MIDTERM 2

Math 3B
2/16/2011

Name: __________________________

Section: __________________________

Signature: __________________________

Read all of the following information before starting the exam:

• Check your exam to make sure all pages are present.

• You may use writing implements and a single 3”x5” notecard.

• NO CALCULATORS!

• Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).

• Circle or otherwise indicate your final answers.

• Good luck!

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>6</td>
<td>16</td>
</tr>
<tr>
<td>Total</td>
<td>110</td>
</tr>
</tbody>
</table>
1. (25 points) Find the following integrals.

(a) \[ \int_{-1/2}^{1/2} e^{\tan x} \sec^2 x \, dx \]

\[ u = \tan x, \quad du = \sec^2 x \, dx, \text{ so } \]

\[ \int_{-1/2}^{1/2} e^{\tan x} \sec^2 x \, dx = \int_{\tan(-1/2)}^{\tan(1/2)} e^u \, du = e^{\tan(1/2)} - e^{\tan(-1/2)} \]

(b) \[ \int \frac{1}{(3x + 5)^2 + 1} \, dx \]

\[ u = 3x + 5, \quad du = 3 \, dx, \text{ so } \]

\[ \int \frac{1}{(3x + 5)^2 + 1} \, dx = \int \frac{1}{u^2 + 1} \, du = \frac{1}{3} \arctan u + C = \frac{1}{3} \arctan(3x + 5) + C. \]

(c) \[ \int \frac{x^5}{\sqrt{x^2 - 7}} \, dx \]

\[ u = x^2 - 7, \quad du = 2x \, dx, x^4 = (u - 7)^2 = u^2 - 14u + 49, \text{ so } \]

\[ \int \frac{x^5}{\sqrt{x^2 - 7}} \, dx = \frac{1}{2} \int (u^2 - 14u + 49)u^{-1/3} \, du \]

\[ = \frac{1}{2} \int u^{5/3} - 14u^{2/3} + 49u^{-1/3} \, du \]

\[ = \frac{1}{2} \left[ \frac{3}{8} u^{8/3} - 14 \frac{5}{3} u^{5/3} + 49 \frac{2}{3} u^{2/3} \right] + C \]

\[ = \frac{1}{2} \left[ \frac{3}{8} (x^2 - 7)^{8/3} - 14 \frac{5}{3} (x^2 - 7)^{5/3} + 49 \frac{2}{3} (x^2 - 7)^{2/3} \right] + C \]
(d) \[
\int \frac{x^2}{x^2 + 1} \, dx
\]
\[
\int \frac{x^2}{x^2 + 1} \, dx = \int \frac{x^2 + 1 - 1}{x^2 + 1} \, dx = \int 1 - \frac{1}{x^2 + 1} \, dx = x - \arctan x + C.
\]

(e) \[
\int \ln(x^2 + 1) \, dx
\]
We try integration by parts:
\[
u = \ln(x^2 + 1) \quad v = x
\]
\[
\mathrm{du} = \frac{2x}{x^2 + 1} \quad \mathrm{dv} = \mathrm{dx}
\]
So
\[
\int \ln(x^2 + 1) \, dx = x \ln(x^2 + 1) - \int \frac{2x^2}{x^2 + 1} \, dx = x \ln(x^2 + 1) - 2x + 2 \arctan x + C
\]
because we can use part (d) to integrate \[\int \frac{2x^2}{x^2 + 1} \, dx = 2 \int \frac{x^2}{x^2 + 1} \, dx.\]
2. (18 points) $y_0$ and $y_1$ are two different solutions to the equation $\frac{dy}{dt} = \tan(y + 1)$.

(a) $y_0(0) = 1$. What is $\lim_{t \to \infty} y_0(t)$?
When $y_0(0) = 1$, $\tan(y_0 + 1) = \tan(1 + 1) = \tan 2$. $\tan \theta$ is negative when $\theta$ is more than $\pi/2$ and less than $\pi$, so $y_0'(0)$ is negative, so $y_0$ is decreasing. $y_0$ decreases towards $\pi/2 - 1$, going faster and faster as it gets close. However $\lim_{y \to (\pi/2 - 1)_+} \tan(y + 1)$ does not exist, so $\lim_{t \to \infty} y_0(t)$ does not exist.

(b) $y_1(0) = -1$. What is $\lim_{t \to \infty} y_1(t)$?
When $y_1(0) = -1$, $y_1'(0) = \tan(y_1(0) + 1) = \tan 0 = 0$. So $y_1$ is constant, so $\lim_{t \to \infty} y_1(0) = -1$.

3. (17 points) Consider the differential equation $\frac{dy}{dt} = e^t y \ln y$.

(a) What is the general solution of this equation?

\[
\frac{dy}{y \ln y} = e^t dt \quad \text{or} \quad y = 0 \quad \text{or} \quad y = 1
\]

so

\[
\ln |\ln y| = e^t + C \quad \text{or} \quad y = 0 \quad \text{or} \quad y = 1
\]

so

\[
y = e^{e^t + c} \quad \text{or} \quad y = e^{-e^t + c} \quad \text{or} \quad y = 0 \quad \text{or} \quad y = 1.
\]

(b) What is the particular solution of this equation such that $y(0) = 2$?

\[
2 = e^{e^0 + c} = e^{e^0 + c}
\]

so

\[
C = \ln \ln 2 - 1.
\]

Therefore $y = e^{e^t + \ln \ln 2 - 1}$.

(c) What is the particular solution of this equation such that $y(0) = 1$?

This is an equilibrium, so $y = 1$. 
4. (20 points)  (a) Find the partial fraction decomposition for
\[
\frac{1}{(x - 1)^3(4x + 1)(x^2 + 2)^2}.
\]
You do not need to solve for the values!
\[
\frac{1}{(x - 1)^3(4x + 1)(x^2 + 2)^2} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3} + \frac{D}{4x + 1} + \frac{Ex + F}{x^2 + 2} + \frac{Gx + H}{(x^2 + 2)^2}.
\]

(b) Find and solve the partial fraction decomposition for
\[
\frac{7x}{(x + 2)(2x - 3)}
\]
\[
\frac{7x}{(x + 2)(2x - 3)} = \frac{A}{x + 2} + \frac{B}{2x - 3}
\]
\[
7x = 2Ax - 3A + Bx + 2B
\]
\[
7 = 2A + B, \ 0 = -3A + 2B
\]
Doubling the first equation, 14 = 4A + 2B, and adding this to the second gives 14 = A. Setting this in the first, B = 7 - 2A = 7 - 28 = -21.

(c) Solve
\[
\int \frac{2}{(2x + 1)^2} + \frac{3}{x^2 + 2x + 4} \, dx
\]
Setting \( u = 2x + 1, du = 2 \, dx \) and \( v = \frac{x + 1}{\sqrt{3}}, dv = \frac{1}{\sqrt{3}} \, dx \), we have
\[
\int \frac{2}{(2x + 1)^2} + \frac{3}{x^2 + 2x + 4} \, dx = \int \frac{1}{u^2} \, du + \int \frac{3}{(x + 1)^2 + 3} \, dx
\]
\[
= -\frac{1}{u} + \int \frac{1}{\frac{(x+1)^2}{3} + 3} \, dx
\]
\[
= -\frac{1}{2x + 1} + \int \frac{1}{\frac{(x+1)^2}{3} + 1} \, dx
\]
\[
= -\frac{1}{2x + 1} + \sqrt{3} \int \frac{1}{\sqrt{2} + 1} \, dx
\]
\[
= -\frac{1}{2x + 1} + \sqrt{3} \arctan v + C
\]
\[
= -\frac{1}{2x + 1} + \sqrt{3} \arctan \frac{x + 1}{\sqrt{3}} + C
\]
5. (14 points) Find the following integrals if they exist, otherwise state that they diverge.

(a) \[ \int_{-\infty}^{\infty} \frac{x}{1 + x^4} \, dx \]

(Hint: use the substitution \( u = x^2 \).)

\[ u = x^2, \quad du = 2x \, dx, \quad \text{so} \]

\[ \int_{-\infty}^{\infty} \frac{x}{1 + x^4} \, dx = \int_{0}^{\infty} \frac{x}{1 + x^4} \, dx + \int_{-\infty}^{0} \frac{x}{1 + x^4} \, dx \]

\[ = \lim_{a \to -\infty} \int_{a}^{0} \frac{x}{1 + x^4} \, dx + \lim_{b \to \infty} \int_{0}^{b} \frac{x}{1 + x^4} \, dx \]

\[ = \lim_{a \to -\infty} \frac{1}{2} \int_{a^2}^{0} \frac{1}{1 + u^2} \, du + \lim_{b \to \infty} \frac{1}{2} \int_{0}^{b^2} \frac{1}{1 + u^2} \, du \]

\[ = \lim_{a \to -\infty} \frac{1}{2} \arctan u_{a^2} + \lim_{b \to \infty} \frac{1}{2} \arctan u_{b^2}^{2} \]

\[ = \frac{1}{2} \left[ \lim_{a \to -\infty} \arctan 0 - \arctan a^2 + \lim_{b \to \infty} \arctan b^2 - \arctan 0 \right] \]

\[ = \frac{1}{2} \left[ \lim_{a \to -\infty} -\arctan a^2 + \lim_{b \to \infty} \arctan b^2 \right] \]

\[ = \frac{1}{2} \left[ -\frac{\pi}{2} + \frac{\pi}{2} \right] \]

\[ = 0 \]

Note that it’s very important to handle the limits correctly in this problem. We have to remember that the lower bound of the first integral is \( u = a^2 \)—in particular, it’s always positive—and therefore goes to \( \pi/2 \).

(b) \[ \int_{1}^{\infty} \frac{3}{x^3} \, dx \]

\[ \int_{1}^{\infty} \frac{3}{x^3} \, dx = \lim_{a \to \infty} \int_{1}^{a} \frac{3}{x^3} \, dx = \lim_{a \to \infty} -3 \frac{1}{2a^2} \big|_{1}^{a} = \lim_{a \to \infty} \frac{-3}{2a^2} - \frac{-3}{2} = \frac{3}{2} \]

6. (16 points) The function \( \frac{e^t}{t} \) cannot be integrated in terms of functions you know. \( Ei \) is a new function defined by

\[ Ei(x) = \int_{1}^{x} \frac{e^t}{t} \, dt. \]

Evaluate \( \int_{e}^{x} \frac{1}{\ln t} \, dt \) in terms of the function \( Ei \) (and other functions you know).

\[ u = \ln t, \quad du = \frac{1}{t} \, dt, \quad \text{and therefore} \quad t = e^u \quad \text{and} \quad e^u \, du = dt, \] so

\[ \int_{e}^{x} \frac{1}{\ln t} \, dt = \int_{1}^{\ln x} \frac{e^u}{u} \, du \]

\[ = Ei(\ln x) \]

6 of 6