MIDTERM 1

Math 3A
10/19/2009

Name: ______________________

Signature: ______________________

Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Whenever you invoke a theorem to justify a result, make sure to clearly identify all premises of the theorem, show that they are true, and specify which theorem you are using.
- Circle or otherwise indicate your final answers.
- Good luck!

1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	10	
Total	100	

1. (*15 points*) Find the following limits if they exist; if not, state that they do not exist, and indicate if they go to ∞ or $-\infty$, indicate this as well. You may use any method you like, but clearly indicate intermediate steps and how you obtain your answer.

(a) $\lim_{x\to\infty} \tan x$

$\tan x$ oscillates as $x \to \infty$, so $\lim_{x\to\infty} \tan x$ DNE

(b) Find $\lim_{x\to\pi/2^-} e^{\tan x}$

$\lim_{x\to\pi/2^-} \tan x = \infty$, and $\lim_{y\to\infty} e^y = \infty$, so $\lim_{x\to\pi/2^-} e^{\tan x} = \infty$ (and therefore DNE)

(c) Find $\lim_{x\to\infty} \frac{5x^4-7x^2+2}{3x^2+4}$

Since the highest power on the top is x^4, while the highest on the bottom is only x^2, $\lim_{x\to\infty} \frac{5x^4-7x^2+2}{3x^2+4} = \infty$, and therefore DNE.

(d) Find $\lim_{x\to 1} \frac{1}{x+1}$

$\frac{1}{1+1} = \frac{1}{2}$, and since $\frac{1}{x+1}$ is continuous whereever it is defined, $\lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$.

(e) Consider the sequence $a_n = \sin n\pi$. What is $\lim_{n\to\infty} a_n$?

For every n, $\sin n\pi = 0$, so $\lim_{n\to\infty} a_n = 0$.

2. (*15 points*) Compute

$$\lim_{x\to 0^-} \sin(2x)\cos\frac{2}{x}$$

$-1 \le \cos\frac{2}{x} \le 1$ for all x
Since, when $x < 0$, $\sin x < 0$, $\sin 2x \le \sin(2x)\cos\frac{2}{x} \le -\sin 2x$ for all $x < 0$
$\lim_{x\to 0^-} -\sin 2x = 0 = \lim_{x\to 0^-} \sin 2x$
By the sandwich theorem, $\lim_{x\to 0^-} \sin(2x)\cos\frac{2}{x} = 0$.

3. (*15 points*) Show that the polynomial

$$x^3 - 100x^2 + 10$$

has at least two roots.

Let $f(x) = x^3 - 100x^2 + 10$. f is continuous everywhere. Since $f(-1) = -91 < 0 < 10 = f(0)$, by IVT there is some c with $-1 < c < 0$ such that $f(c) = 0$.

Since $f(0) = 10 > 0 > -89 = f(1)$, by IVT there is some d with $0 < d < 1$ such that $f(d) = 0$. So c and d are two roots of the polynomial.

4. (*15 points*) For what value of a is the function

$$f(x) = \begin{cases} \frac{x^2-x-20}{x-5} & \text{if } x \neq 5 \\ a & \text{if } x = 5 \end{cases}$$

continuous everywhere?

$\lim_{x\to 5} \frac{x^2-x-20}{x-5} = \lim_{x\to 5} \frac{(x-5)(x+4)}{x-5} = \lim_{x\to 5} x + 4 \lim_{x\to 5} \frac{x-5}{x-5} = 9$. So if $a = 9$, the function will be continuous at 5. For $x \neq 5$, $f(x) = x + 4$, which is continuous everywhere, so when $a = 9$, f is continuous everywhere.

5. (*15 points*) Given an example of three sequences, a_n, b_n, and c_n, such that:

- None of the limits $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} \frac{a_n}{b_n}$, $\lim_{n\to\infty} \frac{a_n}{c_n}$ exist
- The limit $\lim_{n\to\infty} \frac{a_n}{b_n c_n} = 1$

$a_n = n^2$, $b_n = c_n = n$. Then $\lim_{n\to\infty} a_n = \infty$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n}{c_n} = \lim_{n\to\infty} n = \infty$, so these three limits do not exist. But $\lim_{n\to\infty} \frac{a_n}{b_n c_n} = \lim_{n\to\infty} 1 = 1$.

6. (*15 points*) (**a**) Find the derivative of $\sqrt{2x+1}$ using the chain rule.

$\frac{d}{dx}\sqrt{2x+1} = \frac{1}{2\sqrt{2x+1}} \cdot \frac{d}{dx}(2x+1) = \frac{1}{\sqrt{2x+1}}$

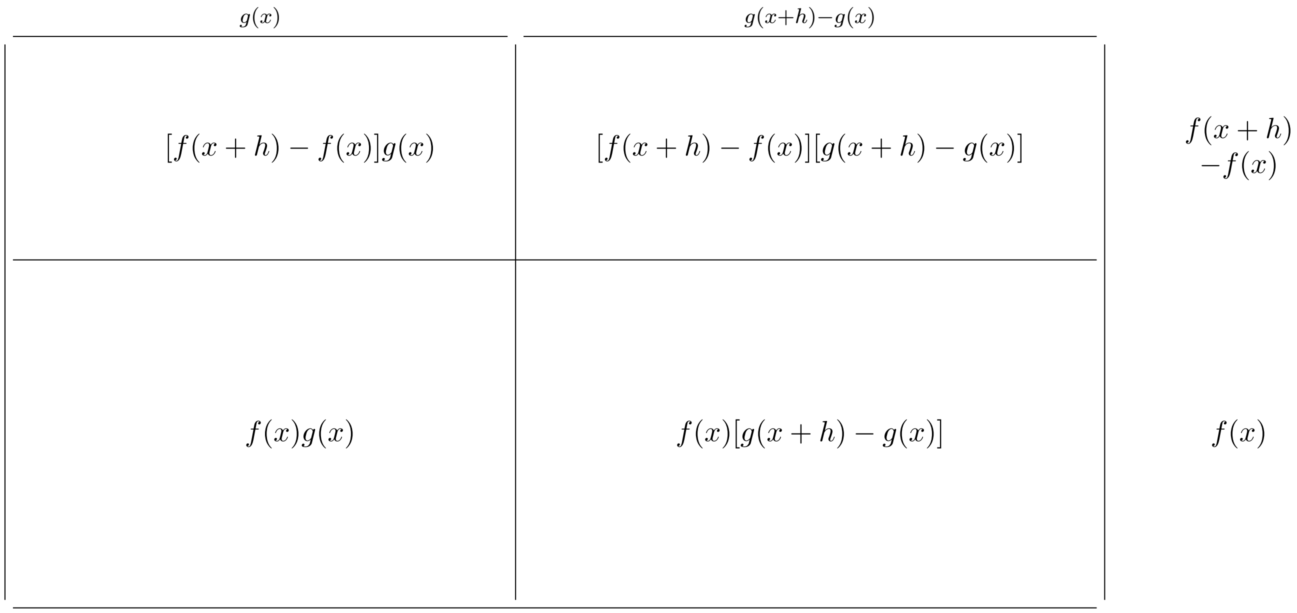
(**b**) Find the derivative of $\sqrt{2x+1}$ using the definition of the derivative.

$$\lim_{h\to\infty} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} = \lim_{h\to\infty} \frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h} \frac{\sqrt{2(x+h)+1}+\sqrt{2x+1}}{\sqrt{2(x+h)+1}+\sqrt{2x+1}}$$
$$= \lim_{h\to\infty} \frac{2(x+h)+1-(2x+1)}{h(\sqrt{2(x+h)+1}+\sqrt{2x+1})}$$
$$= \lim_{h\to\infty} \frac{2h}{h(\sqrt{2(x+h)+1}+\sqrt{2x+1})}$$
$$= \lim_{h\to\infty} \frac{2}{\sqrt{2(x+h)+1}+\sqrt{2x+1}}$$
$$= \frac{2}{\sqrt{2x+1}+\sqrt{2x+1}}$$
$$= \frac{1}{\sqrt{2x+1}}$$

7. (*10 points*) Consider the equality

$$f(x+h)g(x+h)-f(x)g(x) = f(x)[g(x+h)-g(x)]+g(x)[f(x+h)-f(x)]+[f(x+h)-f(x)][g(x+h)-g(x)].$$

Draw a diagram illustrating this equality; each of the quantities in the equation should be represented as the area of some shape in this diagram.



There are five terms in the equation, and five areas in the diagram: the area of the big rectangle, and the areas of the four little rectangles.