Math 3A, Fall 2010 — Homework 2 [due Oct 4th] — Solutions

Section 2.2

3. Determine the values of the sequence \( \{a_n\} \), \( a_n = \frac{1}{n+2} \), for \( n = 0, 1, 2, 3, 4, 5 \).

Solution.

\[
\begin{array}{c|cccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 \\
 a_n & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\
\end{array}
\]

4*. Determine the values of the sequence \( \{a_n\} \), \( a_n = \frac{1}{1+n^2} \), for \( n = 0, 1, 2, 3, 4, 5 \).

Solution.

\[
a_0 = \frac{1}{1+0^2} = \frac{1}{1} = 1, \quad a_1 = \frac{1}{1+1^2} = \frac{1}{2}, \quad a_2 = \frac{1}{1+2^2} = \frac{1}{5}, \quad a_3 = \frac{1}{1+3^2} = \frac{1}{10}, \quad a_4 = \frac{1}{1+4^2} = \frac{1}{17}, \quad a_5 = \frac{1}{1+5^2} = \frac{1}{26}
\]

\[
\begin{array}{c|cccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 \\
 a_n & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{10} & \frac{1}{17} & \frac{1}{26} \\
\end{array}
\]

31. Find an expression for \( a_n \) on the basis of the values of \( a_0, a_1, a_2, \ldots \)

\[-1, 2, -3, 4, -5, \ldots\]

Solution.

The signs are alternating, so each term will be a positive number multiplied by \((-1)^{n+1}\) [or \((-1)^{n-1}\), which is the same]. Absolute values of the terms are \(1, 2, 3, 4, 5, \ldots\), and since the sequence starts with 0th term, we recognize them as \( |a_n| = n + 1 \). Thus, the formula is \( a_n = (-1)^{n+1}(n + 1) \), and it is easy to verify it for \( n = 0, 1, \ldots, 5 \).

32*. Find an expression for \( a_n \) on the basis of the values of \( a_0, a_1, a_2, \ldots \)

\[2, -4, 6, -8, 10 \ldots\]

Solution.

The signs are alternating, so each term will be a positive number multiplied by \((-1)^n\). Absolute values of the terms are \(2, 4, 6, 8, 10 \ldots\), and since the sequence starts with 0th term, we recognize them as \( |a_n| = 2n + 2 \). Thus, the formula is \( a_n = (-1)^n(2n + 2) \), and it is easy to verify it for \( n = 0, 1, \ldots, 5 \).

41. Write the first five terms of the sequence \( \{a_n\} \), \( a_n = \frac{1}{n^2+1} \), and find \( \lim_{n \to \infty} a_n \).

Solution.

\[
\begin{array}{c|cccc}
 n & 0 & 1 & 2 & 3 \\
 a_n & 1 & \frac{1}{2} & \frac{1}{5} & \frac{1}{10} \\
\end{array}
\]
It seems that the sequence converges to 0, i.e. \( \lim_{n \to \infty} a_n = 0 \).

If we wanted to show it rigorously, we would use the limit laws and write:

\[
\lim_{n \to \infty} \frac{1}{n^2 + 1} = \lim_{n \to \infty} \frac{1}{n^2} \cdot \frac{1}{1 + \frac{1}{n^2}} = \lim_{n \to \infty} \frac{1}{n^2} \cdot \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n^2}} = \lim_{n \to \infty} \frac{1}{n^2} \cdot \lim_{n \to \infty} \frac{1}{1 + 0} = \lim_{n \to \infty} \frac{1}{n^2} \cdot 0 = 0.
\]

42*. Write the first five terms of the sequence \( \{a_n\} \), \( a_n = \frac{1}{\sqrt{n} + 1} \), and find \( \lim_{n \to \infty} a_n \).

\[
\begin{array}{c|c|c|c|c|c}
 n & 0 & 1 & 2 & 3 & 4 \\
\hline
 a_n & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{5}} \\
\end{array}
\]

It seems that the sequence converges to 0, i.e. \( \lim_{n \to \infty} a_n = 0 \).

If we wanted to show it rigorously, we would use the limit laws and write:

\[
\lim_{n \to \infty} \frac{1}{\sqrt{n} + 1} = \lim_{n \to \infty} \sqrt{\frac{1}{n + 1}} = \sqrt{\lim_{n \to \infty} \frac{1}{n + 1}} = \sqrt{\lim_{n \to \infty} \frac{1}{n + 1} \cdot \frac{n}{n}} = \sqrt{\lim_{n \to \infty} \frac{1}{n + 1} \cdot \lim_{n \to \infty} \frac{n}{n}} = \sqrt{0 \cdot 1 + 0} = \sqrt{0} = 0.
\]

Above we used the property \( \lim_{n \to \infty} \sqrt{b_n} = \sqrt{\lim_{n \to \infty} b_n} \), which we didn’t learn yet.

47. Write the first five terms of the sequence \( \{a_n\} \), \( a_n = \sqrt{n} \), and determine whether \( \lim_{n \to \infty} a_n \) exists. If the limit exists, find it.

\[
\begin{array}{c|c|c|c|c|c}
 n & 0 & 1 & 2 & 3 & 4 \\
\hline
 a_n & 0 & 1 & \sqrt{2} & \sqrt{3} & 2 \\
\end{array}
\]

It seems that the sequence “diverges to \( \infty \)”, i.e. the limit \( \lim_{n \to \infty} a_n \) does not exist.

48*. Write the first five terms of the sequence \( \{a_n\} \), \( a_n = n^2 \), and determine whether \( \lim_{n \to \infty} a_n \) exists. If the limit exists, find it.

\[
\begin{array}{c|c|c|c|c|c}
 n & 0 & 1 & 2 & 3 & 4 \\
\hline
 a_n & 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 = 0
\]
It seems that the sequence “diverges to $\infty$”, i.e. the limit $\lim_{n \to \infty} a_n$ does not exist.

75. Use the limit laws to determine the limit.

$$\lim_{n \to \infty} \frac{n^2 + 1}{n^2}$$

**Solution.**

$$\lim_{n \to \infty} \frac{n^2 + 1}{n^2} = \lim_{n \to \infty} \left(1 + \frac{1}{n^2}\right) = \lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{1}{n^2}$$

$$= 1 + \lim_{n \to \infty} \left(\frac{1}{n} \cdot \frac{1}{n}\right) = 1 + \lim_{n \to \infty} \frac{1}{n} \cdot \lim_{n \to \infty} \frac{1}{n} = 1 + 0 \cdot 0 = 1$$

76*. Use the limit laws to determine the limit.

$$\lim_{n \to \infty} \frac{3n^2 - 5}{n^2}$$

**Solution.**

$$\lim_{n \to \infty} \frac{3n^2 - 5}{n^2} = \lim_{n \to \infty} \left(3 - \frac{5}{n^2}\right) = \lim_{n \to \infty} 3 - \lim_{n \to \infty} \frac{5}{n^2} = 3 - 5 \lim_{n \to \infty} \frac{1}{n^2}$$

$$= 3 - 5 \lim_{n \to \infty} \left(\frac{1}{n} \cdot \frac{1}{n}\right) = 3 - 5 \lim_{n \to \infty} \frac{1}{n} \cdot \lim_{n \to \infty} \frac{1}{n} = 3 - 5 \cdot 0 \cdot 0 = 3$$

V.K.