2. Let $T_1 \subset T_2$ be theories. Assume $T_1$ is complete and $T_2$ is satisfiable. Let $\phi \in T_2$. Since $T_1$ is complete, $\phi \in T_1$ or $\neg \phi \in T_1$. If $\phi \in T_1$, we're done. If not, $\neg \phi \in T_1 \subset T_2$, so $\phi \land \neg \phi \in T_2$ (since $T_2$ is closed under deduction). But this contradicts our assumption that $T_2$ is satisfiable.

7. Let $T = Th(\mathfrak{N})$. Expand the language $\{<\}$ to $\{<, c_0, c_1, \ldots\}$ by adding countably many fresh constants. Let $\sigma_i$ be the sentence $c_{i+1} < c_i$.

**Claim 1.** $T \cup \{\sigma_i | i \in \mathbb{N}\}$ is consistent.

**Proof.** Let $\Delta \subset T \cup \{\sigma_i | i \in \mathbb{N}\}$ be finite. Let $m$ be the max such that $\sigma_m \in \Delta$. Then $\Delta \subset T \cup \{\sigma_0, \ldots, \sigma_m\}$, so to show that $\Delta$ is satisfiable, it is enough to show that $T \cup \{\sigma_0, \ldots, \sigma_m\}$ is satisfiable. Let $\mathfrak{N}' = (\mathbb{N}, <, c_0, c_1, \ldots)$ be a structure in the language $\{<, c_0, c_1, \ldots\}$, with $<_{\mathfrak{N}'}$ as the usual orderin, $c_i' = m - i$ for $0 \leq i \leq m + 1$, and $c_j' = j$ for $j > m + 1$. Then, $|=_{\mathfrak{N}'}T$ by a previous homework exercise (since $T$ does not contain any of the symbols in the expanded language, and $\mathfrak{N}$ and $\mathfrak{N}'$ are the same apart from those symbols). For $0 \leq i \leq m$, $|=_{\mathfrak{N}'} c_{i+1} < c_i$. Thus, $|=_{\mathfrak{N}'} T \cup \{\sigma_0, \ldots, \sigma_m\}$. So $T \cup \{\sigma_0, \ldots, \sigma_m\}$ is satisfiable. Thus, $\Delta$ is satisfiable, and hence, is consistent. So by compactness, $T \cup \{\sigma_0, \sigma_1, \ldots\}$ is consistent. \hfill \box

Let $\mathfrak{A}'$ satisfy $T \cup \{\sigma_0, \sigma_1, \ldots\}$, and let $\mathfrak{A} = (|\mathfrak{A}'|, <)$ be the reduction of $\mathfrak{A}'$ to the language $\{<\}$. Since every formula in $T$ is in this language, $|=_{\mathfrak{A}}T$. So $|=_{\mathfrak{N}} \phi \Rightarrow |=_{\mathfrak{A}} \phi$. Then, if $|=_{\mathfrak{A}} \phi$, then either $|=_{\mathfrak{N}} \phi$ or $\not|=_{\mathfrak{N}} \phi$, that is $|=_{\mathfrak{N}} \neg \phi$. If the latter case holds, then $\neg \phi \in Th(\mathfrak{N}) = T$, so $|=_{\mathfrak{N}} \neg \phi$, which is a contradiction. Hence, $|=_{\mathfrak{N}} \phi$. Thus, $\mathfrak{N}$ and $\mathfrak{A}$ are elementarily equivalent.

Finally, since $c_0^\mathfrak{A} > c_1^\mathfrak{A} > c_2^\mathfrak{A} > \ldots$ (and this ordering is strict), we have an infinite descending chain.

8. Assume that $\sigma$ is true in all infinite models of a theory $T$. Suppose for contradiction that for every finite $k$, there is a finite model $\mathfrak{A}$ of $T$ with at least $k$ elements in its universe in which $\sigma$ is not true. Let $\phi_n$ be the sentence $\exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} \neg x_i = x_j$ for $n \geq 2$. We see that a structure $\mathfrak{A}$ satisfies $\phi_n$ if and only if $\mathfrak{A}$ has at least $n$ elements. Consider the theory $T \cup \{\phi_n | n \geq 2\} \cup \{\neg \phi\}$. Let $\Delta$ be a finite subset and $k$ be the greatest such that $\phi_k \in \Delta$. Then $\Delta \subset T \cup \{\phi_2, \ldots, \phi_k\} \cup \{\neg \sigma\}$. Let $\mathfrak{A}$ be a models of $T$ with at least $k$ elements in which $\sigma$ is not true (this exists by assumption). Then clearly $|=_{\mathfrak{A}} \Delta$, so $\Delta$ is consistent.

Hence, by compactness, $T \cup \{\phi_2, \phi_3, \ldots\} \cup \{\neg \sigma\}$, so it has a model $\mathfrak{A}$. Suppose $\mathfrak{A}$ has exactly $k$ elements where $k$ is finite. But $|=_{\mathfrak{A}} \phi_{k+2}$, and thus has at least $k + 2$ elements, so this is a contradiction. So $\mathfrak{A}$ is infinite, but $|=_{\mathfrak{A}} \neg \sigma$, contradicting our original assumption.

9. By Theorem 26D, since the language is finite, the set of sentences satisfiable in a finite model is effectively enumerable. Since the satisfiable members of $\Sigma$ are exactly the members of $\Sigma$ satisfiable in a finite model, the members of $\Sigma$ satisfiable in a finite model are effectively enumerable. Conversely, the set of unsatisfiable sentences is effectively
enumerate, since $\sigma$ is unsatisfiable iff $\neg \sigma$ is valid, iff $\vdash \neg \sigma$, and the set of deductions from $\emptyset$ is enumerable. Since the satisfiable members of $\Sigma$ are effectively enumerable, and the complement is also effectively enumerable, the satisfiable members of $\Sigma$ are decidable.

10a. By Exercise 19 in Section 2.2, the sentences from $\exists_2$ in a language without function symbols have the finite model property. Since the language is finite, problem 9 applies, so the set of satisfiable $\exists_2$ sentences is decidable.

10b. A sentence $\sigma$ is valid iff $\neg \sigma$ is not satisfiable. Given a $\forall_2$ sentence $\sigma$, $\neg \sigma$ is $\exists_2$. By the previous part, we may decide whether $\neg \sigma$ is satisfiable; if so, $\sigma$ is not valid. If not, $\sigma$ is valid.