1. Propositional Logic

1.1. Basic Definitions.

Definition 1.1. The alphabet of propositional logic consists of

- Infinitely many *propositional variables* $p_0, p_1, \ldots$,
- The *logical connectives* $\wedge, \vee, \rightarrow, \perp$, and
- Parentheses (and).

We usually write $p, q, r, \ldots$ for propositional variables. $\perp$ is pronounced "bottom".

Definition 1.2. The formulas of propositional logic are given inductively by:

- Every propositional variable is a formula,
- $\perp$ is a formula,
- If ϕ and ψ are formulas then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$.

We omit parentheses whenever they are not needed for clarity. We write $\neg\phi$ as an abbreviation for $\phi \rightarrow \perp$. We occasionally use $\phi \leftrightarrow \psi$ as an abbreviation for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. To limit the number of inductive cases we need to write, we sometimes use $\circledast$ to mean "any of $\wedge, \vee, \rightarrow$".

The propositional variables together with $\perp$ are collectively called *atomic formulas.*

1.2. The Sequent Calculus.

We want to study proofs of statements in propositional logic. Naturally, in order to do this we will introduce a completely formal definition of a proof. To help distinguish between ordinary mathematical proofs, written in (perhaps slightly stylized) natural language, and our formal notion, we will call the formal objects "deductions".

Following standard usage, we will write $\vdash \phi$ to mean "there is a deduction of ϕ (in some particular formal system)". We will also want to study which statements can be deduced from finitely many or infinitely many assumptions. (Particularly in the case of first-order logic, these assumptions might be particular axioms we are interested in. These might also indicate intermediate stages in a deduction—for instance, to deduce $\phi \rightarrow \psi$, we might wish to assume ϕ and show that we can deduce ψ using this assumption.)

Definition 1.3. A *sequent* is a pair of sets of formulas, written

$$\Gamma \Rightarrow \Sigma,$$

such that Σ is finite.

The intended interpretation is that if all the assumptions Γ are true than some element of Σ must also be true.

Often one or both sets are small, and we list the elements without set braces: $\phi, \psi \Rightarrow \gamma, \delta$ or $\Gamma \Rightarrow \phi$. We will also use juxtaposition to abbreviate union; that is

$$\Gamma\Delta \Rightarrow \Sigma\Upsilon$$

abbreviates $\Gamma \cup \Delta \Rightarrow \Sigma \cup \Upsilon$ and similarly

$$\Gamma, \phi, \psi \Rightarrow \Sigma, \gamma$$

abbreviates $\Gamma \cup \{\phi, \psi\} \Rightarrow \Sigma \cup \{\gamma\}$.

It is quite important to note that Γ and Σ are *sets*: they do not have an order, and they do not distinguish multiple copies of the same element. For instance, if $\phi \in \Gamma$ we may still write $\Gamma, \phi \Rightarrow \Sigma$. When Γ is empty, we simply write $\Rightarrow \Sigma$, or (when it is clear from context that we are discussing a sequent) sometimes just Σ.

Note that Γ is interpreted *conjunctively* while Δ is interpreted *disjunctively*. When $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is finite and $\Sigma = \{\sigma_1, \ldots, \sigma_k\}$, we will write $\bigwedge \Gamma \rightarrow \bigvee \Sigma$ for the formula $(\gamma_1 \wedge \cdots \wedge \gamma_n) \rightarrow (\sigma_1 \vee \cdots \vee \sigma_n)$.

We refer to $\Rightarrow$ as *metalanguage implication* to distinguish it from $\rightarrow$.

Before we introduce deductions, we need one more notion: an *inference rule*, or simply a *rule*. An inference rule represents a single step in a deduction; it says that from the truth its premises we may immediately infer the truth of its conclusion. (More precisely, an inference rule will say that if we have deductions of all its premises, we also have a deduction of its conclusion.) For instance, we expect an inference rule which says that if we know both ϕ and ψ then we also know $\phi \wedge \psi$.

A rule is written like this:

Name $\dfrac{\Gamma_0 \Rightarrow \Delta_0 \qquad \cdots \qquad \Gamma_n \Rightarrow \Delta_n}{\Sigma \Rightarrow \Upsilon}$

This rule indicates that from the sequents $\Gamma_i \Rightarrow \Delta_i$ we may deduce $\Sigma \Rightarrow \Upsilon$.

Definition 1.4. Let R be a set of rules. We define $R \vdash \Sigma \Rightarrow \Upsilon$ inductively by:

- If for every $i \leq n$, $R \vdash \Gamma_i \Rightarrow \Delta_i$, and the rule above belongs to R, then $R \vdash \Sigma \Rightarrow \Upsilon$.

We will omit a particular set of rules R if it is clear from context.

Our most important collection of inference rules for now will be classical propositional logic, which we will call $\mathbf{P}_c$. $\mathbf{P}_c$ will include two axioms (rules with no premises):

Ax $\dfrac{}{\Gamma, \phi \Rightarrow \Sigma, \phi}$ where ϕ is atomic.

L$\perp$ $\dfrac{}{\Gamma, \perp \Rightarrow \Sigma}$

$\mathbf{P}_c$ includes inference rules for each connective, neatly paired:

$$\text{L}\wedge\ \dfrac{\Gamma, \phi_i \Rightarrow \Sigma}{\Gamma, \phi_0 \wedge \phi_1 \Rightarrow \Sigma} \qquad\qquad \text{R}\wedge\ \dfrac{\Gamma \Rightarrow \Sigma, \phi_0 \qquad \Gamma' \Rightarrow \Sigma', \phi_1}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma', \phi_0 \wedge \phi_1}$$

$$\text{L}\vee\ \dfrac{\Gamma, \phi_0 \Rightarrow \Sigma \qquad \Gamma', \phi_1 \Rightarrow \Sigma'}{\Gamma\Gamma', \phi_0 \vee \phi_1 \Rightarrow \Sigma\Sigma'} \qquad \text{R}\vee\ \dfrac{\Gamma \Rightarrow \Sigma, \phi_i}{\Gamma \Rightarrow \Sigma, \phi_0 \vee \phi_1}$$

$$\text{L}\to\ \dfrac{\Gamma \Rightarrow \Sigma, \phi \qquad \Gamma', \psi \Rightarrow \Sigma'}{\Gamma\Gamma', \phi \to \psi \Rightarrow \Sigma\Sigma'} \qquad \text{R}\to\ \dfrac{\Gamma, \phi \Rightarrow \Sigma, \psi}{\Gamma \Rightarrow \Sigma, \phi \to \psi}$$

Finally, the *cut rule* is

$$\textbf{Cut}\ \dfrac{\Gamma \Rightarrow \Sigma, \phi \qquad \Gamma', \phi \Rightarrow \Sigma'}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'}$$

These nine rules collectively are the system $\mathbf{P}_c$. The precise definitions of these rules conceal a number of subtleties. We have allowed some small redundencies to support modifications we will consider later. Each of these rules other than **Cut** has a distinguished formula in the conclusion; we call this the *main formula* of that inference rule.

In order to illustrate with a few examples, take the following lemma for granted:

Lemma 1.5. *$\mathbf{P}_c \vdash \Gamma, \phi \Rightarrow \Sigma, \phi$ for any formula ϕ.*

Example 1.6. $\mathbf{P}_c \vdash (\phi \wedge \psi) \to (\phi \wedge \psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \Rightarrow \phi}}{\phi \wedge \psi \Rightarrow \phi} \qquad \dfrac{\dfrac{}{\psi \Rightarrow \psi}}{\phi \wedge \psi \Rightarrow \psi}}{\phi \wedge \psi \Rightarrow \phi \wedge \psi}}{\Rightarrow (\phi \wedge \psi) \to (\phi \wedge \psi)}$$

Example 1.7. $\mathbf{P}_c \vdash (\phi \to (\phi \to \psi)) \to (\phi \to \psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \Rightarrow \phi} \qquad \dfrac{\dfrac{}{\phi \Rightarrow \phi} \qquad \dfrac{}{\phi, \psi \Rightarrow \psi}}{\phi, \phi \to \psi \Rightarrow \psi}}{\phi, \phi \to (\phi \to \psi) \Rightarrow \psi}}{\phi \to (\phi \to \psi) \Rightarrow (\phi \to \psi)}}{\Rightarrow (\phi \to (\phi \to \psi)) \to (\phi \to \psi)}$$

Example 1.8 (Pierce's Law)**.** $\mathbf{P}_c \vdash ((\phi \to \psi) \to \phi) \to \phi$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \Rightarrow \psi, \phi}}{\Rightarrow \phi \to \psi, \phi} \qquad \dfrac{}{\phi \Rightarrow \phi}}{(\phi \to \psi) \to \phi \Rightarrow \phi}}{\Rightarrow ((\phi \to \psi) \to \phi) \to \phi}$$

Example 1.9 (Excluded Middle)**.** Recall that $\neg\phi$ is an abbreviation for $\phi \to \bot$. $\mathbf{P}_c \vdash \phi \vee \neg\phi$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \Rightarrow \phi, \bot}}{\Rightarrow \phi, \neg\phi}}{\Rightarrow \phi, \phi \vee (\neg\phi)}}{\Rightarrow \phi \vee (\neg\phi)}$$

We now prove the lemma:

Lemma 1.10. *$\mathbf{P}_c \vdash \Gamma, \phi \Rightarrow \Sigma, \phi$ for any formula ϕ.*

Proof. By induction on formulas. If ϕ is a atomic, this is simply an application of the axiom.

If ϕ is $\phi_0 \wedge \phi_1$ then we have

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma, \phi_0 \Rightarrow \phi_0\end{matrix}}{\Gamma, \phi_0 \wedge \phi_1 \Rightarrow \phi_0} \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma, \phi_1 \Rightarrow \phi_1\end{matrix}}{\Gamma, \phi_0 \wedge \phi_1 \Rightarrow \phi_1}}{\Gamma, \phi_0 \wedge \phi_1 \Rightarrow \phi_0 \wedge \phi_1}$$

If ϕ is $\phi_0 \vee \phi_1$ then we have

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma, \phi_0 \Rightarrow \Sigma, \phi_0\end{matrix}}{\Gamma, \phi_0 \Rightarrow \Sigma, \phi_0 \vee \phi_1} \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma, \phi_1 \Rightarrow \Sigma, \phi_1\end{matrix}}{\Gamma, \phi_1 \Rightarrow \Sigma, \phi_0 \vee \phi_1}}{\Gamma, \phi_0 \vee \phi_1 \Rightarrow \Sigma, \phi_0 \vee \phi_1}$$

If ϕ is $\phi_0 \to \phi_1$ then we have

$$\dfrac{\dfrac{\dfrac{\cdots}{\Gamma, \phi_0 \Rightarrow \Sigma, \phi_0} \qquad \dfrac{\cdots}{\Gamma, \phi_1 \Rightarrow \Sigma, \phi_1}}{\Gamma, \phi_0 \to \phi_1, \phi_0 \Rightarrow \Sigma, \phi_1}}{\Gamma, \phi_0 \to \phi_1 \Rightarrow \Sigma, \phi_0 \to \phi_1}$$

□

It is also helpful to have the following, proven by straightforward induction on deductions:

Lemma 1.11 (Weakening)**.** *If $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$ then $\mathbf{P}_c \vdash \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$.*

Lemma 1.12 (Compactness)**.** *If $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$ then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\mathbf{P}_c \vdash \Gamma_0 \Rightarrow \Sigma$.*

Proof. The idea is that since $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$, there is a deduction in the sequent calculus witnessing this, and we may therefore restrict Γ to those formulas which actually get used in the deduction.

Formally, if d is a deduction of $\Gamma \Rightarrow \Sigma$, define Γ_0^d to be the subset of Γ consisting of those formulas which appear as the main formula of one of the inference rules $L\wedge, L\vee, L\to, L\bot, Ax$ anywhere in the deduction. (These are essentially the left-side rules, viewing Ax as both a left-side and a right-side rule.) Since d is finite, there are only finitely many inference rules, each of which has only one main formula, so Γ_0^d is finite.

There are two odd technicalities about this definition that should be noted. First, Γ_0^d is always a subset of Γ; it may be that there are formulas which appear as the main rule of the left-hand side of a sequent in d, but not in Γ, as in

$$\dfrac{\overline{p \Rightarrow p}}{\emptyset \Rightarrow p \to p}$$

But $\emptyset^d_0 = \emptyset$: formulas no longer in Γ are not part of Γ^d_0.

The second technicality is that Γ^d_0 does not perfectly capture the intuition of "formulas actually used in the deduction", since it may be that the "instance" of a formula ϕ appearing in Γ is not the same as the "instance" which is the main formula of a left-side rule, as in

$$\dfrac{\begin{array}{c} d_0 \\ \vdots \\ \Gamma, \phi \Rightarrow \psi \wedge \theta \end{array} \qquad \begin{array}{c} \dfrac{\begin{array}{c} d_1 \\ \vdots \\ \Delta, \phi \Rightarrow \theta \end{array} \qquad \begin{array}{c} \vdots \\ \Delta \Rightarrow \phi \end{array}}{\Delta \Rightarrow \theta} \\ \vdots \\ \Gamma' \Rightarrow \theta \end{array}}{\Gamma\Gamma', \phi \Rightarrow \psi \wedge \theta}$$

where it might be that ϕ does not appear in any left-side rules in the subdeduction d_0, but does in d_1. Our definition would then place ϕ in $(\Gamma\Gamma', \phi)^d_0$ even though this is unnecessary. Fortunately, this situation will cause us no harm.[1]

We prove the following by induction on d:

> Let $\Gamma \Rightarrow \Sigma$ be the conclusion of d. Then there is a deduction of $\Gamma^d_0 \Rightarrow \Sigma$.

We will do one of the base cases, for Ax, and three of the inductive cases, $L\wedge$ and $L \to$. The right side rules are quite simple, and the $L\vee$ and Cut cases are similar to $L \to$.

- Suppose d is simply an instance of Ax,
  $$\overline{\Gamma, p \Rightarrow \Sigma, p}$$
  for p atomic. Then $(\Gamma, p)^d_0$ is simply p, and
  $$\overline{\Gamma, p \Rightarrow \Sigma, p}$$
  is a valid deduction.
- Suppose the final inference rule of d is an application of $L\wedge$ to a deduction d',
  $$\dfrac{\begin{array}{c} d_0 \\ \vdots \\ \Gamma, \phi_i \Rightarrow \Sigma \end{array}}{\Gamma, \phi_0 \wedge \phi_1 \Rightarrow \Sigma}$$
  If $\phi_i \notin (\Gamma, \phi_i)^{d'}_0$ then $(\Gamma, \phi_0 \wedge \phi_1)^d_0 = (\Gamma, \phi_i)^{d'}_0 \cup \{\phi_0 \wedge \phi_1\}$, IH gives a deduction of $(\Gamma, \phi_i)^{d'}_0 \Rightarrow \Sigma$, and weakening gives a deduction of $(\Gamma, \phi_0 \wedge \phi_1)^d_0 \Rightarrow \Sigma$.

[1] Sometimes it really is necessary to exactly track which formulas are getting used. This can get involved to do carefully, but one approach would be to "label" formulas, and then track labels through the proof.

If $\phi_i \in (\Gamma, \phi_i)_0^{d'}$ then let us write $\Gamma' = (\Gamma, \phi_i)_0^{d'} \setminus \{\phi_i\}$. Then $(\Gamma, \phi_i)_0^{d'} = \Gamma', \phi_i$ while $(\Gamma, \phi_0 \wedge \phi_1)_0^d = \Gamma', \phi_0 \wedge \phi_1$. By IH, there is a deduction d^* of $\Gamma', \phi_i \Rightarrow \Sigma$, and therefore

$$\dfrac{\begin{matrix} d^* \\ \vdots \\ \Gamma', \phi_i \Rightarrow \Sigma \end{matrix}}{\Gamma', \phi_0 \wedge \phi_1 \Rightarrow \Sigma}$$

is the desired deduction.

- Suppose the final inference rule of d is an application of $L \rightarrow$ to deductions d_0, d_1,

$$\dfrac{\begin{matrix} d_0 \\ \Gamma \Rightarrow \Sigma, \phi \end{matrix} \qquad \begin{matrix} d_1 \\ \Gamma', \psi \Rightarrow \Sigma' \end{matrix}}{\Gamma\Gamma', \phi \rightarrow \psi \Rightarrow \Sigma\Sigma'}$$

If $\psi \notin (\Gamma', \psi)_0^{d_1}$ then $(\Gamma\Gamma', \phi \rightarrow \psi)_0^d = \Gamma_0^{d_0} \cup (\Gamma', \psi)_0^{d_1} \cup \{\phi \rightarrow \psi\}$, IH gives a deduction of $(\Gamma', \psi)_0^{d_1} \Rightarrow \Sigma'$, and weakening then gives a deduction of $(\Gamma\Gamma', \phi \rightarrow \psi)_0^d \Rightarrow \Sigma\Sigma'$.

If $\psi \in (\Gamma', \psi)_0^{d_1}$ then write Γ^* for $(\Gamma', \psi)_0^{d_1} \setminus \{\psi\}$, so $(\Gamma', \psi)_0^{d_1} = \Gamma^*, \psi$ while $(\Gamma\Gamma', \phi \rightarrow \psi)_0^d = \Gamma_0^{d_0}\Gamma^*, \phi \rightarrow \psi$. IH gives deductions d_0^* of $\Gamma_0^{d_0} \Rightarrow \Sigma, \phi$ and d_1^* of $\Gamma^*, \psi \Rightarrow \Sigma'$, so the claim is demonstrated by the deduction

$$\dfrac{\begin{matrix} d_0^* \\ \Gamma_0^{d_0} \Rightarrow \Sigma, \phi \end{matrix} \qquad \begin{matrix} d_1 \\ \Gamma^*, \psi \Rightarrow \Sigma' \end{matrix}}{\Gamma_0^{d_0}\Gamma^*, \phi \rightarrow \psi \Rightarrow \Sigma\Sigma'}$$

- Suppose the final inference rule o d is an application of Cut to deductions d_0, d_1,

$$\dfrac{\begin{matrix} d_0 \\ \Gamma \Rightarrow \Sigma, \phi \end{matrix} \qquad \begin{matrix} d_1 \\ \Gamma', \phi \Rightarrow \Sigma \end{matrix}}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'}$$

If $\phi \notin (\Gamma, \phi)_0^{d_0}$ then $(\Gamma\Gamma')_0^d \supseteq (\Gamma, \phi)_0^{d_0}$, and since IH gives a deduction of $(\Gamma, \phi)_0^{d_0} \Rightarrow \Sigma$, weakening gives a deduction of $(\Gamma\Gamma')_0^d \Rightarrow \Sigma$.

If $\phi \in (\Gamma, \phi)_0^{d_0}$, write Γ^* for $(\Gamma, \phi)_0^{d_0} \setminus \{\phi\}$, so $(\Gamma, \phi)_0^{d_0} = \Gamma^*, \phi$ while $(\Gamma\Gamma')_0^d = \Gamma^*(\Gamma')_0^{d_1}$. IH gives deduction d_0^* of $\Gamma^*, \phi \Rightarrow \Sigma$ and d_1^* of $(\Gamma')_0^{d_1} \Rightarrow \Sigma', \phi$, so the claim is demonstrated by the deduction

$$\dfrac{\begin{matrix} d_0^* \\ \Gamma^*, \phi \Rightarrow \Sigma \end{matrix} \qquad \begin{matrix} d_1 \\ (\Gamma')_0^{d_1} \Rightarrow \Sigma', \phi \end{matrix}}{\Gamma^*(\Gamma')_0^{d_1} \Rightarrow \Sigma\Sigma'}$$

□

1.3. **Variants on Rules.**

Definition 1.13. A rule C is *admissible* for R if whenever $R + C \vdash \phi$, already $R \vdash \phi$.

For example, we have shown that the weakening rule

$$\frac{\Gamma \Rightarrow \Sigma}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'}$$

is admissible for $\mathcal{P}_c$.

It is easy to see that the following rules are admissible for $\mathcal{P}_c$:

$$\frac{\Gamma \Rightarrow \Sigma, \phi_0, \phi_1}{\Gamma \Rightarrow \Sigma, \phi_0 \vee \phi_1}$$

$$\frac{\Gamma, \phi_0, \phi_1 \Rightarrow \Sigma}{\Gamma, \phi_0 \wedge \phi_1 \Rightarrow \Sigma}$$

We have taken sequents to be sets, meaning we don't pay attention to the order formulas appear in or how many times a formula appears. Some people take sequents to be multisets (which do count the number of times a formula appears) or sequences (which also track the order formulas appear in). One then needs to add *contraction* rules, which combine multiple copies of a formula into one copy, and *exchange* rules, which alter the order of formulas. If we omit or restrict these rules we obtain *substructural* logics. (Substructural logics might also have restricted weakening or no weakening at all, which would require changes to the presentation of other rules.)

The most important substructural logic is Linear Logic, which is appropriate for situations where formulas represent resources. Linear logic does not have full contraction, and many inference rules "use up" their premises. For example, in linear logic, $\wedge$ is replaced by two connectives, $\otimes$, which represents having both resources, and &, which represents having the choice between the two resources. The corresponding sequent calculus rules are

$$\frac{\Gamma, \phi_0, \phi_1 \Rightarrow \Sigma}{\Gamma, \phi_0 \otimes \phi_1 \Rightarrow \Sigma} \qquad \frac{\Gamma \Rightarrow \phi_0, \Sigma \qquad \Gamma' \Rightarrow \phi_1, \Sigma'}{\Gamma\Gamma' \Rightarrow \phi_0 \otimes \phi_1, \Sigma\Sigma'}$$

$$\frac{\Gamma, \phi_i \Rightarrow \Sigma}{\Gamma, \phi_0 \& \phi_1 \Rightarrow \Sigma} \qquad \frac{\Gamma \Rightarrow \phi_0, \Sigma \qquad \Gamma \Rightarrow \phi_1, \Sigma}{\Gamma \Rightarrow \phi_0 \& \phi_1, \Sigma}$$

1.4. **Completeness.** We recall the usual semantics for the classical propositional calculus, in which formulas are assigned the values T and F, corresponding to the intended interpration of formulas as either *true* or *false*, respectively.

Definition 1.14. A *truth assignment* for ϕ is a function ν mapping the propositional variables which appear in ϕ to $\{T, F\}$. Given such a ν, we define $\overline{\nu}$ by:

- $\overline{\nu}(p) = \nu(p)$,
- $\overline{\nu}(\perp) = F$,
- $\overline{\nu}(\phi_0 \wedge \phi_1) = 1$ if $\overline{\nu}(\phi_0) = \overline{\nu}(\phi_1) = 1$ and 0 otherwise,
- $\overline{\nu}(\phi_0 \wedge \phi_1) = 0$ if $\overline{\nu}(\phi_0) = \overline{\nu}(\phi_1) = 0$ and 1 otherwise,
- $\overline{\nu}(\phi_0 \to \phi_1) = 0$ if $\overline{\nu}(\phi_0) = 1$ and $\overline{\nu}(\phi_1) = 0$, and 1 otherwise.

We write $\vDash \phi$ if for every truth assignment ν, $\overline{\nu}(\phi) = T$.

A straightforward induction on deductions gives:

Theorem 1.15 (Soundness)**.** *If* $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$ *with* Γ *finite then* $\vDash \bigwedge \Gamma \rightarrow \bigvee \Sigma$.

Theorem 1.16 (Completeness)**.** *Suppose there is no deduction of* $\Rightarrow \phi$ *in* $\mathbf{P}_c$*. Then there is an assignment of truth values* T *and* F *to the propositional variables of* ϕ *making* ϕ *false.*

In fact, the proof is slightly stronger: it makes no use of the cut rule.

Proof. We prove the corresponding statement about sequents: given a finite sequent $\Gamma \Rightarrow \Sigma$, if there is no proof of this sequent then there is such an assignment of truth values for the formula $\bigwedge \Gamma \rightarrow \bigvee \Sigma$. We proceed by induction on the number of connectives $\wedge, \vee, \rightarrow$ appearing in $\Gamma \Rightarrow \Sigma$.

Suppose there are no connectives in $\Gamma \Rightarrow \Sigma$. If $\perp$ appeared in Γ or any propositional variable appeared in both Σ and Γ then there would be a one-step proof of this sequent, so neither of these can happen. Therefore neither of these occur, and we define a truth assignment ν by making every propositional variable in Γ false and every propositional variable in Σ true.

Suppose $\Sigma = \Sigma', \phi_0 \wedge \phi_1$ where $\phi_0 \wedge \phi_1$ is not in Σ'. If there were proofs of both $\Gamma \Rightarrow \Sigma', \phi_0$ and $\Gamma \Rightarrow \Sigma', \phi_1$ then there would be a proof of $\Gamma \Rightarrow \Sigma$. Since this is not the case, there is some i such that there is not a proof of $\Gamma \Rightarrow \Sigma', \phi_i$, and therefore by IH a truth assignment demonstrating the falsehood of $\bigwedge \Gamma \Rightarrow \bigvee \Sigma' \vee \phi_i$, and therefore the falsehood we desire.

Suppose $\Gamma = \Gamma', \phi_0 \wedge \phi_1$. Then apply IH to $\Gamma, \phi_0, \phi_1 \Rightarrow \Sigma$.

The $\vee$ cases are the same, applied to the opposite sides, and similarly for $\rightarrow$. □

An immediate consequence is that the cut rule is redundent: if $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$ then there is already a proof not including **Cut**—a *cut-free* proof. Let us write $\mathbf{P}_c^{cf}$ for the eight rules of $\mathbf{P}_c$ other than **Cut**. We will have more to say about cut-free proofs later, but for now we note an important property:

Definition 1.17. A proof system has the *subformula property* if every formula in every sequent in any proof of $\Gamma \Rightarrow \Sigma$ is a subformula of some formula in $\Gamma \cup \Sigma$.

It is easy to see by inspection of the proof rules that:

Lemma 1.18. *The cut-free portion of* $\mathcal{P}_c^{cf}$ *has the subformula property.*

1.5. **Intuitionistic Logic.** We will want to study an important fragment of classical logic: intuitionistic logic. Intuitionistic logic was introduced to satisfy the philosophical concerns introduced by Brouwer, but our concerns will be purely practical: on the one hand, intuitionistic logic has important properties that classical logic lacks, and on the other, intuitionistic logic is very close to classical logic—in fact, we will be able to translate results in classical logic into intuitionistic logic in a formal way.

We will also introduce minimal logic, which restricts intuitionistic logic even further by dropping the rule L$\perp$. In other words, minimal logic treats

$\bot$ as just another propositional variable, rather than having the distinctive property of implying all other statements.

Definition 1.19. $\mathbf{P}_i$ is the fragment of $\mathbf{P}_c$ in which we require that the right-hand part of each sequent consist of 1 or 0 formulas.

$\mathbf{P}_m$ is the fragment of $\mathbf{P}_i$ omitting L$\bot$.

$\mathbf{P}_i^{cf}$ and $\mathbf{P}_m^{cf}$ are the rules of $\mathbf{P}_i$ and $\mathbf{P}_m$, respectively, other than **Cut**.

We call these intuitionistic and minimal logic, respectively. We will later show that the cut-rule is admissible over both $\mathbf{P}_i^{cf}$ and $\mathbf{P}_m^{cf}$.

Many, though not all, of the properties we have shown for classical logic still hold for intuitionistic and minimal logic.

Lemma 1.20. (1) *For any ϕ, $\mathbf{P}_m \vdash \Gamma, \phi \Rightarrow \phi$,*
(2) *If $\mathbf{P}_m \vdash \Gamma \Rightarrow \phi$ then $\mathbf{P}_m \vdash \Gamma\Gamma' \Rightarrow \phi$, and similarly for $\mathbf{P}_i$.*

The main property of intuitionistic logic we will find useful is the following:

Theorem 1.21 (Disjunction Property)**.** *If $\mathbf{P}_i^{cf} \vdash\Rightarrow \phi \vee \psi$ then either $\mathbf{P}_i \vdash\Rightarrow \phi$ or $\mathbf{P}_i \vdash\Rightarrow \psi$.*

Proof. Consider the last step of such a proof. The only thing the last step can be is $R\vee$. Since the sequence can only have one element, the previous line is either $\Rightarrow \phi$ or $\Rightarrow \psi$. $\square$

Corollary 1.22. $\mathbf{P}_i^{cf} \not\vdash p \vee \neg p$.

1.6. **Double Negation.** We now show that minimal and intuitionistic logic are not so far, either from each other or from classical logic, by creating various translations between these systems.

Theorem 1.23. *If ϕ does not involve $\bot$ and $\mathbf{P}_i^{cf} \vdash \phi$ then also $\mathbf{P}_m^{cf} \vdash \phi$.*

We say $\mathbf{P}_i^{cf}$ is *conservative* over $\mathbf{P}_m^{cf}$ for formulas involving $\bot$.

Proof. Follows immediately from the subformula property. $\square$

Definition 1.24. If ϕ is a formula, we define a formula ϕ^* recursively by:

- p^* is $p \vee \bot$,
- $\bot^*$ is $\bot$,
- $(\phi \circledast \psi)^*$ is $\phi^* \circledast \psi^*$.

When Γ is a set of formulas, we write $\Gamma^* = \{\gamma^* \mid \gamma \in \Gamma\}$.

Theorem 1.25. (1) $\mathbf{P}_i \vdash \phi \leftrightarrow \phi^*$,
(2) *For any ϕ, $\mathbf{P}_m \vdash \bot \to \phi^*$,*
(3) *If $\mathbf{P}_i \vdash \phi$ then $\mathbf{P}_m \vdash \phi^*$.*

Proof. We will only prove the third part. Naturally, this is done by induction on deductions of sequents $\Gamma \Rightarrow \phi$. We need to be a bit careful, because $\mathbf{P}_i$ allows the right hand side of a sequent to be empty, while $\mathbf{P}_m$ does not ($\mathbf{P}_m$ does not explicitly prohibit it, but an easy induction shows that it cannot occur). So more precisely, what we want is:

> Suppose $\mathbf{P}_i \vdash \Gamma \Rightarrow \Sigma$. If $\Sigma = \{\phi\}$ then $\mathbf{P}_m \vdash \Gamma^* \Rightarrow \phi^*$, and if $\Sigma = \emptyset$ then $\mathbf{P}_m \vdash \Gamma^* \Rightarrow \bot$.

We prove this by induction on the the deduction of $\Gamma \Rightarrow \Sigma$. If this is an application of $\mathbf{Ax}$, so $\Gamma = \Gamma', p$, $\Sigma = p$, we have a deduction of $(\Gamma')^*, p \vee \bot \Rightarrow p \vee \bot$.

If the deduction is a single application of $L\bot$, the right side is either empty, in which case it suffices to note that $\Gamma^*, \bot \Rightarrow \bot$ is an axiom of minimal logic, or we have a deduction of $\Gamma, \bot \Rightarrow \phi$, in which case the previous part shows that there is a deduction of $\Gamma^*, \bot \Rightarrow \phi^*$.

The other cases follow easily from IH. □

Definition 1.26. We define the double negation interpretation of ϕ, ϕ^N, inductively by:

- $\bot^N$ is $\bot$,
- p^N is $\neg\neg p$,
- $(\phi_0 \wedge \phi_1)^N$ is $\phi_0^N \wedge \phi_1^N$,
- $(\phi_0 \vee \phi_1)^N$ is $\neg(\neg\phi_0^N \wedge \neg\phi_1^N)$,
- $(\phi_0 \to \phi_1)^N$ is $\phi_0^N \to \phi_1^N$.

Again $\Gamma^N = \{\gamma^N \mid \gamma \in \Gamma\}$.

It will save us some effort to observe that the following rule is admissible, even in minimal logic:

$$\frac{\Gamma \Rightarrow \phi}{\Gamma, \neg\phi \Rightarrow \bot}$$

This is because it abbreviates the deduction

$$\frac{\Gamma \Rightarrow \phi \qquad \dfrac{}{\Gamma, \bot \Rightarrow \bot}}{\Gamma, \neg\phi \Rightarrow \bot}$$

The key point is that the formula ϕ^N is in a fairly special form—for example, ϕ^N does not contain $\vee$. In particular, while $\neg\neg\phi \to \phi$ is not, in general, provable in $\mathbf{P}_m$, or even $\mathbf{P}_i$, this is provable for formulas in the special form ϕ^N. (It is not hard to see that statements like this are not provable without use of the cut-rule; non-provability in general will follow from the fact that the cut-rule is admissible.)

Lemma 1.27. $\mathbf{P}_m \vdash \neg\neg\phi^N \Rightarrow \phi^N$.

Proof. By induction on ϕ. When $\phi = \bot$, we have

$$\cfrac{\cfrac{\cfrac{}{\bot \Rightarrow \bot}}{\Rightarrow \neg\bot}}{\neg\neg\bot \Rightarrow \bot}$$

When $\phi = p$, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \neg p \Rightarrow \neg p\end{array}}{\neg p, \neg\neg p \Rightarrow \bot}}{\neg p \Rightarrow \neg\neg\neg p}}{\neg\neg\neg\neg p, \neg p \Rightarrow \bot}}{\neg\neg\neg\neg p \Rightarrow \neg\neg p}$$

For $\phi = \phi_0 \wedge \phi_1$, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \phi_0^N \Rightarrow \phi_0^N\end{array}}{\phi_0^N \wedge \phi_1^N \Rightarrow \phi_0^N}}{\neg\phi_0^N, \phi_0^N \wedge \phi_1^N \Rightarrow \bot}}{\neg\phi_0^N \Rightarrow \neg(\phi_0^N \wedge \phi_1^N)}}{\neg\neg(\phi_0^N \wedge \phi_1^N), \neg\phi_0^N \Rightarrow \bot}}{\neg\neg(\phi_0^N \wedge \phi_1^N) \Rightarrow \neg\neg\phi_0^N} \quad \begin{array}{c}\vdots\\ \neg\neg\phi_0^N \Rightarrow \phi_0^N\end{array}}{\neg\neg(\phi_0^N \wedge \phi_1^N) \Rightarrow \phi_0^N} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \phi_1^N \Rightarrow \phi_1^N\end{array}}{\phi_0^N \wedge \phi_1^N \Rightarrow \phi_1^N}}{\neg\phi_1^N, \phi_0^N \wedge \phi_1^N \Rightarrow \bot}}{\neg\phi_1^N \Rightarrow \neg(\phi_0^N \wedge \phi_1^N)}}{\neg\neg(\phi_0^N \wedge \phi_1^N), \neg\phi_1^N \Rightarrow \bot}}{\neg\neg(\phi_0^N \wedge \phi_1^N) \Rightarrow \neg\neg\phi_1^N} \quad \begin{array}{c}\vdots\\ \neg\neg\phi_1^N \Rightarrow \phi_1^N\end{array}}{\neg\neg(\phi_0^N \wedge \phi_1^N) \Rightarrow \phi_1^N}}{\neg\neg(\phi_0^N \wedge \phi_1^N) \Rightarrow \phi_0^N \wedge \phi_1^N}$$

For $\phi = \phi_0 \vee \phi_1$, we have

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \neg\phi_0^N \wedge \neg\phi_1^N, \neg(\neg\phi_0^N \wedge \neg\phi_1^N) \Rightarrow \bot\end{array}}{\neg\phi_0^N \wedge \neg\phi_1^N \Rightarrow \neg\neg(\neg\phi_0^N \wedge \neg\phi_1^N)}}{\neg\neg\neg(\neg\phi_0^N \wedge \neg\phi_1^N), \neg\phi_0^N \wedge \neg\phi_1^N \Rightarrow \bot}}{\neg\neg\neg(\neg\phi_0^N \wedge \neg\phi_1^N) \Rightarrow \neg(\neg\phi_0^N \wedge \neg\phi_1^N)}$$

For $\phi = \phi_0 \to \phi_1$, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \phi_0^N \Rightarrow \phi_0^N\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \phi_1^N \Rightarrow \phi_1^N\end{array}}{\neg\phi_1^N, \phi_1^N \Rightarrow \bot}}{\phi_0^N, \neg\phi_1^N, \phi_0^N \to \phi_1^N \Rightarrow \bot}}{\phi_0^N, \neg\phi_1^N \Rightarrow \neg(\phi_0^N \to \phi_1^N)}}{\neg\neg(\phi_0^N \to \phi_1^N), \phi_0^N, \neg\phi_1^N \Rightarrow \bot}}{\neg\neg(\phi_0^N \to \phi_1^N), \phi_0^N \Rightarrow \neg\neg\phi_1^N} \quad \begin{array}{c}\vdots\\ \neg\neg\phi_1^N \Rightarrow \phi_1^N\end{array}}{\dfrac{\neg\neg(\phi_0^N \to \phi_1^N), \phi_0^N \Rightarrow \phi_1^N}{\neg\neg(\phi_0^N \to \phi_1^N) \Rightarrow \phi_0^N \to \phi_1^N}}$$

□

Lemma 1.28. *If* $\mathbf{P}_m \vdash \Gamma, \neg\phi^N \Rightarrow \bot$ *then* $\mathbf{P}_m \vdash \Gamma \Rightarrow \phi^N$.

Proof.
$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \neg\phi^N \Rightarrow \bot\end{array}}{\Rightarrow \neg\neg\phi^N} \qquad \begin{array}{c}\vdots\\ \neg\neg\phi^N \Rightarrow \phi^N\end{array}}{\Rightarrow \phi^N}$$
□

Theorem 1.29. *Suppose $\mathbf{P}_c \vdash \phi$. Then $\mathbf{P}_m \vdash \phi^N$.*

Proof. We have to chose our inductive hypothesis a bit carefully. We will show by induction on deductions

If $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$ then $\mathbf{P}_m \vdash \Gamma^N, \neg(\Sigma^N) \Rightarrow \bot$.

If our deduction consists only of **Ax**, we have either $\bot$ in Γ^N or $\neg\neg p$ in both Γ^N and Σ^N. In the former case the inference is still an axiom. In the latter case we have

$$\dfrac{\begin{array}{c}\vdots\\ \neg\neg p \Rightarrow \neg\neg p\end{array}}{\neg\neg p, \neg\neg\neg p \Rightarrow \bot}$$

If the deduction is an application of $L\bot$ then $\bot$ is in Γ^N, so the the claim is an application of **Ax**.

If the last inference rule is $L\wedge$, we have $\phi_0 \wedge \phi_1$ in Γ, and therefore $\phi_0^N \wedge \phi_1^N$ in Γ^N, so the claim follows from $L\wedge$ applied to IH.

If the last inference rule is $R\wedge$, we have $\phi_0 \wedge \phi_1$ in Σ, and therefore $\neg(\phi_0^N \wedge \phi_1^N)$ in $\neg\Sigma^N$ and IH gives us deductions of $\Gamma^N, \neg\Sigma^N, \neg\phi_0^N \Rightarrow \bot$ and $\Gamma^N, \neg\Sigma^N, \neg\phi_1^N \Rightarrow \bot$. Then we have, by the previous lemma, $\Gamma^N, \neg\Sigma^N \Rightarrow \phi_0^N$ and $\Gamma^N, \neg\Sigma^N \Rightarrow \phi_1^N$, and the claim follows by $R\wedge$ and $L \rightarrow$.

If the last inference rule is $L\vee$, we have $\phi_0 \vee \phi_1$ in Γ, and therefore $\neg(\neg\phi_0^N \wedge \neg\phi_1^N)$ in Γ^N. By IH, we have $\Gamma^N, \phi_0^N, \neg\Sigma^N \Rightarrow \bot$ and $\Gamma^N, \phi_1^N, \neg\Sigma^N \Rightarrow \bot$. We therefore have $\Gamma^N, \neg\Sigma^N \Rightarrow \neg\phi_0^N \wedge \neg\phi_1^N$, which gives $\Gamma^N, \neg(\neg\phi_0^N \wedge \neg\phi_1^N), \neg\Sigma^N \Rightarrow \bot$ as desired.

If the last inference is $R\vee$ then we have $\neg\neg(\neg\phi_0^N \wedge \neg\phi_1^N)$ in $\neg\Sigma^N$ and, by IH, $\Gamma^N, \neg\Sigma^N, \neg\phi_i^N \Rightarrow \bot$. Then we also have $\Gamma^N, \neg\Sigma^N, \neg\phi_0^N \wedge \neg\phi_1^N \Rightarrow \bot$, and since $\neg\neg(\neg\phi_0^N \wedge \neg\phi_1^N) \Rightarrow \neg\phi_0^N \wedge \neg\phi_1^N$, a cut gives the desired deduction.

If the last inference is $L \rightarrow$ then we have $\phi_0^N \rightarrow \phi_1^N$ in Γ^N, IH gives $\Gamma^N, \neg\Sigma^N, \neg\phi_0^N \Rightarrow \bot$ and $\Gamma^N, \phi_1^N, \neg\Sigma^N \Rightarrow \bot$. But by the previous lemma, the former implies that also $\Gamma^N, \neg\Sigma^N \Rightarrow \phi_0^N$, so an application of $L \rightarrow$ suffices.

If the last inference is $R \rightarrow$ then we have $\neg(\phi_0^N \rightarrow \phi_1^N)$ in $\neg\Sigma^N$ and, by IH, $\Gamma^N, \phi_0^N, \neg\Sigma^N, \neg\phi_1^N \Rightarrow \bot$. This implies $\Gamma^N, \phi_0^N, \neg\Sigma^N \Rightarrow \phi_1^N$, which easily gives us the desired deduction.

If the last inference is **Cut** then we have $\Gamma^N, \phi^N, \neg\Sigma^N \Rightarrow \bot$ and $\Gamma^N, \neg\Sigma^N, \neg\phi^N \Rightarrow \bot$, and therefore $\Gamma^N, \neg\Sigma^N \Rightarrow \phi^N$, so the claim follows by a **Cut**.

This completes the induction. In particular, if $\mathbf{P}_c + Cut \vdash \phi$ then $\mathbf{P}_m + Cut \vdash \neg\phi^N \Rightarrow \bot$, and so by the previous lemma, $\mathbf{P}_m + Cut \vdash \phi^N$. □

We also have

Theorem 1.30. (1) $\mathbf{P}_m \vdash \phi \to \phi^N$,
(2) $\mathbf{P}_c \vdash \phi^N \leftrightarrow \phi$.
(3) $\mathbf{P}_i \vdash \phi^N \leftrightarrow \neg\neg\phi$

The last part of this theorem, together with the theorem above, gives:

Theorem 1.31 (Gilvenko's Theorem)**.** *If* $\mathbf{P}_c \vdash \phi$ *then* $\mathbf{P}_i \vdash \neg\neg\phi$.

1.7. **Cut Elimination.** Finally we come to our main structural theorem about propositional logic:

Theorem 1.32. Cut *is admissible for* $\mathbf{P}_c$, $\mathbf{P}_i$, *and* $\mathbf{P}_m$.

We have already given an indirect proof of this for $\mathbf{P}_c$, but the proof we give now will be *effective*: we will give an explicit method for transforming a deduction in $\mathbf{P}_c$ into a deduction in $\mathbf{P}_c^{cf}$.

To avoid repetition, we will write $\mathbf{P}_\epsilon$ to indicate any of the three systems $\mathbf{P}_c, \mathbf{P}_i, \mathbf{P}_m$.

In the cut-rule
$$\frac{\Gamma \Rightarrow \Sigma, \phi \qquad \Gamma', \phi \Rightarrow \Sigma'}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'}$$
we call ϕ the *cut formula*.

Definition 1.33. We define the *rank* of a formula inductively by:

- $rk(p) = rk(\bot) = 0$,
- $rk(\phi \circledast \psi) = \max\{rk(\phi), rk(\psi)\} + 1$.

We write $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma$ if there is a deduction of $\Gamma \Rightarrow \Sigma$ in $\mathbf{P}_\epsilon$ such that all cut-formulas have rank $< r$.

Clearly $\mathbf{P}_\epsilon \vdash_0 \Gamma \Rightarrow \Sigma$ iff $\mathbf{P}_\epsilon^{cf} \vdash \Gamma \Rightarrow \Sigma$.

Lemma 1.34 (Inversion)**.**
- *If* $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1$ *then for each* i, $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_i$.
- *If* $\mathbf{P}_\epsilon \vdash_r \Gamma, \phi_0 \vee \phi_1 \Rightarrow \Sigma$ *then for each* i, $\mathbf{P}_\epsilon \vdash_r \Gamma, \phi_i \Rightarrow \Sigma$.
- *If* $\mathbf{P}_c \vdash_r \Gamma, \phi_0 \to \phi_1 \Rightarrow \Sigma$ *then* $\mathbf{P}_c \vdash_r \Gamma \Rightarrow \Sigma, \phi_0$ *and* $\mathbf{P}_c \vdash_r \Gamma, \phi_1 \Rightarrow \Sigma$.

Proof. Both parts are proven by induction on deductions. We will only prove the first part, since the other is similar.

We have a deduction of $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1$. If $\phi_0 \wedge \phi_1$ is not the main formula of the last inference rule, the claim immediately follows from IH. For example, suppose the last inference is
$$\frac{\Gamma \Rightarrow \Sigma, \psi_0, \phi_0 \wedge \phi_1}{\Gamma \Rightarrow \Sigma, \psi_0 \vee \psi_1, \phi_0 \wedge \phi_1}$$

Then $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \psi_0, \phi_0 \wedge \phi_1$, and so by IH, $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \psi_0, \phi_i$, and therefore
$$\frac{\Gamma \Rightarrow \Sigma, \psi_0, \phi_i}{\Gamma \Rightarrow \Sigma, \psi_0 \vee \psi_1, \phi_i}$$
demonstrates that $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \psi_0 \vee \psi_1, \phi_i$.

The interesting case is when $\phi_0 \wedge \phi_1$ is the main formula of the last inference. Then we must have

$$\dfrac{\begin{matrix} \vdots & & \vdots \\ \Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1, \phi_0 & & \Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1, \phi_1 \end{matrix}}{\Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1}$$

In particular, $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1, \phi_i$, and therefore by IH, $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_i$. (Of course, if $\epsilon \neq c$, we do not need to worry about the case where $\phi_0 \wedge \phi_1$ appears again in the premises, but we need to worry about this in all three systems in the $\vee$ and $\rightarrow$ cases.) $\square$

Lemma 1.35.
- *Suppose* $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1$ *and* $\mathbf{P}_\epsilon \vdash_r \Gamma', \phi_0 \wedge \phi_1 \Rightarrow \Sigma'$*,* $rk(\phi_0 \wedge \phi_1) \leq r$*, and if* $\epsilon \in \{i, m\}$ *then* $|\Sigma\Sigma'| \leq 1$*. Then* $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$*.*
- *Suppose* $\mathbf{P}_\epsilon \vdash_r \Gamma, \phi_0 \vee \phi_1 \Rightarrow \Sigma$ *and* $\mathbf{P}_\epsilon \vdash_r \Gamma' \Rightarrow \Sigma', \phi_0 \vee \phi_1$*,* $rk(\phi_0 \vee \phi_1) \leq r$*, and if* $\epsilon \in \{i, m\}$ *then* $|\Sigma\Sigma'| \leq 1$*. Then* $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$*.*
- *Suppose* $\mathbf{P}_c \vdash_r \Gamma, \phi_0 \rightarrow \phi_1 \Rightarrow \Sigma$ *and* $\mathbf{P}_c \vdash_r \Gamma' \Rightarrow \Sigma', \phi_0 \rightarrow \phi_1$ *and* $rk(\phi_0 \rightarrow \phi_1) \leq r$*. Then* $\mathbf{P}_c \vdash_r \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$*.*

Proof. Again, we only prove the first case, since the other is similar. We proceed by induction on the deduction of $\Gamma', \phi_0 \wedge \phi_1 \Rightarrow \Sigma'$. If this deduction is an axiom then the same axiom can also derive $\Gamma' \Rightarrow \Sigma'$, and therefore $\Gamma\Gamma' \Rightarrow \Sigma\Sigma'$.

If $\phi_0 \wedge \phi_1$ is not the main formula of the last inference rule of this deduction, the claim follows immediately from IH. If $\phi_0 \wedge \phi_1$ is the main formula of the last inference rule of this deduction, we must have

$$\dfrac{\Gamma', \phi_0 \wedge \phi_1, \phi_i \Rightarrow \Sigma'}{\Gamma', \phi_0 \wedge \phi_1 \Rightarrow \Sigma'}$$

Since $\mathbf{P}_\epsilon \vdash_r \Gamma', \phi_0 \wedge \phi_1, \phi_i \Rightarrow \Sigma'$, IH shows that $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma', \phi_i \Rightarrow \Sigma\Sigma'$. By Inversion, we know that $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_i$, so we may apply a cut:

$$\dfrac{\Gamma\Gamma', \phi_i \Rightarrow \Sigma\Sigma' \qquad \Gamma \Rightarrow \Sigma, \phi_i}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'}$$

Since $rk(\phi_i) < rk(\phi_0 \wedge \phi_1) \leq r$, the cut formula has rank $< r$, so we have shown that $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$. $\square$

The $\rightarrow$ case needs to be handled slightly differently in intuitionistic and minimal logic.

Lemma 1.36. *Suppose* $\mathbf{P}_\epsilon \vdash_r \Gamma, \phi_0 \rightarrow \phi_1 \Rightarrow \Sigma$ *and* $\mathbf{P}_\epsilon \vdash_r \Gamma' \Rightarrow \phi_0 \rightarrow \phi_1$ *and* $rk(\phi_0 \rightarrow \phi_1) \leq r$*. Then* $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma' \Rightarrow \Sigma$*.*

Proof. We handle this case by simultaneous induction on both deductions. (If this seems strange, we may think of this as being by induction on the sum of the sizes of the two deductions.)

If $\phi_0 \rightarrow \phi_1$ is not the main formula of the last inference of either deduction, the claim follows immediately from IH. So we may assume that $\phi_0 \rightarrow \phi_1$ is the last inference of both deductions. We therefore have deductions of

- $\Gamma, \phi_0 \rightarrow \phi_1, \phi_1 \Rightarrow \Sigma$,
- $\Gamma, \phi_0 \rightarrow \phi_1 \Rightarrow \phi_0$,

- $\Gamma', \phi_0 \Rightarrow \phi_1$.

Applying IH gives deductions of

- $\Gamma, \phi_1 \Rightarrow \Sigma$,
- $\Gamma \Rightarrow \phi_0$,
- $\Gamma', \phi_0 \Rightarrow \phi_1$.

We may then obtain a deduction of $\Gamma\Gamma' \Rightarrow \Sigma$ using two cuts, one with cut-formula ϕ_0 and one with cut-formula ϕ_1. Since both these formulas have rank $< rk(\phi_0 \to \phi_1) \leq r$, the resulting deduction demonstrates $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma' \Rightarrow \Sigma$. □

Lemma 1.37. *Suppose ϕ is atomic, $\mathbf{P}_\epsilon \vdash_0 \Gamma \Rightarrow \Sigma, \phi$, $\mathbf{P}_\epsilon \vdash_0 \Gamma', \phi \Rightarrow \Sigma'$, and if $\epsilon \in \{i, m\}$ then $|\Sigma\Sigma'| \leq 1$. Then $\mathbf{P}_\epsilon \vdash_0 \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$.*

Proof. By induction on the deduction of $\Gamma \Rightarrow \Sigma, \phi$. If the deduction is anything other than an axiom, the claim follows by IH. If the deduction is an instance of **Ax** introducing a propositional variable other than ϕ, $\Gamma \Rightarrow \Sigma$ is an instance of the same axiom.

If the deduction is an instance of L$\perp$ then $\perp$ is in Γ, and therefore an instance of L$\perp$ gives $\Gamma\Gamma' \Rightarrow \Sigma\Sigma'$.

If the deduction is an instance of **Ax** introducing ϕ then ϕ is in Γ, and therefore the deduction of $\Gamma', \phi \Rightarrow \Sigma'$ is also a deduction of $\Gamma\Gamma' \Rightarrow \Sigma\Sigma'$. □

Theorem 1.38. *Suppose $\mathbf{P}_\epsilon \vdash_{r+1} \Gamma \Rightarrow \Sigma$. Then $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma$.*

Proof. By induction on deductions. If the last inference is anything other than a cut over a formula of rank r, the claim follows immediately from IH. If the last inference is a cut over a formula of rank r, we have

$$\frac{\Gamma \Rightarrow \Sigma, \phi \qquad \Gamma', \phi \Rightarrow \Sigma'}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'}$$

Note that if $\epsilon \in \{i, m\}$ then $|\Sigma\Sigma'| \leq 1$. (Indeed, $|\Sigma| \leq 1$ and $|\Sigma'| = 0$.)

Therefore $\mathbf{P}_\epsilon \vdash_{r+1} \Gamma \Rightarrow \Sigma, \phi$ and $\mathbf{P}_\epsilon \vdash_{r+1} \Gamma', \phi \Rightarrow \Sigma'$, and by IH, $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi$ and $\mathbf{P}_\epsilon \vdash_r \Gamma', \phi \Rightarrow \Sigma'$.

If ϕ is $\phi_0 \circledast \phi_1$, Lemma 1.35 (or Lemma 1.36 when $\circledast = \to$ and $\epsilon \in \{i, m\}$) shows that $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$. If ϕ is atomic, we obtain the same result by Lemma 1.37. □

If we began with a deduction of size m ending in a single cut, we might have to divide this into two deductions with sizes m_0, m_1 so that $m_0 + m_1 + 1 = m$, and then end up with a deduction of size roughly $m_0 m_1$. In the worst case, this deduction could be slightly smaller than $m^2/4$.

Theorem 1.39. *Suppose $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma$. Then $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma$.*

Proof. By induction on r, applying the previous theorem. □

We have already mentioned the subformula property. We observe an explicit consequence of this, which will be particularly interesting when we move to first-order logic:

Theorem 1.40. *Suppose* $\mathbf{P}_\epsilon \vdash \Gamma \Rightarrow \Sigma$*. Then there is a formula* ψ *such that:*

- $\mathbf{P}_\epsilon \vdash \Gamma \Rightarrow \psi$,
- $\mathbf{P}_\epsilon \vdash \psi \Rightarrow \Sigma$,
- *Every propositional variable appearing in* ψ *appears in both* Γ *and* Σ.

Proof. We know that there is a cut-free deduction of $\Gamma \Rightarrow \Sigma$, and we proceed by induction on this deduction. We need a stronger inductive hypothesis:

> Suppose $\mathbf{P}_\epsilon \vdash \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$ (where if $\epsilon \in \{i, m\}$ then $\Sigma = \emptyset$).
> Then there is a formula ψ such that:
> - $\mathbf{P}_\epsilon \vdash \Gamma \Rightarrow \Sigma, \psi$,
> - $\mathbf{P}_\epsilon \vdash \Gamma', \psi \Rightarrow \Sigma'$,
> - Every propositional variable appearing in ψ appears in both $\Gamma\Sigma$ and $\Gamma'\Sigma'$.

If the deduction is **Ax** (we assume for a propositional variable; the $\bot$ case is identical), there are four possibilities. If $p \in \Gamma$ and $p \in \Sigma'$ then $\psi = p$ suffices. If $p \in \Gamma'$ and $p \in \Sigma$ then $\psi = \neg p$ suffices (note that this case only happens in classical logic). If $p \in \Gamma$ and $p \in \Sigma$ then $\psi = \bot$ (again, this case only happens in classical logic). Finally if $p \in \Gamma'$ and $p \in \Sigma'$ then $\psi = \bot \to \bot$.

If the deduction is L$\bot$, if $\bot \in \Gamma$ then $\psi = \bot$, and if $\bot \in \Gamma'$ then $\psi = \bot \to \bot$.

If the final inference is L$\wedge$, we are deducing $\Gamma\Gamma', \phi_0 \wedge \phi_1 \Rightarrow \Sigma$ from $\Gamma\Gamma', \phi_i \Rightarrow \Sigma$. We apply IH with ϕ_i belonging to Γ if $\phi_0 \wedge \phi_1$ does, and otherwise belonging to Γ', and the ψ given by IH suffices.

If the final inference is R$\wedge$, suppose $\phi_0 \wedge \phi_1 \in \Sigma'$. Then we apply IH to $\Gamma\Gamma' \Rightarrow \Sigma\Sigma', \phi_0$ and $\Gamma \Rightarrow \Sigma\Sigma', \phi_1$, in both cases taking ϕ_i to belong to the Σ' component. This gives ψ_0, ψ_1 and deductions of $\Gamma \Rightarrow \Sigma, \psi_i$ and $\Gamma', \psi_i \Rightarrow \Sigma', \phi_i$. We take $\psi = \psi_0 \wedge \psi_1$, and obtain deductions of $\Gamma \Rightarrow \Sigma, \psi_0 \wedge \psi_1$ and $\Gamma', \psi_0 \wedge \psi_1 \Rightarrow \Sigma', \phi_0 \wedge \phi_1$.

If $\phi_0 \wedge \phi_1 \in \Sigma$ then we apply IH to $\Gamma\Gamma' \Rightarrow \Sigma\Sigma', \phi_0$ and $\Gamma \Rightarrow \Sigma\Sigma', \phi_1$, in both cases taking ϕ_i to belong to the Σ component. We obtain ψ_0, ψ_1 and deductions of $\Gamma \Rightarrow \Sigma, \phi_i, \psi_i$ and $\Gamma', \psi_i \Rightarrow \Sigma'$. We take $\psi = \psi_0 \vee \psi_1$ and obtain deductions of $\Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1, \psi_0 \wedge \psi_1$ and $\Gamma, \psi_0 \vee \psi_1 \Rightarrow \Sigma'$.

The cases for L$\vee$, R$\vee$, R$\to$ are similar. If the final inference is L$\to$, $\phi_0 \to \phi_1$ is in Γ, and $\epsilon = c$, we apply IH to $\Gamma\Gamma' \Rightarrow \Sigma\Sigma', \phi_0$ and $\Gamma\Gamma', \phi_1 \Rightarrow \Sigma\Sigma'$ taking ϕ_0 to belong to the Σ component and ϕ_1 to belong to the Γ component, obtaining deductions of $\Gamma \Rightarrow \Sigma, \phi_0, \psi_0$, $\Gamma', \psi_0 \Rightarrow \Sigma'$, $\Gamma, \phi_1 \Rightarrow \Sigma, \psi_1$, and $\Gamma', \psi_1 \Rightarrow \Sigma'$. We take $\psi = \psi_0 \vee \psi_1$ and obtain deductions of $\Gamma, \phi_0 \to \phi_1 \Rightarrow \Sigma, \psi_0 \vee \psi_1$ and $\Gamma', \psi_0 \vee \psi_1 \Rightarrow \Sigma'$.

If the final inference is L$\to$, $\phi_0 \to \phi_1$ is in Γ, and $\epsilon \in \{i, m\}$ then this argument does not quite work (Σ is required to be empty). But we may apply IH to $\Gamma\Gamma' \Rightarrow \phi_0$ and $\Gamma\Gamma', \phi_1 \Rightarrow \Sigma'$; the key step is that we *swap* the roles of Γ and Γ' in the first of these, and take ϕ_1 to belong to Γ in the second, so IH gives ψ_0, ψ_1, and deductions of $\Gamma' \Rightarrow \psi_0$, $\Gamma, \psi_0 \Rightarrow \phi_0$, $\Gamma, \phi_1 \Rightarrow \psi_1$, and $\Gamma', \psi_1 \Rightarrow \Sigma'$. Then we take $\psi = \psi_0 \to \psi_0$, which gives us deductions of $\Gamma', \psi_0 \to \psi_1 \Rightarrow \Sigma'$ and $\Gamma, \phi_0 \to \phi_1 \Rightarrow \psi_0 \to \psi_1$.

If $\phi_0 \to \phi_1 \in \Gamma'$, we apply IH to $\Gamma\Gamma' \Rightarrow \Sigma\Sigma', \phi_0$ and $\Gamma\Gamma', \phi_1 \Rightarrow \Sigma\Sigma'$ taking ϕ_0 to belong to the Σ' component and ϕ_1 to belong to the Γ' component, obtaining deductions of $\Gamma \Rightarrow \Sigma, \psi_0$, $\Gamma', \psi_0 \Rightarrow \Sigma', \phi_0$, $\Gamma \Rightarrow \Sigma, \psi_1$, and $\Gamma', \phi_1, \psi_1 \Rightarrow \Sigma'$. We take $\psi = \psi_0 \wedge \psi_1$ and obtain deductions of $\Gamma \Rightarrow \Sigma, \psi_0 \wedge \psi_1$ and $\Gamma', \phi_0 \to \phi_1, \psi_0 \wedge \psi_1 \Rightarrow \Sigma'$.

(Note that the $\to$ cases are what force us to use the more general inductive hypothesis: even if we begin with $\Gamma' = \Sigma = \emptyset$, the inductive step at $\to$ inference rules bring in the general situation.) $\square$

The results about cut-free proofs can be sharpened: we can distinguish between positive and negative occurrences of propositional variables in the interpolation theorem, we can improve the disjunction property of intuitionistic logic to include deductions from premises which have restrictions on disjunctions, and so on. In the next section we will need the following strengthened form of the subformula property.

Definition 1.41. If ϕ is a formula we define the *positive* and *negative* subformulas of ϕ, $pos(\phi)$ and $neg(\phi)$, recursively by:

- If ϕ is atomic then $pos(\phi) = \{\phi\}$ and $neg(\phi) = \emptyset$,
- For $\circledast \in \{\wedge, \vee\}$, $pos(\phi \circledast \psi) = pos(\phi) \cup pos(\psi) \cup \{\phi \circledast \psi\}$ and $neg(\phi \circledast \psi) = neg(\phi) \cup neg(\psi)$,
- $pos(\phi \to \psi) = pos(\psi) \cup neg(\phi) \cup \{\phi \to \psi\}$ while $neg(\phi \to \psi) = pos(\phi) \cup neg(\psi)$.

We say ϕ *occurs positively* in a sequent $\Gamma \Rightarrow \Sigma$ if $\phi \in \bigcup_{\gamma \in \Gamma} neg(\gamma) \cup \bigcup_{\sigma \in \Sigma} pos(\sigma)$ and ϕ *occurs negatively* in $\Gamma \Rightarrow \Sigma$ if $\phi \in \bigcup_{\gamma \in \Gamma} pos(\gamma) \cup \bigcup_{\sigma \in \Sigma} neg(\Sigma)$.

Theorem 1.42. *If d is a cut-free deduction of $\Gamma \Rightarrow \Sigma$ then any formula which occurs positively (resp. negatively) in any sequent anywhere in d also occurs positively (resp. negatively) in $\Gamma \Rightarrow \Sigma$.*

Proof. By induction on deductions. For axioms this is trivial. For rules this follows easily from IH; we give one example.

Suppose d is an application of $L \to$ to d_0, d_1:

$$\dfrac{\overset{d_0}{\Gamma \Rightarrow \Sigma, \phi} \qquad \overset{d_1}{\Gamma', \psi \Rightarrow \Sigma'}}{\Gamma\Gamma', \phi \to \psi \Rightarrow \Sigma\Sigma'}$$

Observe that by IH, if θ occurs positively (resp. negatively) in d_0 then θ occurs positively (resp. negatively) in $\Gamma \Rightarrow \Sigma, \phi$, and similarly for d_1, so it suffices to show that any formula which occurs positively (resp. negatively) in $\Gamma \Rightarrow \Sigma, \phi$ or $\Gamma', \psi \Rightarrow \Sigma'$ also occurs positively (resp. negatively) in $\Gamma\Gamma', \phi \to \psi \Rightarrow \Sigma\Sigma'$.

For any formula in $\Gamma\Gamma'\Sigma\Sigma'$ this is immediate. The formula ϕ appears positively in $\Gamma \Rightarrow \Sigma, \phi$, and also in $\Gamma\Gamma', \phi \to \psi \Rightarrow \Sigma\Sigma'$, while ψ appears negatively in $\Gamma', \psi \Rightarrow \Sigma'$ and also $\Gamma\Gamma', \phi \to \psi \Rightarrow \Sigma\Sigma'$. $\square$

1.8. **Optimality of Cut-Elimination.** While cut-elimination is admissible, we can show that it provides a substantial *speed-up* relative to cut-free proofs. More precisely, we will show that there are sequents which have short proofs using cut but only very long cut-free proofs.

Definition 1.43. The *size* of a deduction is the number of inference rules appearing in the deduction.

A crucial observation is that the inversion lemma is actually size preserving: inspection of the proof reveals that we only remove inference rules from the deduction we start with, never add any, so the resulting deduction is no larger than the starting one. The only cases we need are:

Lemma 1.44. (1) *Suppose there is a deduction showing $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1$ and that this deduction has size m. Then for each $i \in \{0,1\}$, there is a deduction showing $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma, \phi_i$ with size $\leq m$.*
(2) *Suppose there is a deduction showing $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma, \phi_0 \vee \phi_1$ and that this deduction has size m. Then there is a deduction showing $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma, \phi_0, \phi_1$ with size $\leq m$.*
(3) *Suppose there is a deduction showing $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma, \bot$ and that this deduction has size m. Then there is a deduction showing $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$ with size $\leq m$.*

Lower case letters will all represent propositional variables. We define the following families of formulas:

- $F_0 = \neg\bot$,
- $F_{n+1} = F_n \wedge (p_{n+1} \vee q_{n+1})$,
- $A_{n+1} = F_n \wedge \neg p_{n+1}$,
- $B_{n+1} = F_n \wedge \neg q_{n+1}$.

We then let Λ_n be the sequent

$$A_1 \wedge B_1, \ldots, A_n \wedge B_n$$

and Γ_n be the sequent

$$\Lambda_n, p_n, q_n.$$

Consider why the sequent Γ_n must be true. F_0 is always true, and so $A_1 \wedge B_1$ is equivalent to $\neg p_1 \wedge \neg q_1$. F_1 is equivalent $p_1 \vee q_1$, so if $A_1 \wedge B_1$ fails, F_1 must be true. When F_1 is true, $A_2 \wedge B_2$ is equivalent to $\neg p_2 \wedge \neg q_2$, and so on.

Lemma 1.45. *S_n has a deduction of size linear in n. (That is, the size of the deduction is at most $cn + d$ for some constants c, d.)*

Proof. (To make the deductions that follow easier to read despite the various named formulas, the formula in a sequent which is the main formula of the previous rule has been placed in bold in some sequents.)

We construct several useful side deductions. First, for every n there is a deduction of $F_n \Rightarrow A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$, and the size of this deduction is linear in n:

$$\dfrac{\dfrac{F_n \Rightarrow F_n \qquad \Rightarrow \neg p_{n+1}, p_{n+1}}{F_n \Rightarrow \mathbf{A_{n+1}}, p_{n+1}} \qquad \dfrac{F_n \Rightarrow F_n \qquad \Rightarrow \neg q_{n+1}, q_{n+1}}{F_n \Rightarrow \mathbf{B_{n+1}}, q_{n+1}}}{F_n \Rightarrow A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}}$$

(Indeed, this is almost constant, except for the deduction of $F_n \Rightarrow F_n$.)

Next, by induction on n we construct a deduction of Λ_n, F_n with size linear in n. For $n = 0$, this is just F_0, which is easily deduced with size 2.

Given a deduction of Λ_n, F_n, we construct a deduction of

$$\Lambda_{n+1}, F_{n+1} = \Lambda_n, A_{n+1} \wedge B_{n+1}, F_{n+1}.$$

It obviously suffices to give a deduction of

$$F_n \Rightarrow A_{n+1} \wedge B_{n+1}, F_{n+1}$$

and then apply a cut.

$$\dfrac{F_n \Rightarrow F_n \qquad \dfrac{\begin{array}{c}\vdots \\ F_n \Rightarrow A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}\end{array}}{F_n \Rightarrow A_{n+1} \wedge B_{n+1}, p_{n+1} \vee q_{n+1}}}{F_n \Rightarrow A_{n+1} \wedge B_{n+1}, \mathbf{F_{n+1}}}$$

From our deduction Λ_n, F_n, we wish to find a deduction of $\Gamma_{n+1} = \Lambda_n, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$. This is given by applying cut to the deduction of Λ_n, F_n and the deduction of $F_n \Rightarrow A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$ given above.

□

We need the following lemma:

Lemma 1.46. *Suppose there is a cut-free deduction of*

$$\Gamma_n \setminus \{A_i \wedge B_i\}, \neg p_i, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$$

for some $i \leq n$. Then there is a cut-free deduction of Γ_{n+1} of the same size.

Proof. Since the deduction is cut-free, only the listed formulas can appear. We define an ad hoc transformation $\cdot^*$ on formulas appearing in this deduction:

- p_i^* is $\bot$,
- $\neg p_i^*$ is $\bot$,
- q_i^* is $\bot$,
- For any other atomic formula or negation of an atomic formula p, p^* is p,
- $(p_i \vee q_i)^*$ is $\bot$,
- For any other disjunction, ϕ^* is ϕ,
- For $k < i$, F_k^* is F_k,

- F_i^* is F_{i-1},
- F_{k+1}^* is $F_k^* \wedge (p_k \vee q_k)$ for $k > i$,
- A_k^* is $F_{k-1}^* \wedge \neg p_k$ for $k \neq i$,
- B_k^* is $F_{k-1}^* \wedge \neg q_k$ for $k \neq i$,
- $(A_k \wedge B_k)^*$ is $A_k^* \wedge B_k^*$.

Let d be a deduction of a sequent $\Delta \Rightarrow \Sigma$ consisting of subformulas of $\Gamma_n \setminus \{A_i \wedge B_i\}, \neg p_i, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$ and with the property that $p_i \vee q_i, p_i$, and q_i do not appear except as subformulas of F_i. (In particular, do not appear themselves.) Let m be the size of d. We show by induction on d that there is a cut-free deduction of $\Delta^* \Rightarrow \Sigma^*$ of size $\leq m$.

Any axiom must be introducing some atomic formula which is unchanged, and so the axiom remains valid.

Suppose d consists of an introduction rule with main formula F_i; then one of the subbranches contains a deduction of $F_i^* = F_{i-1}$, so we simply take the result of IH applied to this subdeduction. (Note that $p_i \vee q_i, p_i$, and q_i may appear above the *other* branch, which we have just discarded, and so do not need to apply IH to it.)

The only other inference rules which can appear are $R \rightarrow, R\wedge, R\vee$ applied to formulas with the property that $(\phi \circledast \psi)^* = \phi^* \circledast \psi^*$, and therefore remain valid, so the claim in those cases followed immediately from IH.

In particular, we began with a deduction of

$$A_1 \wedge B_1, \cdots, A_{i-1} \wedge B_{i-1}, A_{i+1} \wedge B_{i+1}, \cdots, A_{n+1} \wedge B_{n+1}, \neg p_i, p_{n+1}, q_{n+1}.$$

After the transformation, this becomes a deduction of

$$A_1 \wedge B_1, \cdots, A_{i-1} \wedge B_{i-1}, A_{i+1}^* \wedge B_{i+1}^*, \cdots, A_{n+1}^* \wedge B_{n+1}^*, \perp, p_{n+1}, q_{n+1}.$$

Applying inversion to eliminate $\perp$ and renaming the propositional variables p_{k+1} to p_k and q_{k+1} to q_k for $k \geq i$, we obtain a deduction of Γ_n. $\square$

Theorem 1.47. *A cut-free deduction of Γ_n has size $\geq 2^n$ for all $n \geq 1$.*

Proof. By induction on n.

Γ_1 is the sequent $A_1 \wedge B_1, p_1, q_1$. This is not an axiom, and so has size ≥ 2.

Suppose the claim holds for Γ_n. We show that it holds for Γ_{n+1}. The only possible final rule in a cut-free deduction is $R\wedge$. We consider two cases.

In the first case, the final rule introduces $A_{n+1} \wedge B_{n+1}$. We thus have a deduction of $\Lambda_n, A_{n+1}, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$ of size m and a deduction of $\Lambda_n, B_{n+1}, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$ of size m', with $m + m' + 1$ the size of the whole deduction.

We work with the first of these deductions. Applying inversion twice, we have a deduction of

$$\Lambda_n, F_n, p_{n+1}, q_{n+1}$$

of at most size m. Note that p_{n+1}, q_{n+1} do not appear elsewhere in Λ_n, F_n. In particular, since p_{n+1}, q_{n+1} do not appear negatively, there can be no axiom rule for these variables anywhere in the proof. Therefore we have a deduction of

$$\Lambda_n, F_n = \Lambda_n, F_{n-1} \wedge (p_n \vee q_n)$$

of size at most m. Another application of inversion tells us that there must be a deduction of

$$\Lambda_n, p_n, q_n$$

of size at most m. But this is actually Γ_n, and therefore $2^n \leq m$. The same argument applied to the second of these deductions shows that $2^n \leq m'$. Then the size of the deduction of Γ_{n+1} was $m+m'+1 \geq 2^n+2^n+1 = 2^{n+1}+1$.

In the second case, we must introduce $A_i \wedge B_i$ for some $i \leq n$. We thus have a deduction of $\Lambda_n, A_i, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$ of size m, a deduction of $\Lambda_n, B_i, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$ of size m', with $m + m' + 1$ the size of the whole deduction.

Applying inversion to the first of these gives a deduction of

$$\Lambda_n \setminus \{A_i \wedge B_i\}, \neg p_i, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$$

of height at most m. By the previous lemma, it follows that there is a deduction of Γ_n of height at most m. By similar reasoning, there must be a deduction of Γ_n of height at most m', and so again $m+m'+1 \geq 2^{n+1}+1$. $\square$