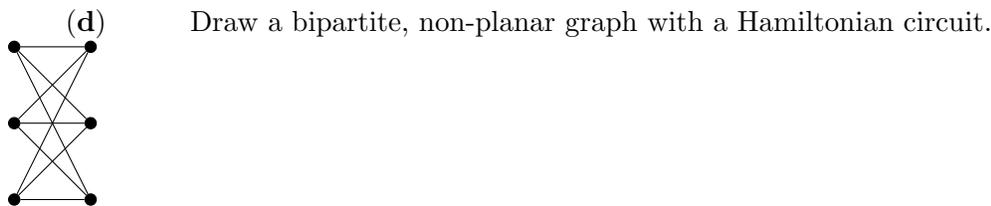
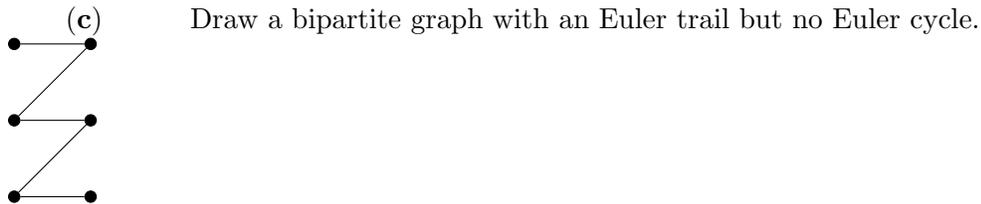
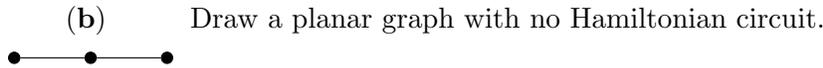
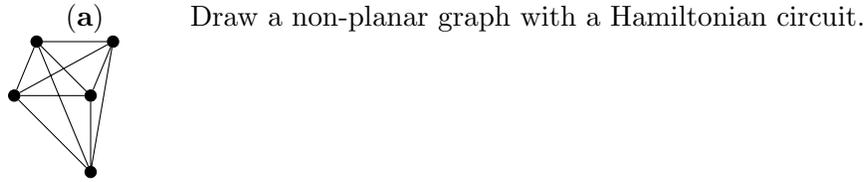


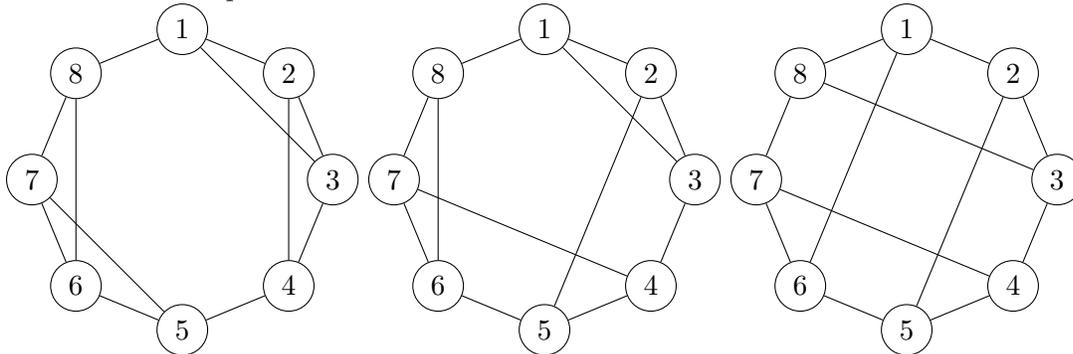
Practice Midterm Solutions

1. (20 points)

You do not need to prove that the graphs have the specified properties; it suffices to draw the graphs.

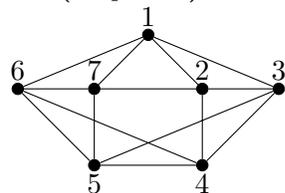


2. (15 points) All three of these graphs have 8 vertices, each of degree 3. Show that no pair of them is isomorphic.

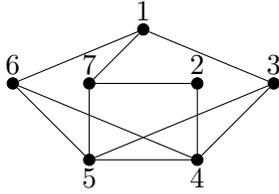


The first two graphs both have subgraphs isomorphic to K_3 (i.e. triangles); the third does not, so the third is not isomorphic to either of the others. The second graph has subgraphs isomorphic to C_4 (the circuit with four vertices---the ‘square’) while the first does not, so the first two are not isomorphic.

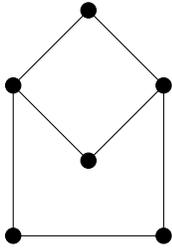
3. (10 points) Show that this graph is non-planar.



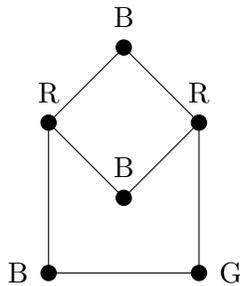
The following graph is a subgraph which is also a subdivision of $K_{3,3}$; by Kuratowski's theorem, the original graph is non-planar.



4. (10 points) Consider the following graph:



(a) Show that this graph can be colored using 3 colors.



(b) Show that this graph cannot be colored using 2 colors.

By theorems we have proven, a graph can be 2 colored iff it is bipartite, which happens iff it has no odd cycles. But there is a cycle of length 5, so this graph cannot be 2 colored.

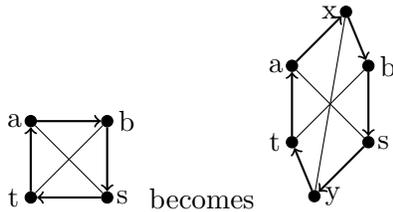
5. (15 points) $G = (V, E)$ is a graph with an Euler cycle. (In order for this to make sense, you may assume that $|V| \geq 3$.) Suppose I add some additional edges to obtain a new graph, $G' = (V, E \cup E')$. (So $E' \cap E = \emptyset$ and $|E'| > 0$; that is, I add at least one edge, and I only add new edges.) If G' also has an Euler cycle, prove that $|E'| \geq 3$. (That is, prove that if we add exactly 1 or exactly 2 edges, we can't get an Euler cycle. It matters that we're in a graph, not a multigraph—there are no loops or multiple edges.)

Since G has an Euler cycle, every vertex has even degree. Since G' also has an Euler cycle, each vertex in G' still has even degree. If $|E'| = 1$ then the two ends of the single new edge each see their degrees increase by 1, so they would have odd degree; that is impossible, so $|E'| > 1$. If $|E'| = 2$ then the total degree increases by 4; that must be spread across at least 3 vertices (it cannot be two vertices getting two new edges each, because G' does not have multiple edges). So $|E'| > 2$ as well, so $|E'| \geq 3$.

6. (10 points) Show that if n is even and $n \geq 4$ then there is a graph with n vertices such that every vertex has degree 3 and the graph has a Hamilton circuit.

When n is 4, this is K_4 . Suppose that we have a graph G with n vertices where every vertex has degree 3 and the graph has a Hamiltonian circuit, and we want to construct a graph with $n+2$ vertices with the same property. Choose two edges in the Hamiltonian circuit, which do not share endpoints, say an edge from a to b and a second edge from s to t . (These exist because the graph is connected and has ≥ 4 vertices---the first and third edges in the Hamiltonian circuit always work.)

We define G' to contain the vertices of G plus two new vertices x, y . We delete the edge between a, b and the edge between s, t , and add edges $x - y$, $a - x$, $b - x$, $s - y$, $t - y$; each vertex has degree 3 in the new graph as well. The Hamiltonian circuit is like the old Hamiltonian circuit except that we replace $a - b$ with $a - x - b$ and $s - t$ with $s - y - t$.



7. (20 points) We want to prove that every finite connected graph has a connected *spanning* (containing every vertex) subgraph with no circuits. Let p_k be the statement “Every connected graph with k vertices has a spanning subgraph with no circuits.”

(a) Prove p_1 , that every connected graph with 1 vertex has a spanning subgraph with no circuits.

If a graph has 1 vertex, it is its own spanning subgraph.

(b) Suppose that p_k is true. Prove p_{k+1} .

Suppose every graph with k vertices has a spanning subgraph. Take a graph G with $k+1$ vertices, and pick any vertex v . The graph $G' = G \setminus \{v\}$ has connected components, and v must have at least one edge to each connected component (because the original graph was connected). By the inductive hypothesis, each component contains a spanning subgraph with no circuits. Combine these spanning subgraphs with the vertex v and exactly one edge from v to the spanning subgraph of each component. This is spanning (it contains every vertex other than v , and also v), connected, and has no circuits (each component of G' has no circuit in the spanning subgraph, and no circuit includes v because once we pass through v into any component, we can never leave that component again.¹)

(c) Using the previous two parts, prove that *any* finite connected graph has a spanning subgraph with no circuits.

By induction on k . The base case is the first part, the inductive case is the second part. So, by induction on k , every finite connected graph has a spanning subgraph with no circuits.

¹I tweaked this problem to talk about circuits instead of cycles because I thought that made it slightly easier. If you solved it using cycles, the same basic argument applies; to see there are no cycles, notice that once you cross the bridge from any component to v , you can't get back to that component because there is only one such “bridge”.