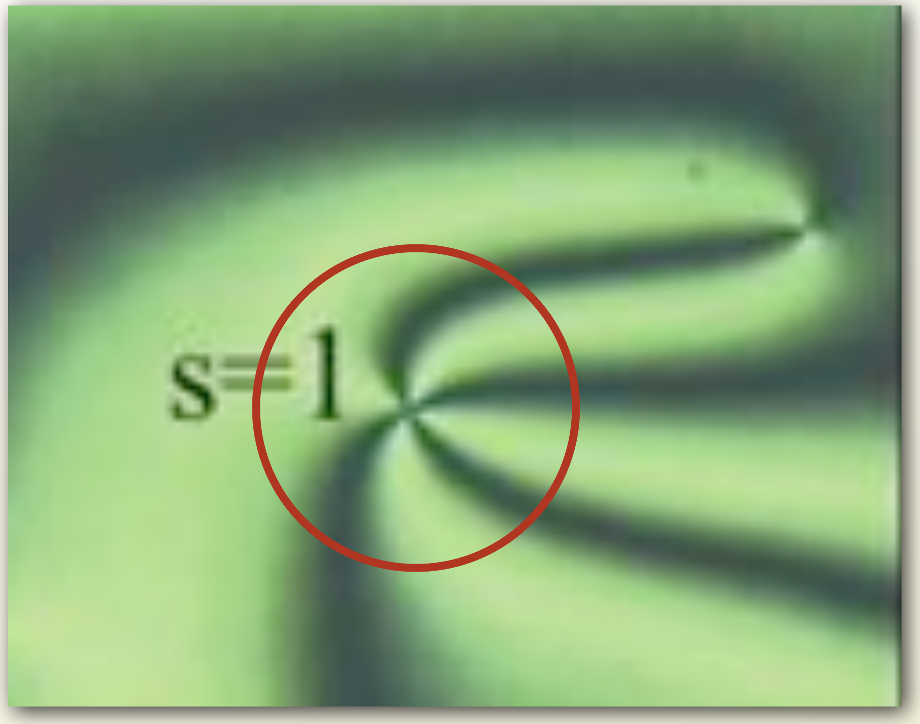
images: Sabetta Matsumoto

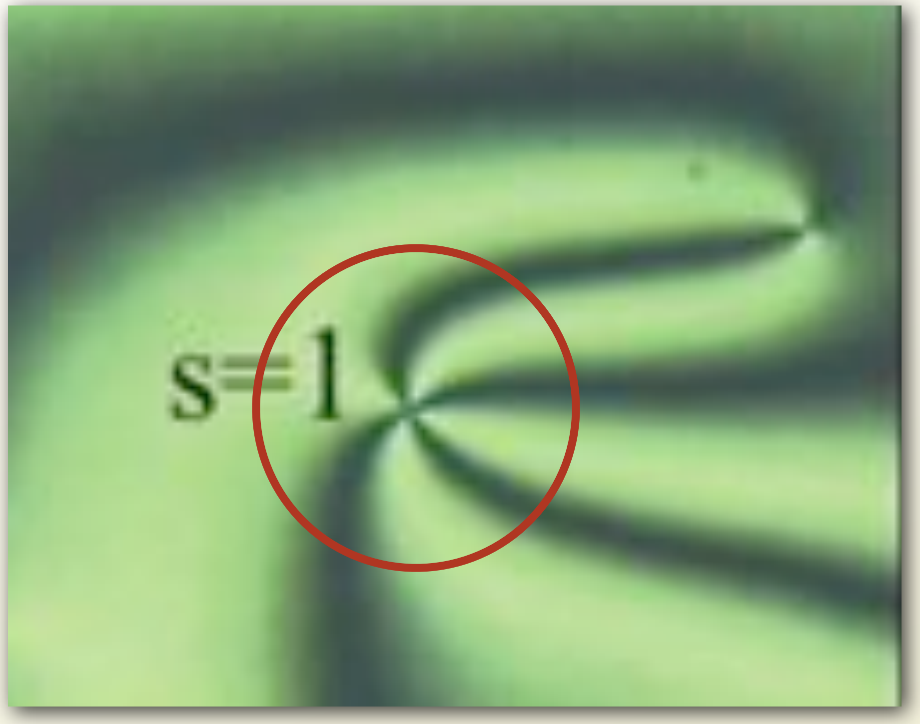
TEXTURES



so far we have thought about:

loops	$\pi_1(X, x_0)$
spheres	$\pi_2(X, x_0)$

Textures



so far we have thought about:

loops	$\pi_1(X, x_0)$
spheres	$\pi_2(X, x_0)$
textures	$\pi_3(X, x_0)$
	$\vdots$

famous result: $\pi_3(\mathbb{RP}^2, x_0) = \mathbb{Z}$

generated by the
Hopf fibration

No textures for nematics

Derrick's theorem

suppose $\theta(x) = \Theta(x)$ satisfies Euler-Lagrange for the energy

$$E = \int \mathrm{d}^d x \left[\frac{1}{2}(\nabla\theta)^2 + U(\theta)\right] = I_1 + I_2$$

consider a scaled version: $\theta_\lambda(x) = \Theta(\lambda x)$

$$E_\lambda = \lambda^{2-d} I_1 + \lambda^{-d} I_2$$

minimise over λ

$$\left.\frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda}\right|_{\lambda=1} = (2-d)I_1 - dI_2 = 0$$

$$\left.\frac{\mathrm{d}^2 E_\lambda}{\mathrm{d}\lambda^2}\right|_{\lambda=1} = (2-d)I_1$$

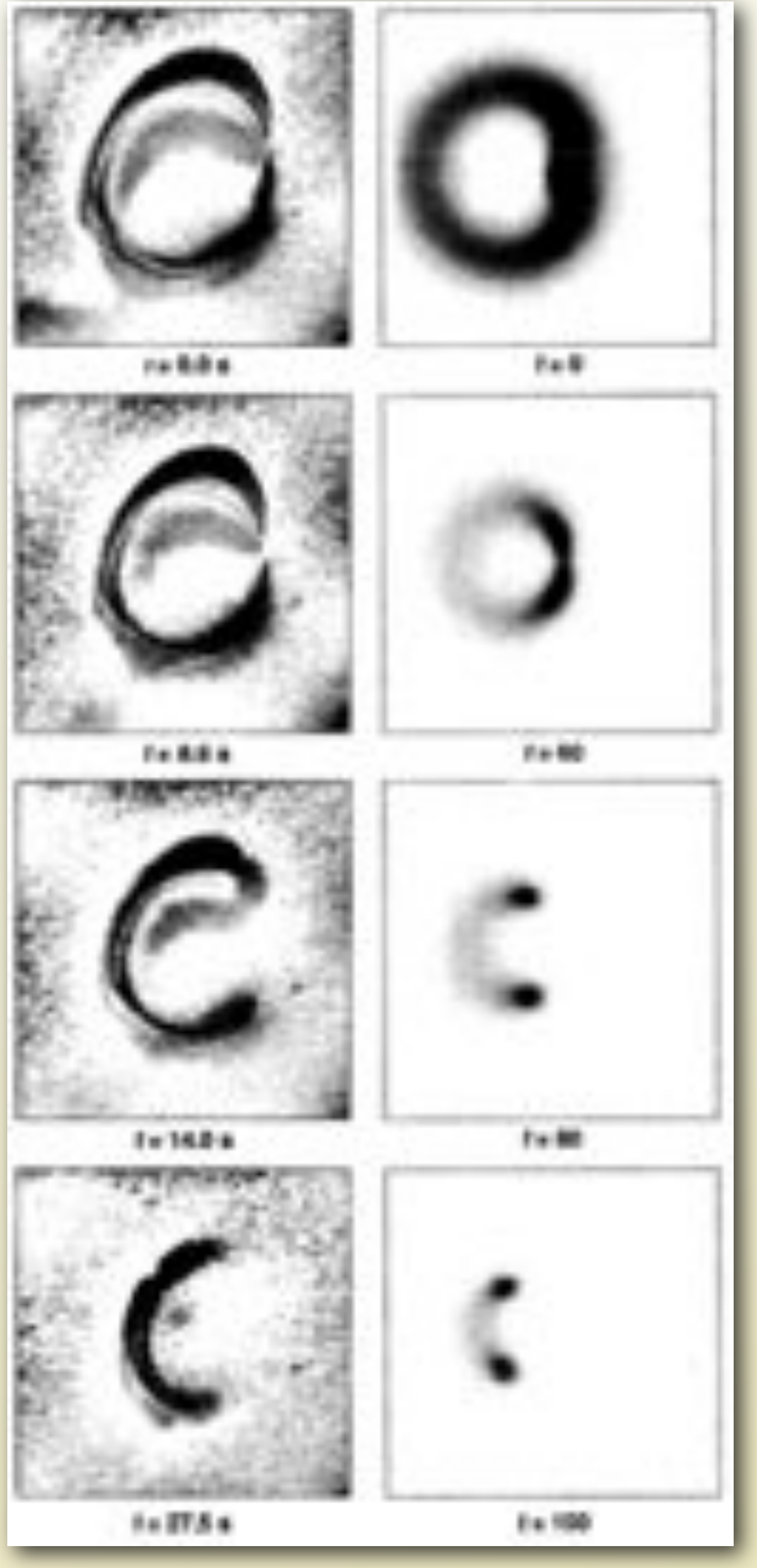
textures are unstable

Penn

Derrick *J. Math. Phys.* **5**, 1252–1254 (1964)

No textures for nematics

Derrick's theorem



textures are unstable

Derrick *J. Math. Phys.* **5**, 1252–1254 (1964)
Chuang, Durrer, Turok & Yurke *Science* **251**, 1336–1342 (1991)

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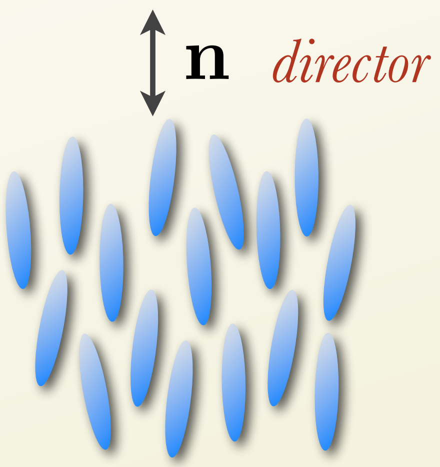
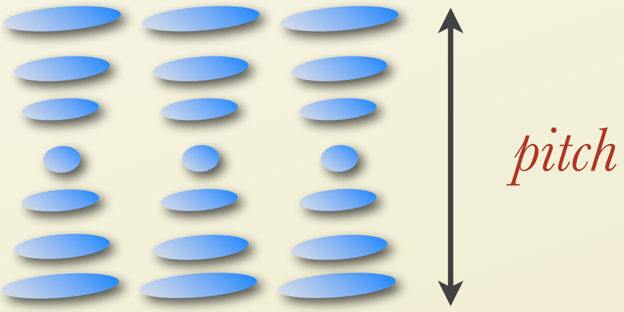
A GET OUT CLAUSE

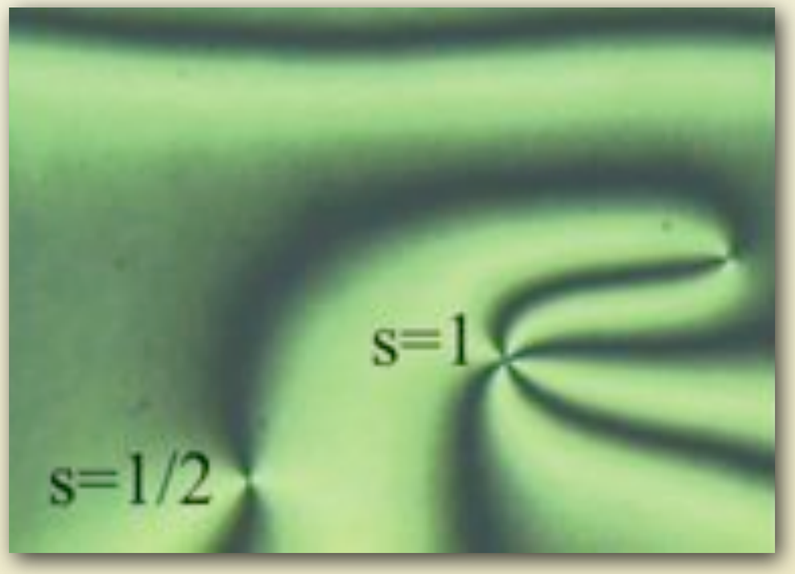
there's no length in this problem because there's no length in this problem



Nematic



Cholesteric



courtesy of Ingo Dierking

Photo by Michi Nakata

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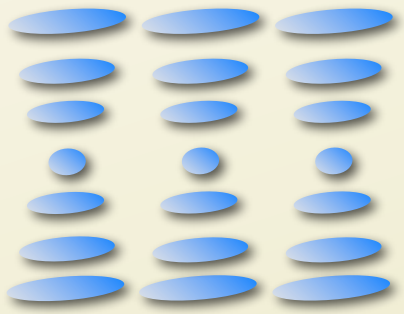
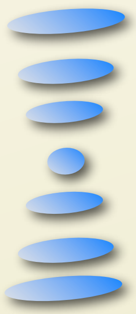
A GET OUT CLAUSE

double twist



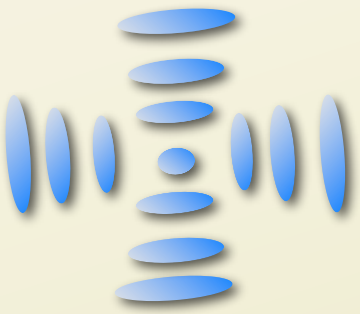
A GET OUT CLAUSE

double twist



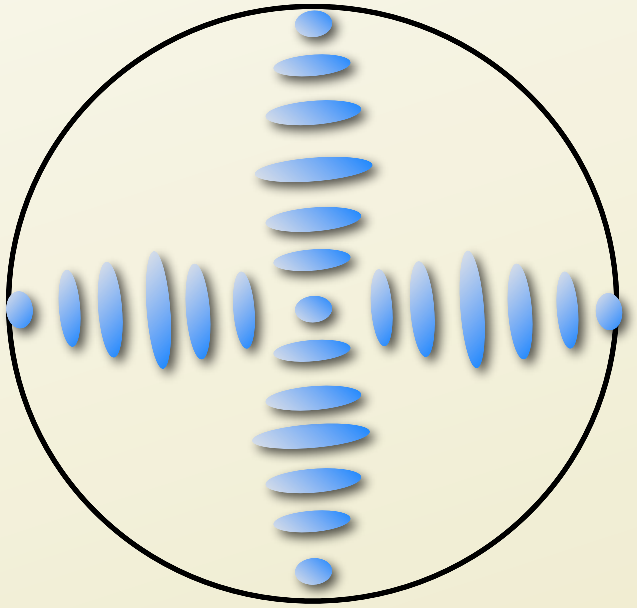
A GET OUT CLAUSE

double twist



A GET OUT CLAUSE

double twist



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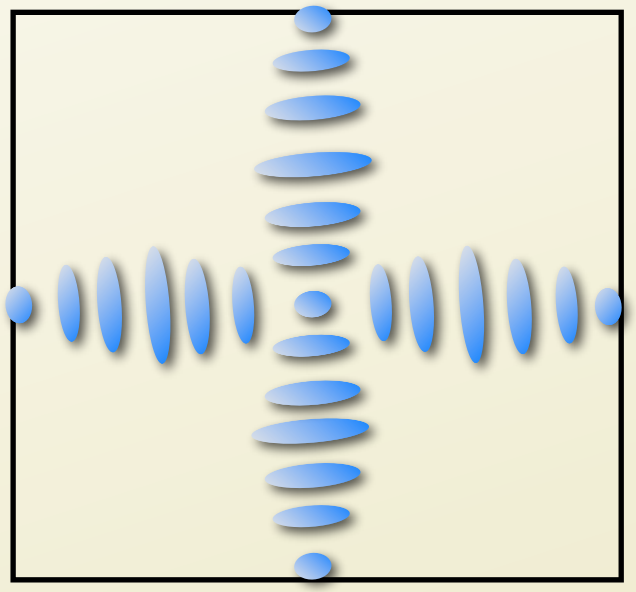
double twist



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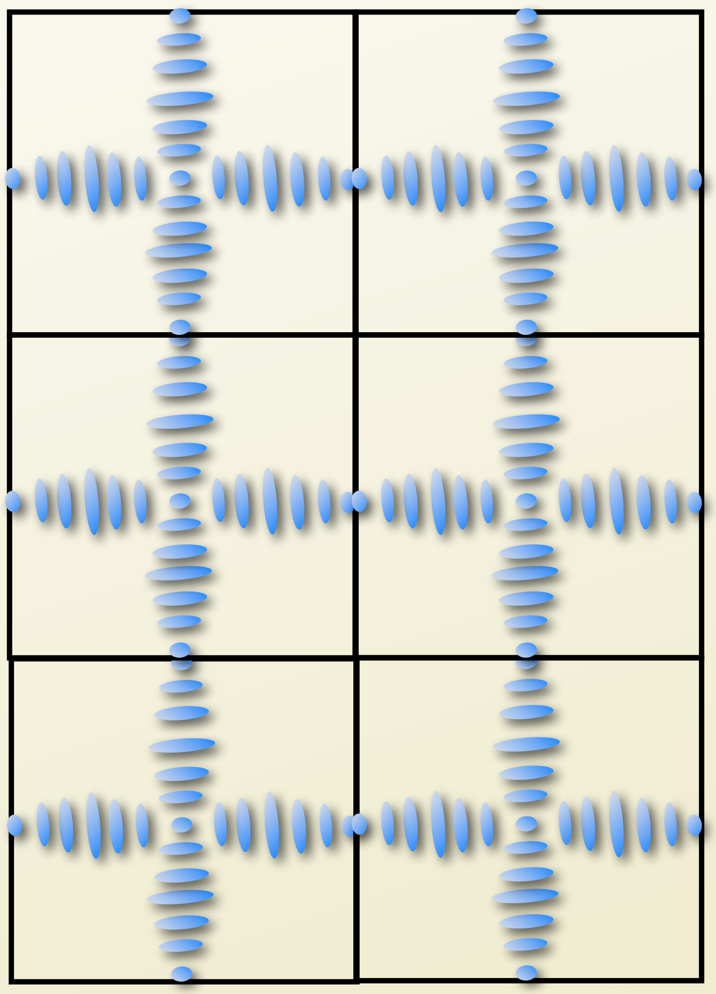
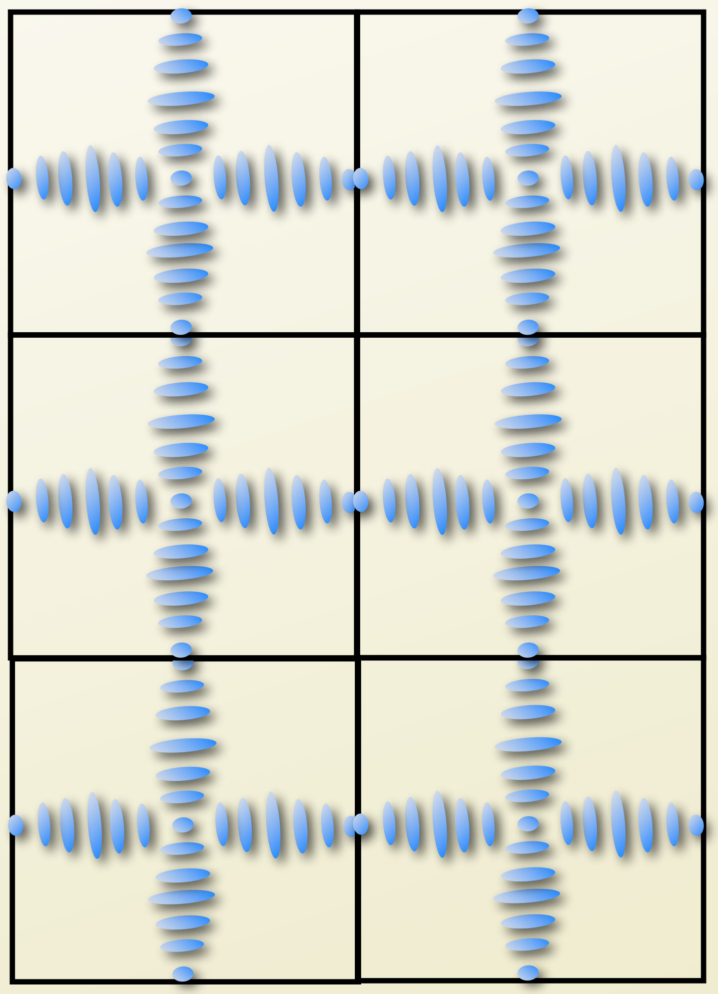
double twist



doubly periodic texture

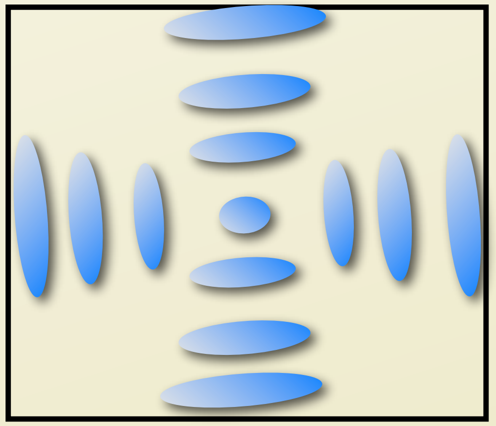
A GET OUT CLAUSE

double twist



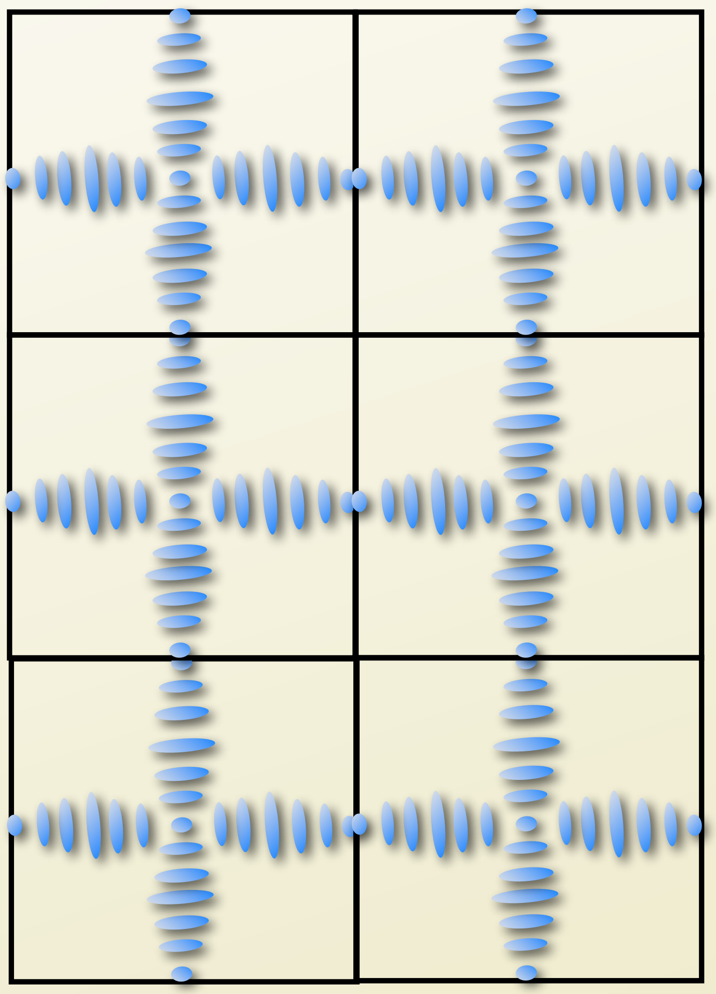
doubly periodic texture

what about this?

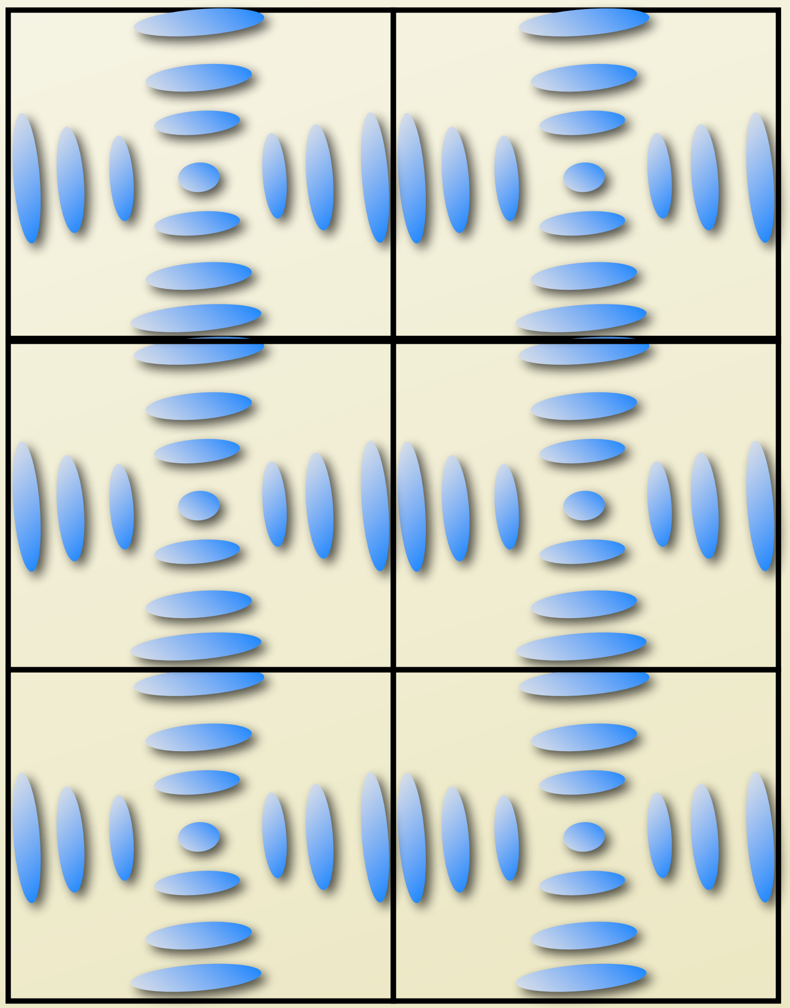


A GET OUT CLAUSE

double twist



doubly periodic texture



doubly periodic texture
with defects

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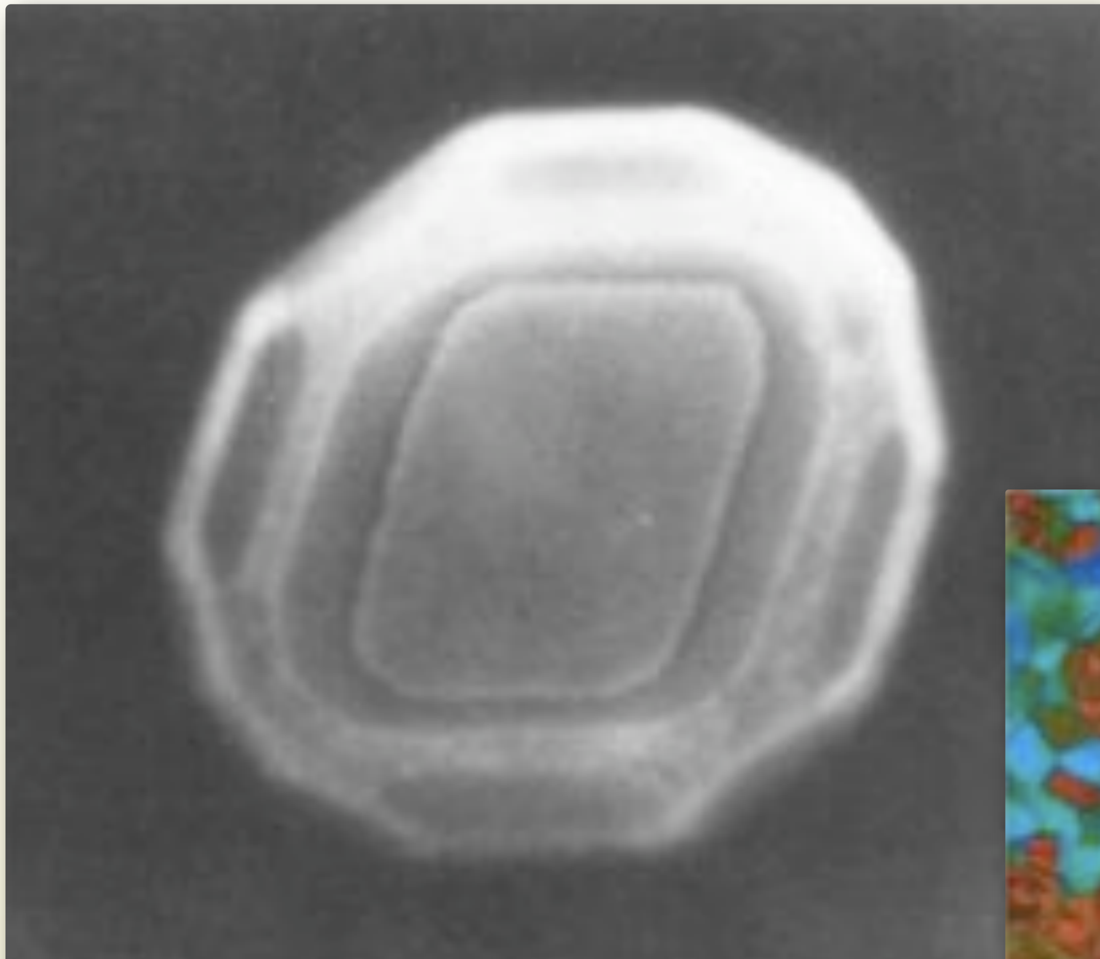
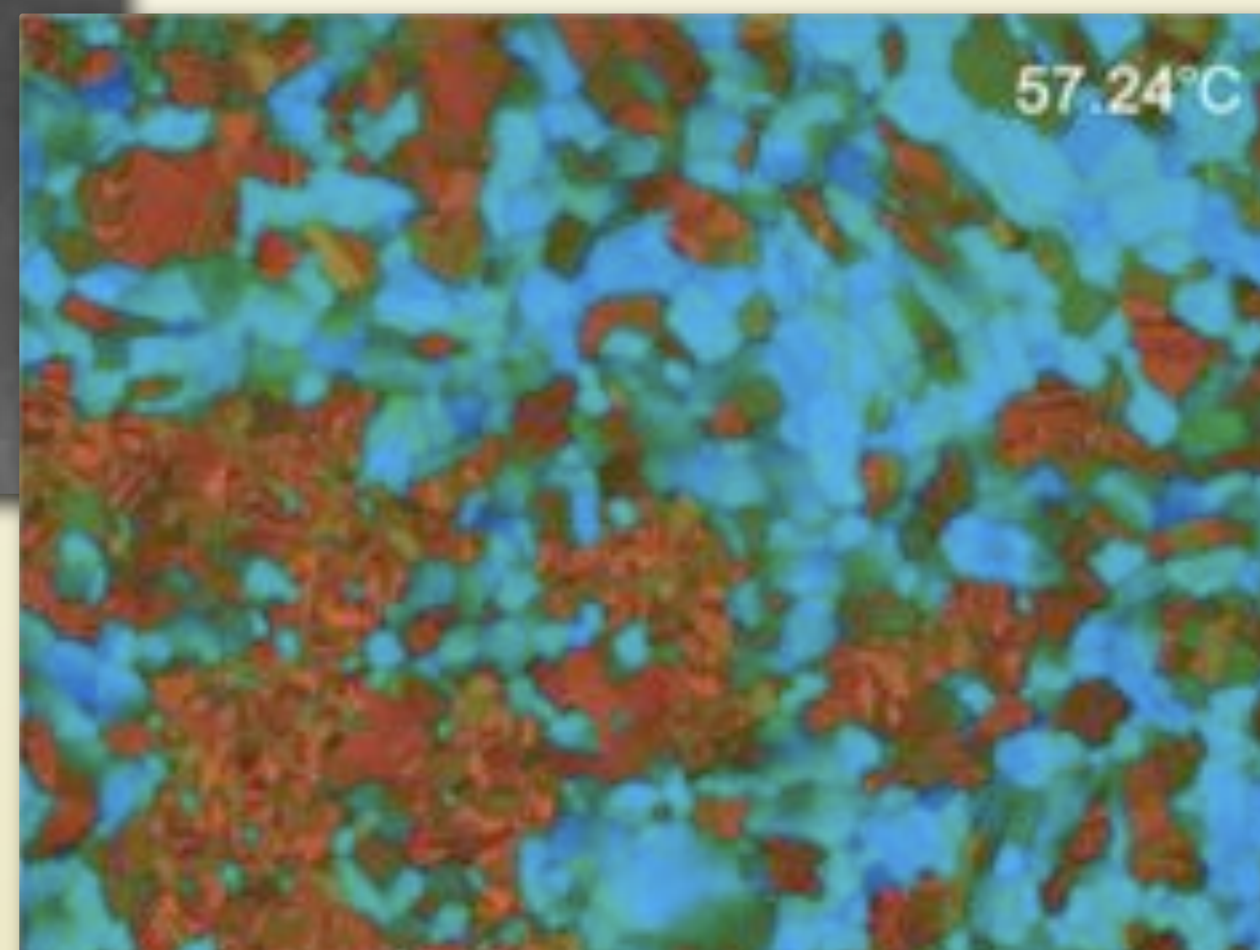
Blue phases



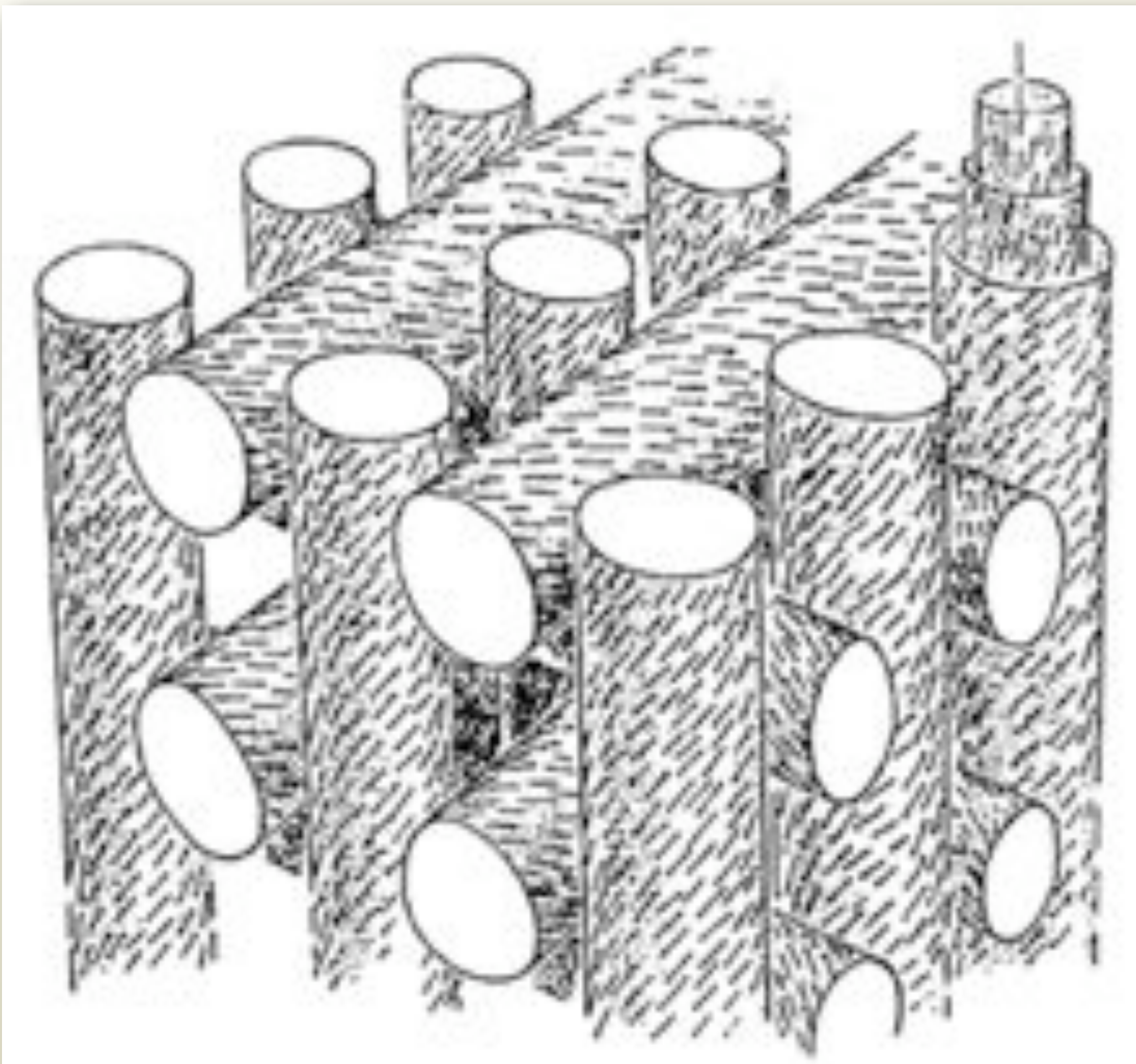
Grow as facetted monocrystals
Bragg scatter in the visible range

indexed by cubic space groups

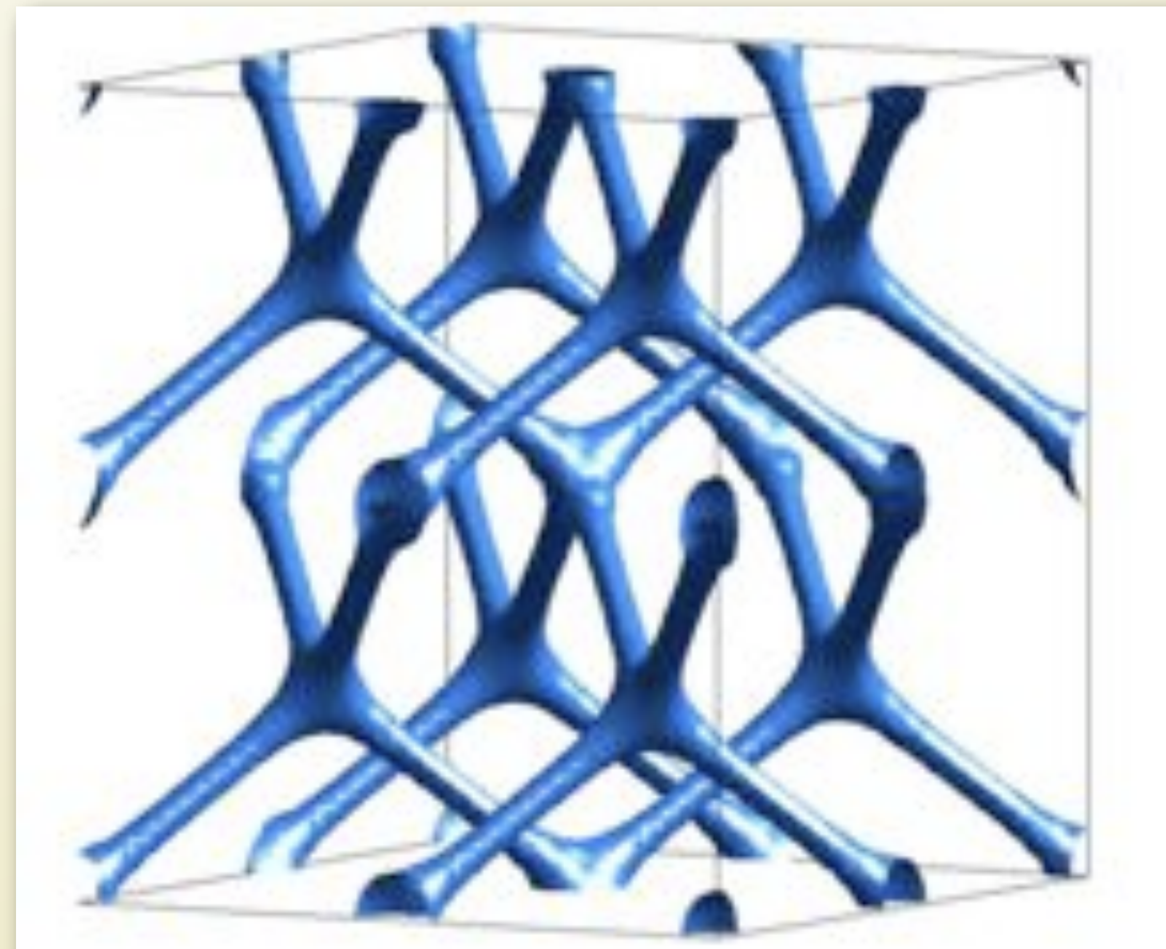


Blue phases

triply periodic stacking of double twist cylinders



Bouligand, Lagerwall, Stebler *Comptes rendus-Chimie* **11**, 212–220 (2008)



network of disclination lines

What Are They Good For?

"They are totally useless, I think, except for one important intellectual use, that of providing tangible examples of topological oddities, and so helping to bring topology into the public domain of science, from being the private preserve of a few abstract mathematicians and particle theorists."

Sir Charles Frank, 1983

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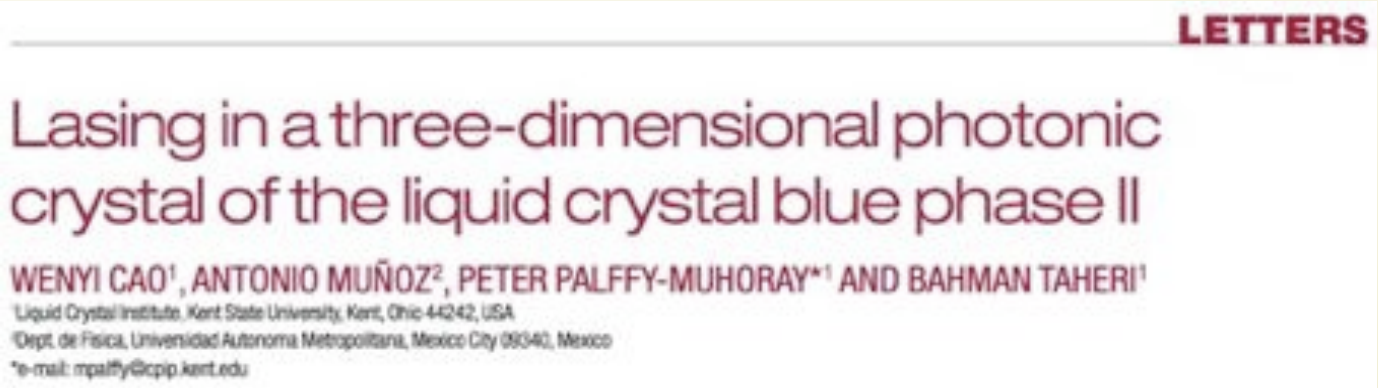
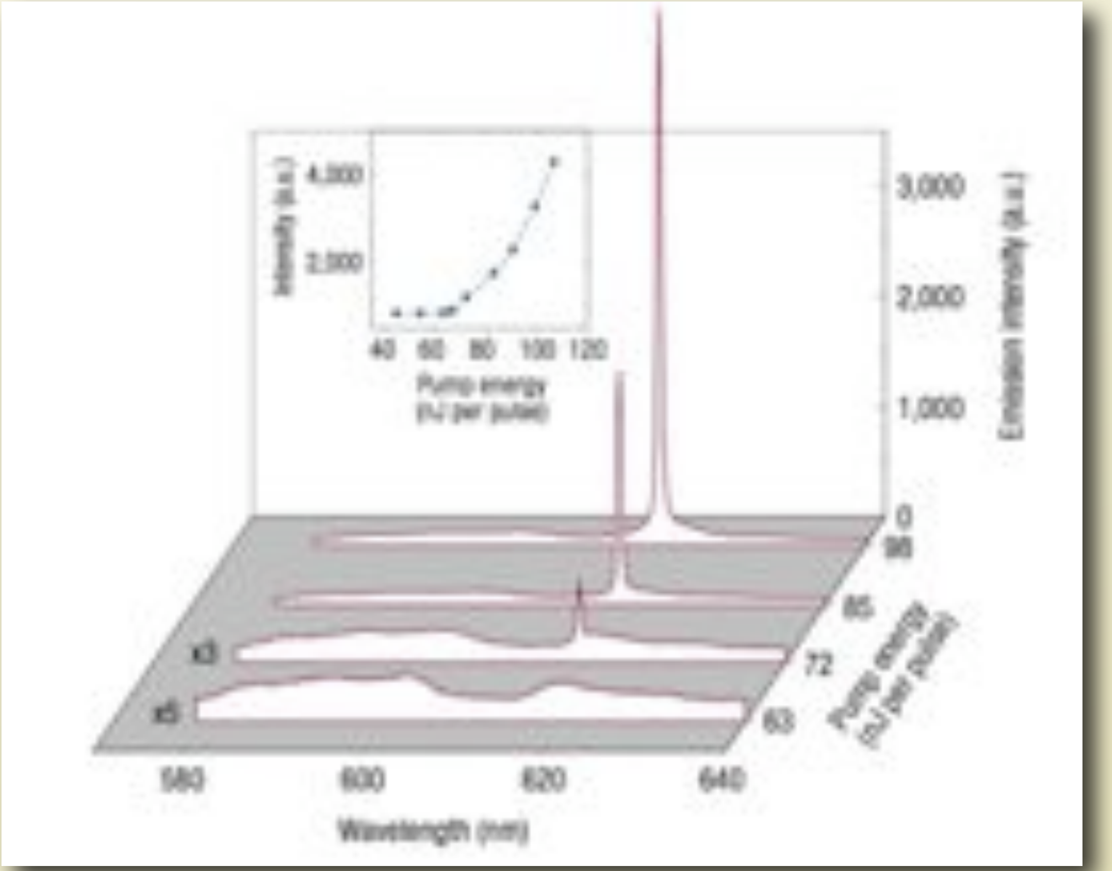


LETTERS

Lasing in a three-dimensional photonic crystal of the liquid crystal blue phase II

WENYI CAO[1], ANTONIO MUÑOZ[2], PETER PALFFY-MUHORAY*[1] AND BAHMAN TAHERI[1]
[1]Liquid Crystal Institute, Kent State University, Kent, Ohio 44242, USA
[2]Dept. de Fisica, Universidad Autonoma Metropolitana, Mexico City 09340, Mexico
*e-mail: mpalffy@cpip.kent.edu



Penn

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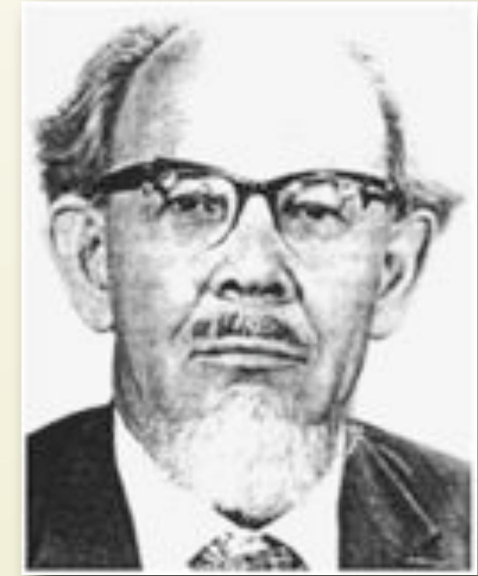
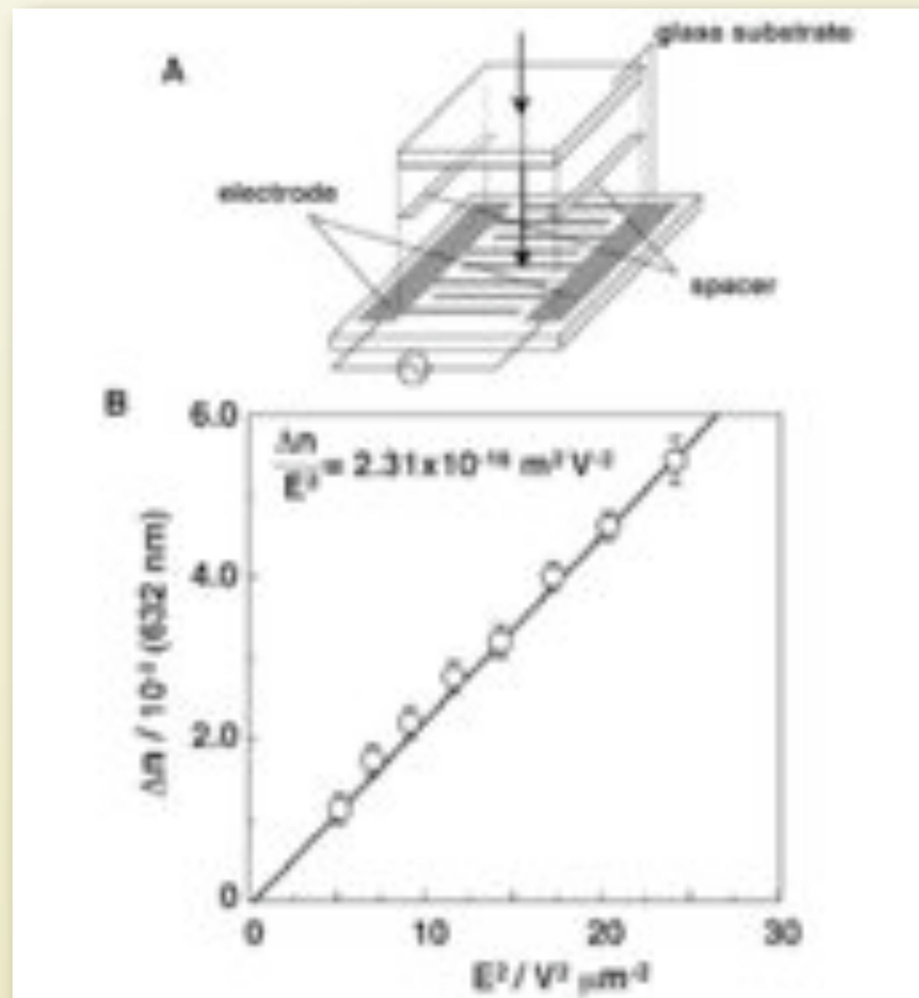


Large Electro-optic Kerr Effect in Polymer-Stabilized Liquid-Crystalline Blue Phases**

By *Yoshiaki Hisakado*, *Hirotsugu Kikuchi*,* *Toshihiko Nagamura*, and *Tisato Kajiyama*

Adv. Mater. **17**, 96 (2005).

"These achievements can contribute to providing fast-response flat-panel liquid-crystal displays that need not undergo a rubbing process during manufacture."



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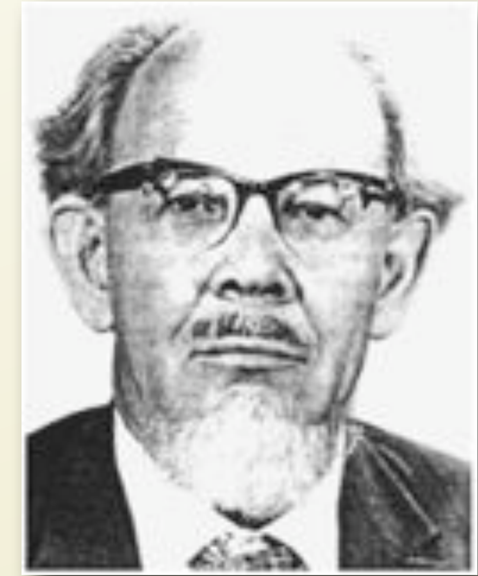

Samsung Electronics, prototype (2008)

What Are They Good For?

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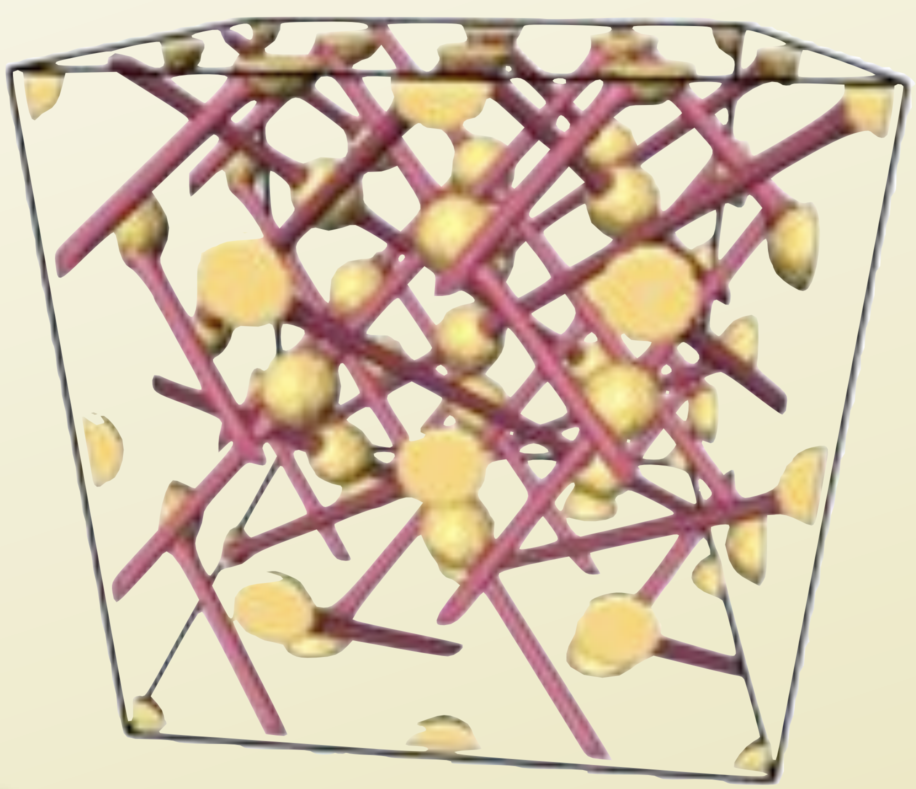
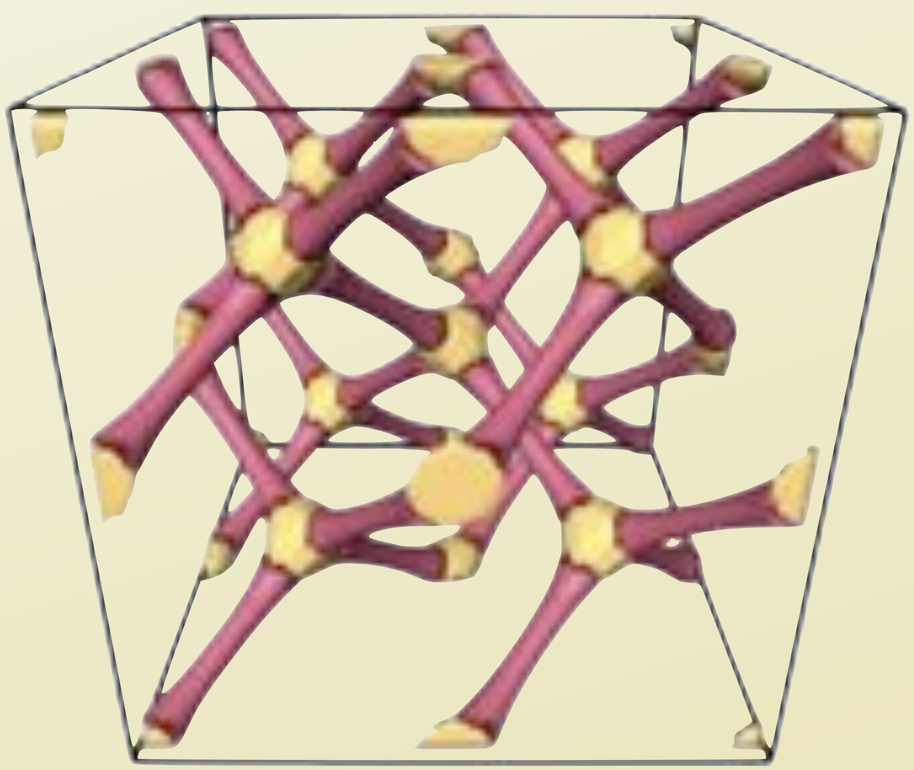
Sir Charles Frank, 1983

Three-dimensional colloidal crystals in liquid crystalline blue phases

Miha Ravnik[a,b], Gareth P. Alexander[b,c], Julia M. Yeomans[b], and Slobodan Žumer[a,d,1]

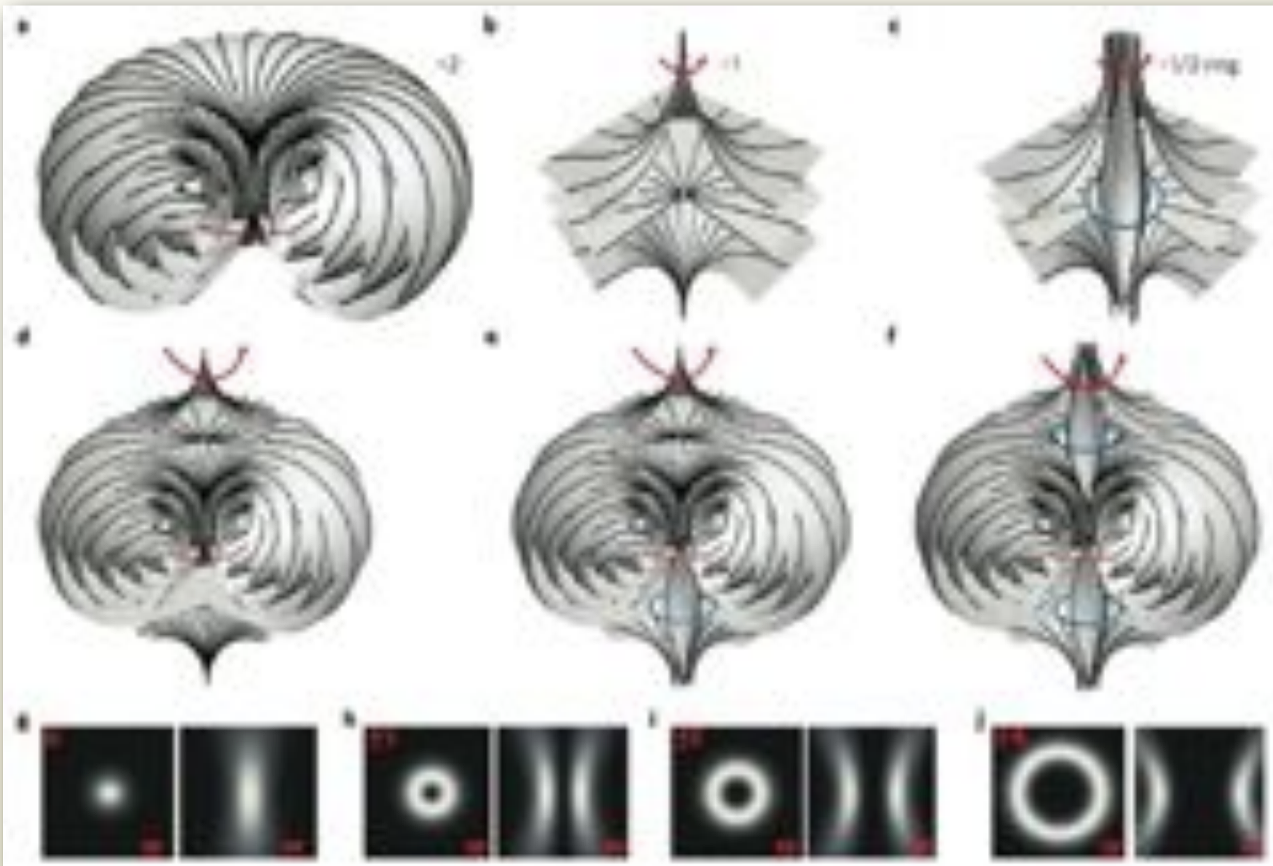


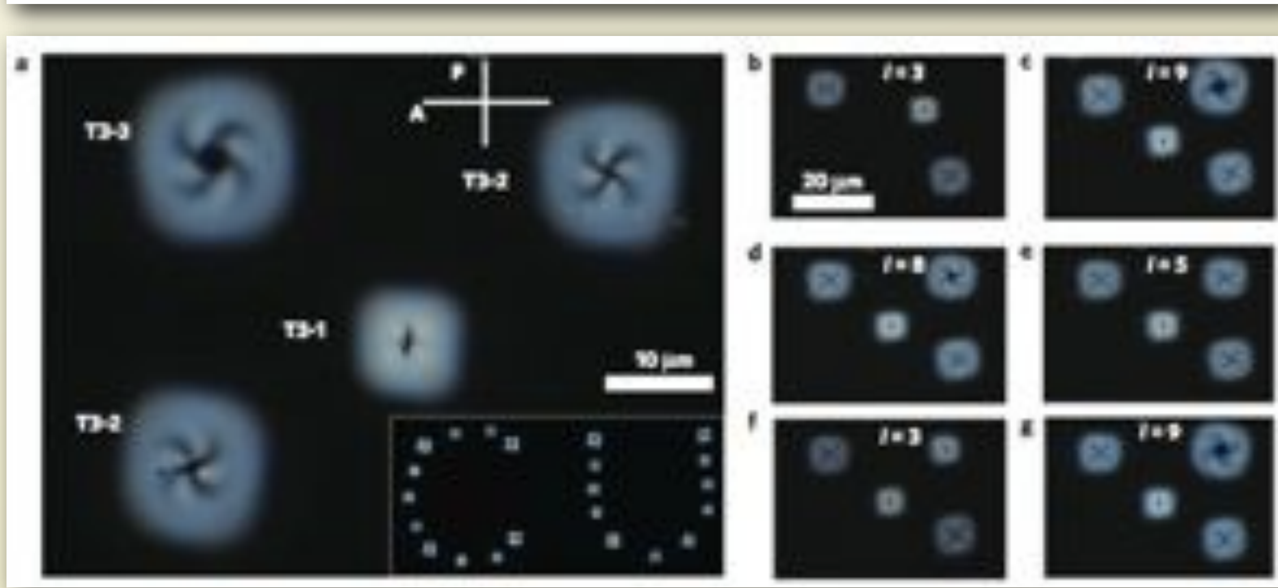


Penn

Inducing topology with cholesterics

torons



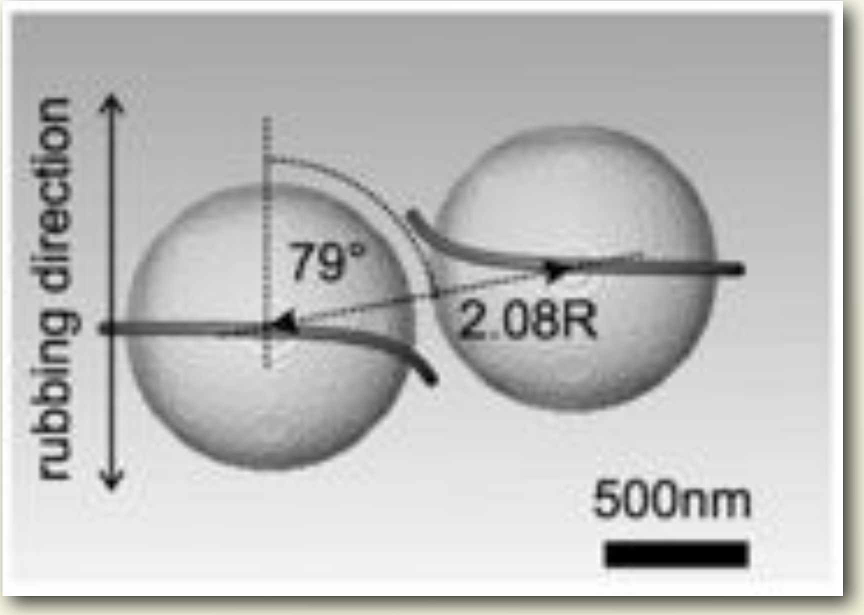


non-trivial localised textures in thin cell cholesterics induced by laser beams

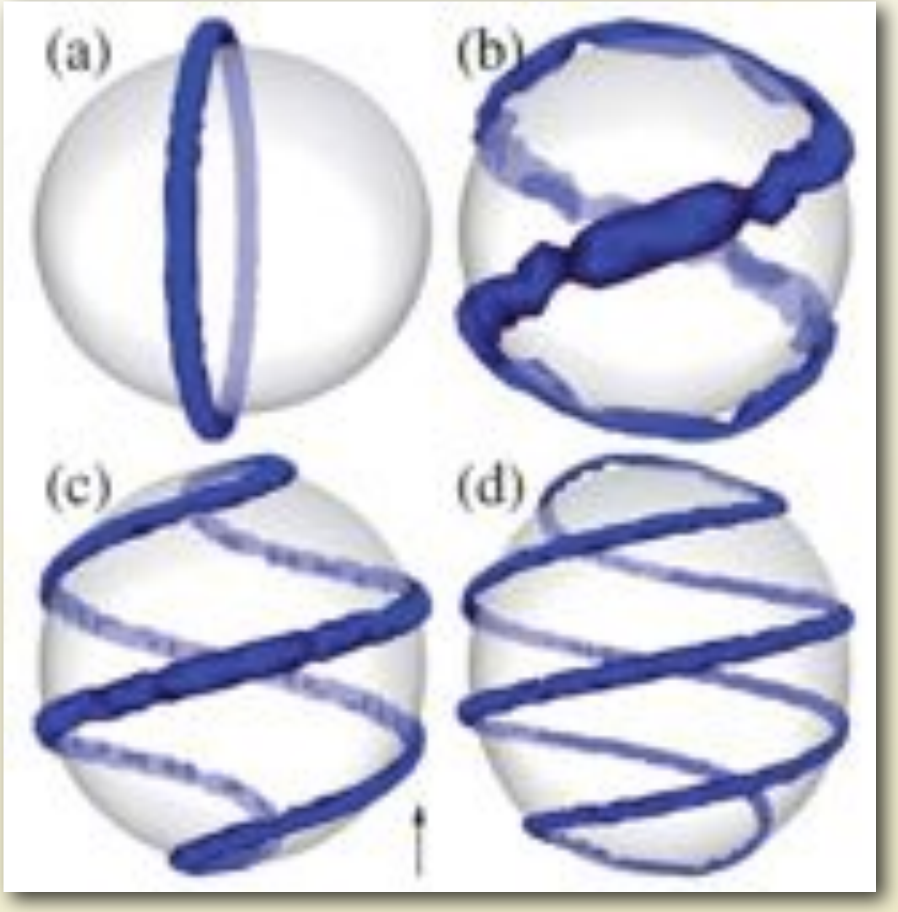
Smalyukh et al *Nat. Mater.* **9**, 1–7 (2009)

Inducing topology with cholesterics

colloids in cholesterics



energetically favourable ***not*** to entangle
only ***two*** "rewiring sites"



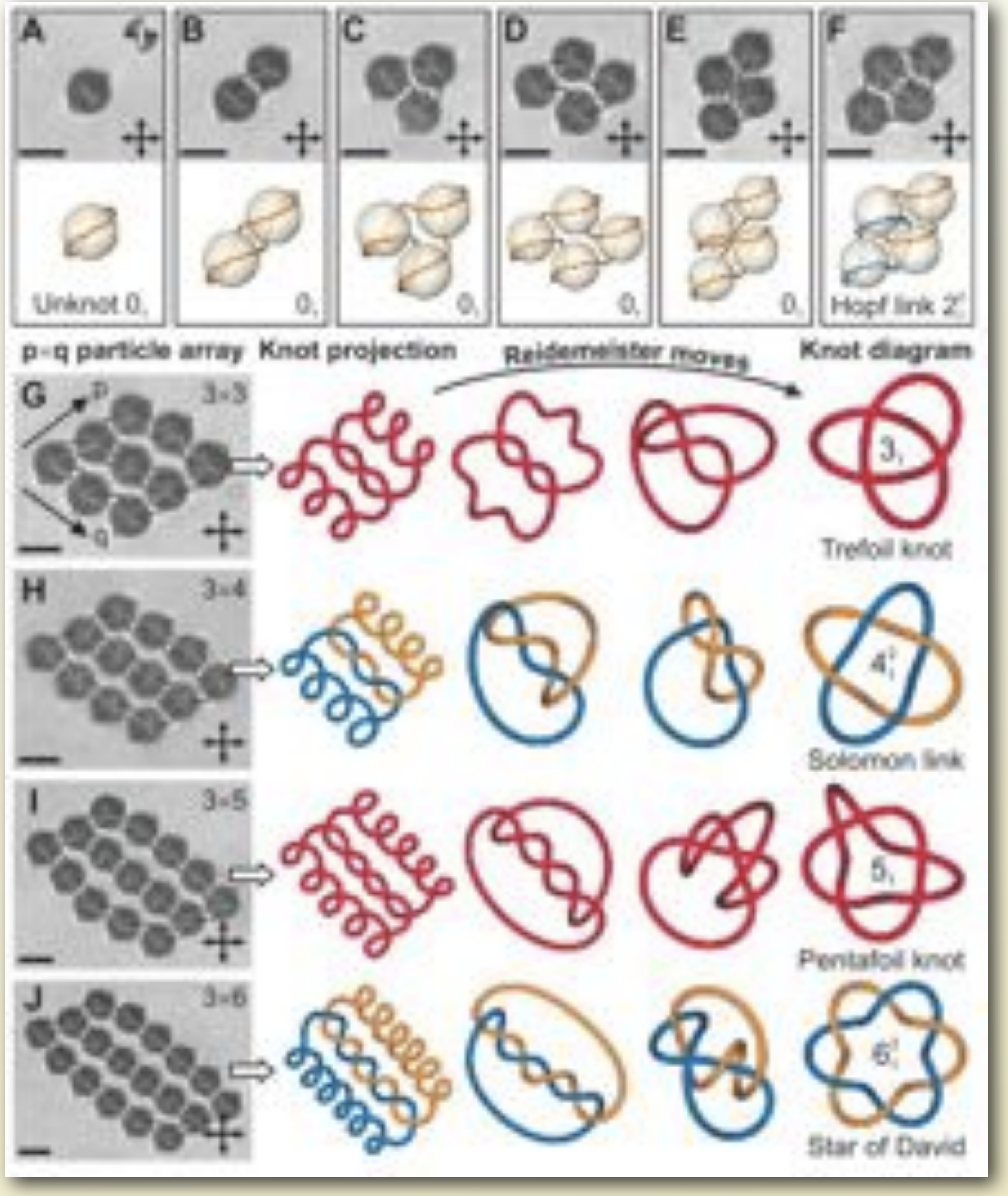
energetically favourable to entangle
number of "rewiring sites" grows with the ratio of the colloid size to the pitch
many more configurations possible

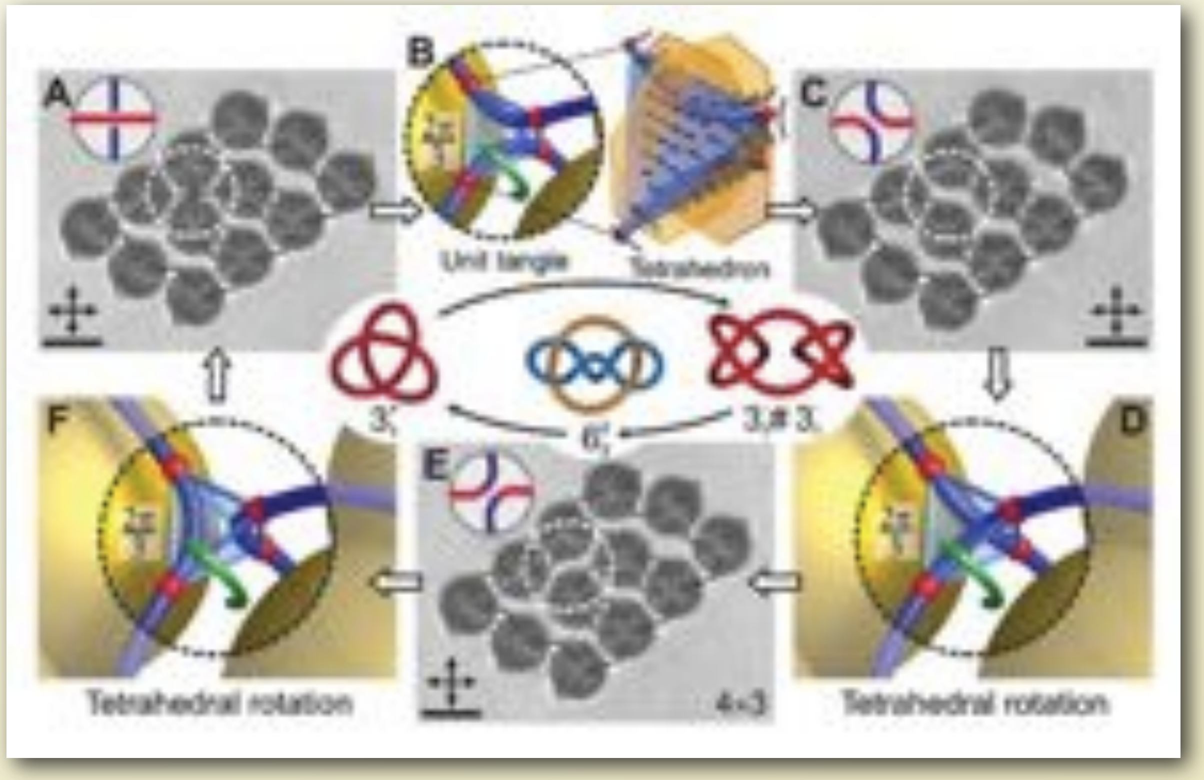
Penn

Lintuvuori, Marenduzzo, Stratford & Cates
J. Mater. Chem **20**, 10547–10552 (2010)

Inducing topology with cholesterics

colloids in cholesterics





every knot up to crossing number seven!

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Thanks!

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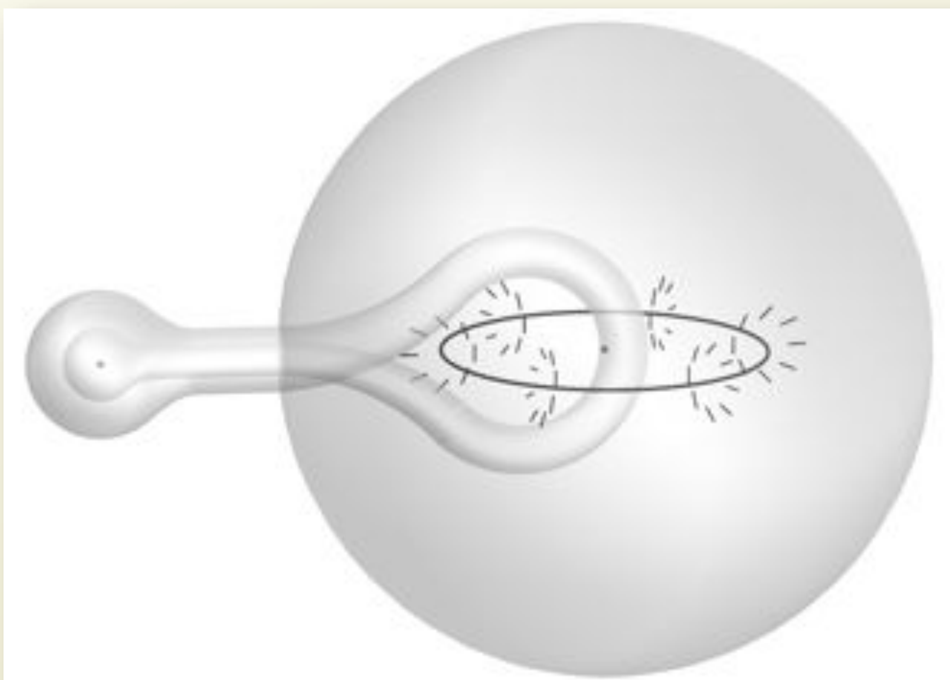

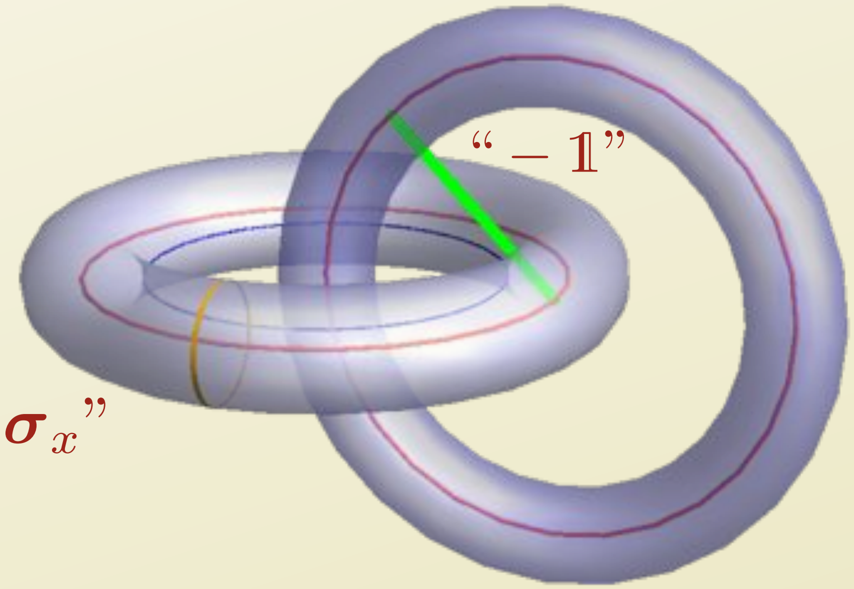

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