3. Differentiate the function with respect to the independent variable.
   \[ f(x) = (1 - 3x^2)^4 \]
   
   \textit{Solution.} One can decompose \( f \) as the composition \( f(x) = f_2(f_1(x)) \), where \( f_1(x) = 1 - 3x^2 \) and \( f_2(y) = y^4 \). Using the chain rule we get:
   
   \[ f'(x) = f'_2(f_1(x))f'_1(x) = 4(1 - 3x^2)^3(-6x) \]
   
   or after simplification
   
   \[ f'(x) = -24x(1 - 3x^2)^3 \]

4*. Differentiate the function with respect to the independent variable.
   \[ f(x) = (5x^2 - 3x)^3 \]
   
   \textit{Solution.} One can decompose \( f \) as the composition \( f(x) = f_2(f_1(x)) \), where \( f_1(x) = 5x^2 - 3x \) and \( f_2(y) = y^3 \). Using the chain rule we get:
   
   \[ f'(x) = f'_2(f_1(x))f'_1(x) = 3(5x^2 - 3x)^2(10x - 3) \]

15. Differentiate the function with respect to the independent variable.
   \[ f(s) = \sqrt{s + \sqrt{s}} \]
   
   \textit{Solution.} One can decompose \( f \) as the composition \( f(s) = f_2(f_1(s)) \), where \( f_1(s) = s + \sqrt{s} = s + s^{1/2} \) and \( f_2(t) = \sqrt{t} = t^{1/2} \). Using the chain rule we get:
   
   \[ f'(s) = f'_2(f_1(s))f'_1(s) = \frac{1}{2}(s + s^{1/2})^{-1/2}(1 + \frac{1}{2}s^{-1/2}) \]
   
   or after simplification
   
   \[ f'(s) = \frac{2\sqrt{s} + 1}{4\sqrt{s} \sqrt{s + \sqrt{s}}} \]

16*. Differentiate the function with respect to the independent variable.
   \[ g(t) = \sqrt{t^2 + \sqrt{t + 1}} \]
   
   \textit{Solution.} One can decompose \( g \) as the composition \( g(t) = g_2(g_1(t)) \), where \( g_1(t) = t^2 + \sqrt{t + 1} = t^2 + (t + 1)^{1/2} \) and \( g_2(u) = \sqrt{u} = u^{1/2} \). Using the chain rule we get:
   
   \[ g'(t) = g'_2(g_1(t))g'_1(t) = \frac{1}{2}(t^2 + (t + 1)^{1/2})^{-1/2}(2t + \frac{1}{2}(t + 1)^{-1/2}) \]
   
   or after simplification
   
   \[ g'(t) = \frac{4t\sqrt{t + 1} + 1}{4\sqrt{t + 1}\sqrt{t^2 + \sqrt{t + 1}}} \]
61. Assume that \( x \) and \( y \) are differentiable functions of \( t \). Find \( \frac{dy}{dt} \) when \( x^2 + y^2 = 1 \), \( \frac{dx}{dt} = 2 \) for \( x = \frac{1}{2} \), and \( y > 0 \).

**Solution.** Observe that \( x = \frac{1}{2} \) and \( y > 0 \) together with the equation give \( \left(\frac{1}{2}\right)^2 + y^2 = 1 \), i.e. \( y = \frac{\sqrt{3}}{2} \). Differentiating the equation \( x^2 + y^2 = 1 \) with respect to \( t \) and using the chain rule we obtain

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0
\]

Plugging in \( x = \frac{1}{2} \), \( y = \frac{\sqrt{3}}{2} \), \( \frac{dx}{dt} = 2 \) we get

\[
2 \cdot \frac{1}{2} + 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{dy}{dt} = 0
\]

and thus \( \frac{dy}{dt} = -\frac{\sqrt{3}}{2} \).

62*. Assume that \( x \) and \( y \) are differentiable functions of \( t \). Find \( \frac{dy}{dt} \) when \( y^2 = x^2 - x^4 \), \( \frac{dx}{dt} = 1 \) for \( x = \frac{1}{2} \), and \( y > 0 \).

**Solution.** Observe that \( x = \frac{1}{2} \) and \( y > 0 \) together with the equation give \( y^2 = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^4 = \frac{3}{16} \), i.e. \( y = \frac{\sqrt{3}}{4} \). Differentiating the equation \( y^2 = x^2 - x^4 \) with respect to \( t \) and using the chain rule we obtain

\[
2y \frac{dy}{dt} = 2x \frac{dx}{dt} - 4x^3 \frac{dx}{dt}
\]

Plugging in \( x = \frac{1}{2} \), \( y = \frac{\sqrt{3}}{4} \), \( \frac{dx}{dt} = 1 \) we get

\[
2 \cdot \frac{\sqrt{3}}{4} \cdot \frac{dy}{dt} = 2 \cdot \frac{1}{2} \cdot 1 - 4 \cdot \left(\frac{1}{2}\right)^3 \cdot 1
\]

i.e.

\[
\frac{\sqrt{3}}{2} \cdot \frac{dy}{dt} = 1
\]

and thus \( \frac{dy}{dt} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \).

69. Suppose that water is stored in a cylindrical tank of radius 5m. If the height of the water in the tank is \( h \), then the volume of the water is \( V = \pi r^2 h = 25\pi h \text{m}^3 \). If we drain the water at a rate of 250 liters per minute, what is the rate at which the water level inside the tank drops?

**Solution.** Differentiating \( V = 25\pi h \) with respect to \( t \) we get

\[
\frac{dV}{dt} = 25\pi \frac{dh}{dt}
\]

Since \( \frac{dV}{dt} = 250 \frac{1}{\text{min}} = 0.25 \frac{\text{m}^3}{\text{min}} \) we get from above

\[
\frac{dh}{dt} = \frac{0.25}{25\pi} = \frac{1}{100\pi}
\]

and so the rate is \( \frac{1}{100\pi} \text{m/\text{min}} \) [meters per minute].
70*. Suppose that we pump water into an inverted right circular conical tank at the rate of 5 cubic feet per minute. The tank has height of 6 ft and the radius on top is 3 ft. What is the rate at which the water level is rising when the water is 2 ft deep? (Note that the volume of a right circular cone of radius \( r \) and height \( h \) is \( V = \frac{1}{3}\pi r^2 h \).

**Solution.** When the water level is \( h \) and the radius on top of the water level is \( r \), we can equal the proportions

\[
\frac{h}{6} = \frac{r}{3}
\]

to get \( r = \frac{h}{2} \). Therefore the formula becomes

\[
V = \frac{1}{3}\pi \left( \frac{1}{2}h \right)^2 h = \frac{\pi}{12} h^3
\]

Differentiating with respect to \( t \) gives

\[
\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}
\]

Since \( \frac{dV}{dt} = 5 \) and we are interested in the moment when \( h = 2 \), we finally obtain

\[
\frac{dh}{dt} = \frac{5}{\frac{\pi}{4}2^2} = \frac{5}{\pi}
\]

and so the rate is \( \frac{5}{\pi} \) ft/min.

75. Find the first and the second derivative of the function. 
\( g(x) = \frac{x-1}{x+1} \)

**Solution.** Using the quotient rule we get:

\[
g'(x) = \frac{1 \cdot (x+1) - (x-1) \cdot 1}{(x+1)^2} = \frac{2}{(x+1)^2}
\]

and once again:

\[
g''(x) = \frac{0 \cdot (x+1)^2 - 2 \cdot 2(x+1)}{(x+1)^3} = \frac{-4}{(x+1)^3}
\]

76*. Find the first and the second derivative of the function. 
\( h(s) = \frac{1}{s^2+2} \)

**Solution.** Using the quotient rule we get:

\[
h'(s) = \frac{0 \cdot (s^2+2) - 1 \cdot 2s}{(s^2+2)^2} = \frac{-2s}{(s^2+2)^2}
\]

and once again, together with the chain rule for \((s^2+2)^2)' = 2(s^2+2) \cdot 2s:

\[
h''(s) = \frac{(-2) \cdot (s^2+2)^2 - (-2s) \cdot 2(s^2+2) \cdot 2s}{(s^2+2)^3} = \frac{6s^2 - 4}{(s^2+2)^3}
\]
83. Find the first 10 derivatives of \( y = x^5 \).

Solution.

\[
\begin{align*}
y' &= 5x^4 \\
y'' &= 5 \cdot 4 \cdot x^3 = 20x^3 \\
y^{(3)} &= 5 \cdot 4 \cdot 3 \cdot x^2 = 60x^2 \\
y^{(4)} &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot x = 120x \\
y^{(5)} &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\
y^{(6)} &= 0 \\
y^{(7)} &= 0 \\
y^{(8)} &= 0 \\
y^{(9)} &= 0 \\
y^{(10)} &= 0
\end{align*}
\]

84*. Find \( f^{(n)}(x) \) and \( f^{(n+1)}(x) \) of \( f(x) = x^n \).

Solution.

\[
\begin{align*}
f'(x) &= nx^{n-1} \\
f''(x) &= n(n - 1)x^{n-2} \\
f^{(3)}(x) &= n(n - 1)(n - 2)x^{n-3} \\
\vdots \\
f^{(n)}(x) &= n(n - 1)(n - 2)\ldots 3 \cdot 2 \cdot 1 \cdot x^0 = n! \\
f^{(n+1)}(x) &= 0
\end{align*}
\]

Section 4.5

5. Find the derivative with respect to the independent variable.
\( f(x) = \tan x - \cot x \)

Solution.

\[
f'(x) = \sec^2 x - (-\csc^2 x) = \sec^2 x + \csc^2 x
\]

6*. Find the derivative with respect to the independent variable.
\( f(x) = \sec x - \csc x \)

Solution.

\[
f'(x) = \sec x \tan x - (-\csc x \cot x) = \sec x \tan x + \csc x \cot x
\]
47. Find the derivative with respect to the independent variable.
\[ g(x) = \csc^3(1 - 5x^2) \]

\textit{Solution.} It is convenient to use \( \csc \theta = \frac{1}{\sin \theta} \) and rewrite the function as
\[ g(x) = \sin^3(1 - 5x^2) = \left( \sin(1 - 5x^2) \right)^3 \]

Now we apply the chain rule:
\[ g'(x) = 3 \left( \sin(1 - 5x^2) \right)^2 (-10x) = -30x \sin^2(1 - 5x^2) \]

48*. Find the derivative with respect to the independent variable.
\[ h(x) = \cot(3x) \csc(3x) \]

\textit{Solution.} We apply the product rule:
\[ h'(x) = -\csc^2(3x) \cdot 3 \cdot \csc(3x) + \cot(3x) \cdot (-\csc(3x) \cot(3x)) \cdot 3 \]

and this result can be simplified as
\[ h'(x) = -3 \csc^3(3x) - 3 \cot^2(3x) \csc(3x) \]

63. Use the quotient rule to show that
\[ \frac{d}{dx} \sec x = \sec x \tan x \]

\textit{Solution.}
\[ \frac{d}{dx} \sec x = \frac{d}{dx} \left( \frac{1}{\cos x} \right) = \frac{0 \cdot \cos x - 1 \cdot (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x \]

64*. Use the quotient rule to show that
\[ \frac{d}{dx} \csc x = -\csc x \cot x \]

\textit{Solution.}
\[ \frac{d}{dx} \csc x = \frac{d}{dx} \left( \frac{1}{\sin x} \right) = \frac{0 \cdot \sin x - 1 \cdot \cos x}{\sin^2 x} = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x \]
Section 4.6

1. Differentiate the function with respect to the independent variable. 
   \( f(x) = e^{3x} \)
   
   \textit{Solution.} Using the chain rule we get:
   \[ f'(x) = e^{3x} \cdot 3 = 3e^{3x} \]

2*. Differentiate the function with respect to the independent variable. 
   \( f(x) = e^{-2x} \)
   
   \textit{Solution.} Using the chain rule we get:
   \[ f'(x) = e^{-2x} \cdot (-2) = -2e^{-2x} \]

45. Differentiate the function with respect to the independent variable. 
   \( f(x) = 2\sqrt{x^2 - 1} \)
   
   \textit{Solution.} One can decompose \( f \) as the composition \( f(x) = f_3(f_2(f_1(x))) \), where \( f_1(x) = x^2 - 1 \), \( f_2(y) = \sqrt{y} \), \( f_3(z) = 2^z \). Using the chain rule we get:
   \[ f'(x) = f'_3(f_2(f_1(x)))f'_2(f_1(x))f'_1(x) = 2\sqrt{x^2 - 1} \ln 2 \cdot \frac{1}{2}(x^2 - 1)^{-1/2} \cdot 2x \]
   or after simplification
   \[ f'(x) = \frac{x \cdot 2\sqrt{x^2 - 1} \ln 2}{\sqrt{x^2 - 1}} \]

46*. Differentiate the function with respect to the independent variable. 
   \( f(x) = 4\sqrt{1 - 2x^3} \)
   
   \textit{Solution.} One can decompose \( f \) as the composition \( f(x) = f_3(f_2(f_1(x))) \), where \( f_1(x) = 1 - 2x^3 \), \( f_2(y) = \sqrt{y} \), \( f_3(z) = 4^z \). Using the chain rule we get:
   \[ f'(x) = f'_3(f_2(f_1(x)))f'_2(f_1(x))f'_1(x) = 4\sqrt{1 - 2x^3} \ln 4 \cdot \frac{1}{2}(1 - 2x^3)^{-1/2} \cdot (-6x^2) \]
   or after simplification
   \[ f'(x) = \frac{-3x^2 \cdot 4\sqrt{1 - 2x^3} \ln 4}{\sqrt{1 - 2x^3}} \]

\textit{V.K.}