Homework 7

5.1, 41. Since \(f(x) = -x^2 + 2\) is continuous on \([-1, 2]\), and differentiable on \((-1, 2)\), therefore by MVT, there exists \(c\) in \((-1, 2)\), such that \(f'(c) = \frac{f(2) - f(1)}{2 - (-1)} = -1\).

42. Since \(f(x) = x^3\) is continuous on \([-1, 0]\), and differentiable on \((-1, 0)\), therefore by MVT, there exists \(c\) in \((-1, 0)\) in \((-1, 1)\), such that \(f'(c) = \frac{f(1) - f(0)}{1 - 0} = 1\).

43. We should draw a function that is continuous on \([0, 1]\) and differentiable on \((0, 1)\), the graph is omitted. We should draw a function that is continuous on \([0, 1]\) and differentiable on \((0, 1)\), therefore there exist a point \(c\) in \((0, 1)\), such that \(f'(c) = \frac{f(1) - f(0)}{1 - 0}\).

44. We should draw a function that is continuous on \([0, 1]\) and differentiable on \((0, 1)\), and should have two "peaks". The graph is omitted. For the second statement the reason is that: by MVT, since \(f(x)\) is a function that is continuous on \([0, 1]\) and differentiable on \((0, 1)\), therefore there exist a point \(c\) in \((0, 1)\), such that \(f'(c) = \frac{f(1) - f(0)}{1 - 0}\).

46. Since \(f\) is continuous on \([a, b]\), and differentiable on \((a, b)\), and \(f(b) - f(a) > 0\), then by MVT, there exists \(c\) in \((a, b)\), such that \(f'(c) = \frac{f(b) - f(a)}{b - a} > 0\).

47. Since \(f(x)\) is not constant, then we can find a point \(a\) in \((a, b)\) such that \(f(a) \neq 0\), so \(f(x) > 0\) or \(f(x) < 0\). If \(f(x) > 0\), apply MVT on \([a, d]\), we get there exists a point \(c_1\) in \((a, d)\), such that \(f(c_1) = \frac{f(d) - f(a)}{d - a} > 0\), since \(f(d) - f(a) = f(d) > 0\); and then apply MVT on \([d, b]\), we get that there exists a point \(c_2\) in \((d, b)\), such that \(f(c_2) = \frac{f(b) - f(d)}{b - d} < 0\), since \(f(b) - f(d) = -f(d) < 0\). Similarly if \(f(x) < 0\), we can also find such two points satisfy the required conditions.

5.2, 6. \(y = (x - 2)^3 + 3, x \in R\).
\(y' = 3(X - 2)^2 \geq 0\) for all \(x \in R\), therefore \(f(x)\) is increasing on \(R\).
\(y'' = 6(x - 2),\) then when \(x > 2, y'' > 0\), thus \(y\) concave up; then when \(x < 2, y'' < 0\), thus \(y\) concave down.

7. \(y = \sqrt{x + 1}, x \geq -1\).
\(y' = \frac{1}{2\sqrt{x + 1}} \geq 0,\) for all \(x \geq -1\), thus \(y\) is increasing on \(x \geq -1\).
\(y'' = \frac{-1}{4(x + 1)^{3/2}} < 0,\) for \(x > -1\), therefore \(y\) is concave down for \(x > -1\).

8. \(y = (3x - 1)^{3/2}, x \in R\).
\(y' = \frac{3}{2}(3x - 1)^{-\frac{1}{2}} \geq 0,\) thus the function is increasing on \(R\).
\(y'' = -\frac{3}{4}(3x - 1)^{-\frac{3}{2}};\) when \(x < \frac{1}{3}, y'' > 0\), thus concave up; when \(x > \frac{1}{3}, y'' < 0\), thus concave down.

9. \(y = \frac{1}{x^2}, x \neq 0\).
\(y' = -\frac{1}{x^3} < 0\) for all \(x \neq 0\), thus the function is decreasing.
\(y'' = \frac{2}{x^4};\) when \(x > 0, y'' > 0\), thus concave up; when \(x < 0, y'' < 0\), thus concave down.

31 \(f(P) = e^{-aP}\), then \(f'(P) = -ae^{-aP} < 0\), therefore \(f(P)\) decreases.
32 \(f(P) = (1 + aP)^{-k}\), then \(f'(P) = -a(1 + aP)^{-k-1} < 0\), since \(P\) and \(k\) are both positive constants,
therefore \( f(P) \) decreases.

5.3 2. \( y = \sqrt{x - 1}, \ 1 \leq x \leq 2. \)
\[
y' = \frac{1}{\sqrt{x-1}} > 0, \text{ for } 1 < x < 2, \text{ therefore the function is increasing. And for } 1 \leq x \leq 2, \text{ the local maximum is } (2, f(2)) = (2, 1), \text{ and the local minimum is } (1, f(1)) = (1, 0), \text{ therefore the absolute maximum is } (2, 1), \text{ and absolute minimum is } (1, 0).
\]

3. \( y = \ln(2x - 1), \ 1 \leq x \leq 2. \)
\[
y' = \frac{2}{2x-1} > 0 \text{ for all } 1 \leq x \leq 2, \text{ thus the local maximum is } (2, f(2)) = (2, \ln 3), \text{ and the local minimum is } (1, f(1)) = (1, 0), \text{ therefore the absolute maximum is } (2, \ln 3), \text{ and absolute minimum is } (1, 0).
\]

4. \( y = \ln \frac{x}{x+1}, \ x > 0. \)
\[
y' = \frac{1}{x(x+1)} > 0, \text{ for all } x > 0. \text{ therefore, there is no maximum or minimum.}
\]

(5. )\( y = xe^{-x}, \ 0 \leq x \leq 1 \)
\[
y' = e^{-x}(1-x) \geq 0, \text{ for all } 0 \leq x \leq 1, \text{ therefore the function is increasing. thus the local maximum is } (1, f(1)) = (1, \frac{1}{e}), \text{ and the local minimum is } (0, f(0)) = (0, 0), \text{ therefore the absolute maximum is } (1, \frac{1}{e}), \text{ and absolute minimum is } (0, 0)
\]

19. \( f(x) = x^3 - 2, x \in R. \)
\[
f''(x) = 6x, \text{ let } f''(x) = 0. \text{ we get } x = 0, \text{ and when } x > 0, f''(x) > 0, \text{ and when } x < 0, f''(x) < 0, \text{ therefore, at } x = 0, \text{ the concavity changes, therefore, } x = 0 \text{ is the inflection point of } f(x).
\]

20. \( f(x) = (x - 3)^5, \ x \in R. \)
\[
f''(x) = 20(x - 3)^3, \text{ let } f''(x) = 0. \text{ we get } x = 3, \text{ and when } x > 3, f''(x) > 0, \text{ and when } x < 3, f''(x) < 0, \text{ therefore, at } x = 3, \text{ the concavity changes, therefore } x = 3 \text{ is the inflection point of } f(x).
\]

Draw a function \( f \) so that \( f'(x) \) is negative when \( x \) is negative, \( f'(x) \) is positive when \( x \) is positive, but \( f(0) \) is not a minimum.
The function \( f(x) = -|\frac{1}{x}|\) for \( x \neq 0 \) works.