Question 1. We have four balls, colored Emerald, Teal, Blue, and Red. How many possible orderings of these four balls are there?

There are 4 objects, and we are ordering all 4, so the answer should be \( P(4,4) = 4! = 24 \). We list the possible arrangements, abbreviating each ball by its first letter:

- ETBR
- TEBR
- ETRB
- TERB
- BETR
- BTER
- RETB
- RTEB
- EBTR
- TBER
- ETRB
- RTRB
- BRET
- BRTE
- RBET
- RBE
- EBTR
- TBER
- ERTB
- EBTR
- BERT
- BTRE
- REBT
- RTBE
- EBRT
- TBRE
- ERBT
- RTBE

Question 2. A philistine considers the Emerald and Teal balls to be indistinguishably Green. How many possible orderings does this person recognize?

First approach: Start with the 4! = 24 possible orderings where all the balls are distinct. Group these orderings into collections that all look the same to the philistine. Each of these collections contains exactly 2! = 2 orderings: for example, one collection is ETBR and TEBR, which both look like GGBR to this person. When we group 24 orders into groups of 2 each, there are 24/2 = 12 possible groups, so this person sees 12 orderings.

- GGBR
- GGRB
- BGGR
- RGGB
- BRGG
- RBGG
- GBGR
- GRGB
- BGRG
- RGBG
- GBRG
- GRBG

Second approach: We use the multiplication principle and a two-step process. In the first step, we assign two of the four slots to be green. There are \( \binom{4}{2} = 6 \) ways of doing this:

- GG??
- ??GG
- G??G
- ?G?G
- G??G

As a second step, we take this setup and assign B to one of the blank spots and R to the other. There are \( P(2,2) = 2 \) ways of doing this. For instance, GG?? can be turned into either GGBR or GGRB. This again gives us \( \binom{4}{2} \cdot 2 = 12 \) possible orderings.

Question 3. Now suppose someone is red-green colorblind, and the Emerald, Teal, and Red balls all look indistinguishably Gray. How many possible orderings are there?

First approach: We start with the 24 original orderings, and put them in groups that the colorblind person can’t tell apart. For instance,

- ETBR, TEBR, ERTB, RTBE, REBT, RTBE

are all indistinguishable to this person. There are 3! = 6 ways of permuting the \( E, T, R \) balls, so each of these groups contains 6 of the original orderings. Therefore there are 24/6 = 4 possible orderings:

- GGG
- GGB
- GBR
- BGG

Second approach: We use a two step process. In the first step, we choose 3 of the 4 positions to contain Gray balls. In the final step, we assign the Blue ball to the remaining spot. There are \( \binom{4}{3} = 4 \) ways of doing the first step and only 1 way of doing the second, so there are \( \binom{4}{3} \cdot 1 = 4 \) possible orderings.

Question 4. We now have 10 balls. 3 are Green, 2 are Red, and the remaining 5 are Yellow, Blue, Purple, Mauve, and Chartreuse (and all 5 can be distinguished from each other). How many possible orderings are there?
First approach: If the Green and Red were distinguishable, there would be $10!$ possible orderings. We can then group these into indistinguishable collections. For instance one ordering is

$$G_1G_2YBR_1MR_2PG_3C.$$  

Then all rearrangements of the Green and Red balls are indistinguishable:

- $G_1G_2YBR_1MR_2PG_3C$ 
- $G_2G_1YBR_1MR_2PG_3C$ 
- $G_3G_1YBR_1MR_2PG_3C$ 
- $G_3G_2YBR_1MR_2PG_3C$ 
- $G_1G_2YBR_2MR_1PG_3C$ 
- $G_2G_1YBR_2MR_1PG_3C$ 
- $G_3G_1YBR_2MR_1PG_3C$ 
- $G_3G_2YBR_2MR_1PG_3C$ 
- $G_1G_3YBR_2MR_1PG_3C$ 
- $G_2G_3YBR_2MR_1PG_3C$ 
- $G_3G_3YBR_2MR_1PG_3C$ 

Given one of the $10!$ original colorings, there are $3! = 6$ ways of arranging the Green balls and $2! = 2$ ways of arranging the Red balls, for a total of $12$ possible orderings. So there are $\frac{10!}{3!2!}$ orderings where we can’t distinguish the green balls from each other or the blue balls from each other.

Second approach: A multistep process. In the first step, we choose 3 out of the 10 positions to be green. In the second step, we choose 2 of the remaining 7 positions to be red. In the final step, we pick an arrangement of the 5 remaining balls in the 5 remaining positions. There are

$$\binom{10}{3}\binom{7}{2}5! = \frac{10!}{7!3!5!} \cdot \frac{7!}{5!3!2!} = \frac{10!}{3!2!}$$

possible orderings.

Third approach: A different multistep process. We first place the 5 miscellaneous balls into 5 of the 10 positions in some order ($P(10,5)$ options). In the second step, we choose 3 of the 5 remaining positions to receive green balls ($\binom{5}{3}$ options). There are 2 positions left, and they will both receive red balls ($\binom{2}{2}$ options). There are

$$P(10,5)\binom{5}{3}\binom{2}{2} = \frac{10!}{5!3!} \cdot 1 = \frac{10!}{3!2!}$$

possible orderings.

A question to see if you have the idea:

**Question 5.** A track and field team is attending a competition. There are 8 team members (all distinguishable!) and 14 events. The two team captains will participate in two events each, the other six will only participate in one each. (The team will skip four events.) How many possible arrangements are there?