Below are three question about a helix. They will be graded as if they were multiple choice questions. This means that there will be no partial credit! *Clearly place the result of your final calculation in a box, so that the grader can see which answer you would have picked if this was a mulitple choice test. However, answers without supporting work receive no credit.*

A circular helix is the graph of the vector valued function

$$\mathbf{r}(t) = \cancel{(r\cos t, r\cos t, ct)} \quad (r\cos t, r\sin t, ct)$$

where r, c are constants (r is positive).

1. Derive a general formula for the curvature of the helix in terms of r and c.
2. Derive a formula for the *sine* of the angle ϕ between the binormal vector $\mathbf{B}(t)$ and the vertical vector $\mathbf{k} = (0,0,1)$ in terms of r, c. (Hint: first calculate $\mathbf{B}$, then $\cos\phi$, and then use $\cos^2\phi + \sin^2\phi = 1$.)
3. Derive a formula for the torsion of the helix in terms of r, c. (The general torsion formula is found on page 906 in the book, as part (d) of exercise 51.)

$K(t)$ w.r.t. $r\,c \in \mathbb{R}, r>0$

$$K = \frac{T'}{|T|} \text{ or } \left|\frac{dT}{ds}\right|$$

$$r(t) = (r\cos t, r\sin t, ct) \qquad T = \frac{r'}{|r'|} = \frac{(-r\sin t, r\cos t, c)}{\sqrt{r^2+c^2}}$$

$$r'(t) = (-r\sin t, r\cos t, c)$$

$$\frac{dT}{ds} = \frac{dT}{dt}\cdot\frac{dt}{ds} = T'\cdot\frac{1}{|r'|} = \frac{1}{\sqrt{r^2+c^2}}(-r\cos t, -r\sin t, 0)\cdot\frac{1}{\sqrt{r^2+c^2}}$$

$$= \frac{r}{r^2+c^2}(\cos t, \sin t, 0)$$

$$K = \left|\frac{dT}{ds}\right| = \frac{r}{r^2+c^2}\left|\cos t, \sin t, 0\right| = \frac{r}{r^2+c^2}\sqrt{\cos^2+\sin^2} = \frac{r}{r^2+c^2}$$

$$\boxed{K(r,c) = \frac{r}{r^2+c^2}}$$

find $\angle m$ btw $\mathbb{B}$ and $\hat{k}$. wrt. r, c.

$$\mathbb{B} = \mathbb{T} \times \mathbb{N}$$

$$\mathbb{T} = \frac{(-r\sin t, +r\cos t, c)}{\sqrt{r^2+c^2}}$$

$$\mathbb{N} = \frac{T'}{|T'|} = \frac{(-r\cos t, -r\sin t, 0)}{\sqrt{r^2+c^2}} \Big/ \sqrt{\left(\frac{r^2}{r^2+c^2}\right)}$$

$$\mathbb{N} = (-\cos t, -\sin t, 0)$$

$$\mathbb{B} = (c\sin t, c\cos t, +r\sin^2 + r\cos^2) = \frac{(c\sin t, c\cos t, r)}{\sqrt{c^2+r^2}}$$

$$\frac{\mathbb{B}\cdot\hat{k}}{|\mathbb{B}||\hat{k}|} = \cos\phi \qquad \mathbb{B}\cdot\hat{k} = (0+0+r)$$

$$|\mathbb{B}| = \sqrt{c^2+r^2} \qquad |\hat{k}| = 1$$

$$\therefore \cos\phi = \frac{r}{\sqrt{c^2+r^2}}$$

$$\boxed{\text{Compare / look back @ } K = \frac{r}{r^2+c^2}}$$

$$\sin\phi = \sqrt{1-\cos^2\phi} = \sqrt{1-\frac{r^2}{r^2+c^2}} = \sqrt{\frac{r^2+c^2}{r^2+c^2} - \frac{r^2}{r^2+c^2}} = \sqrt{\frac{c^2}{r^2+c^2}}$$

$$\boxed{\sin\phi = \frac{c}{\sqrt{r^2+c^2}}}$$

$\tau = ?$

$$\tau = \frac{-dB}{ds} \cdot \mathbb{N}$$

$$\mathbb{B} = (c\sin t,\ c\cos t,\ r) / \sqrt{r^2+c^2}$$

$$\mathbb{N} = (-\cos t,\ -\sin t,\ 0)$$

$$\frac{dB}{ds} = \frac{dB}{dt}\,\frac{dt}{ds} = (c\cos t, -c\sin t, 0) \cdot \frac{1}{\sqrt{r^2+c^2}^{\,2}}$$

$$\frac{-dB}{ds} = \frac{(-c\cos t,\ +c\sin t,\ 0)}{\sqrt{r^2+c^2}}$$

$$\tau = \frac{+c\cos^2}{\sqrt{r^2+c^2}^{\,2}} + \frac{+c\sin^2}{\sqrt{r^2+c^2}^{\,2}} + 0 = \frac{+c\,(\cos^2+\sin^2)}{\sqrt{r^2+c^2}^{\,2}} = \boxed{\frac{+c}{r^2+c^2}}$$

$$\tau = \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} \Big/ |r' \times r''|^2$$

$$= \begin{vmatrix} -r\sin t & r\cos t & c \\ -r\cos t & -r\sin t & 0 \\ r\sin t & -r\cos t & 0 \end{vmatrix} \Big/ \left| cr\sin t,\ -cr\cos t,\ r^2\sin^2 + r^2\cos^2 \right|^2$$

$$= \left[(0+0+cr^2\cos^2 t) - (-cr^2\sin^2 t + 0 + 0)\right] \Big/ \sqrt{c^2r^2\sin^2 + c^2r^2\cos^2 + r^4}^{\,2}$$

$$cr^2 / c^2r^2 + r^4 = \boxed{\frac{c}{c^2+r^2}}\ \checkmark\checkmark$$