Favor-trading with Incomplete Information: Designated First Favor Maker

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Abstract

We investigate whether and how individuals who stand to gain from favor-trading can best form cooperative relationships in an environment with private information about each agent's ability and willingness to do favors. For agents with a low discount factor (low types) cooperation is not incentive compatible, for agents with a high discount factor (high types) it is. Both types receive privately observed opportunities to do favors with positive probability each period. We focus on a strategy for separation that involves designating one of the agents to do the first favor. We show high types are always able to separate from low types using this strategy and that separation is implementable as soon as the designated high type agent receives a favor opportunity. We also compare these types of equilibria to symmetric strategy equilibria from a companion paper, Kalla [7], that do not involve designating one of the agents to do the first favor.

Key words: Favor-trading, incomplete information, signaling, separation, cooperation
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1 Introduction

This paper and its companion paper, Kalla [7], study whether and how individuals who stand to gain from trading favors can best form cooperative relationships in an environment with private information about each agent's ability and willingness to do favors. Previous models in the favor-trading literature focused on optimizing favor-trading relationships under complete information. This paper introduces incomplete information about player types. The central question addressed is whether cooperation can be maintained in favor-trading relationships after the introduction of non-cooperative players into the pool of potential trading partners, and if so how can the cooperative players separate themselves most efficiently from the non-cooperative types.

For the purposes of this paper favor-trading is considered to be non-monetary trade in goods, services or opportunities, and favors are assumed to be efficient. The model contains a positive measure of players with a low discount factor (*low types*) who do not find cooperation beneficial, and a positive measure of players with a high discount factor (*high types*) who do. Players receive opportunities to do favors for each other (*favor opportunities*) according to either a mutually exclusive or independent distribution, but these opportunities are private information.

As an example, consider a firm with several parallel divisions that function independently under separate managers. Suppose two new managers have been recruited to head the marketing and finance divisions, respectively. Each manager's job is to maximize productivity within her own division, but occasionally one of the managers receives a new idea or opportunity that would be beneficial for her division but even more beneficial for the neighboring division. Monetary side payments are not allowed, but reciprocation in similar favors can provide a basis for mutual gains if both managers are sufficiently patient. However, the managers do not know each other's discount factor, which in this example could be interpreted as the likelihood of staying with the firm long-term. So how should the managers proceed?

The main result in this paper is that by designating one of the agents to do the first favor, or *designated first favor maker (DFFM)* strategies, high type players are always able to separate themselves from low type players at the time of the first favor opportunity received by the designated agent. Separation can be achieved by the use of an *equality matching (EM)* mechanism. EM simply means that each agent waits for reciprocation for a previous favor before doing the next one. Strategies for separation that do not involve designating one of the agents to go first, or *symmetric separating (SS)* strategies, are analyzed in a companion paper, Kalla [7]. In the present paper, we compare the DFFM and SS equilibria. We show with numerical examples that either one can dominate the other depending on the parameter values.

For purposes of the present analysis the distribution of favors may be set aside because when one of the agents is designated to go first she does not receive any informative signals from the other agent regardless of the favor opportunity distribution. For the mathematics in this paper we use the mutually exclusive distribution, but we maintain that our results would hold even were the analysis repeated with an otherwise equivalent independent favor opportunity distribution.[1]

The rest of the paper is organized as follows: Section 2 describes the model and our equilibrium

[1]Kalla [7] addresses the significance of the favor opportunity distribution in detail. It is shown to play an important role in the effectiveness of SS strategies.

concepts including how key concepts from Abdulkadiroğlu and Bagwell (*AB* for short) [1] translate to our streamlined version of their model and how other favor-trading literature relates to this paper. In section 3 we analyze the case of a patient agent facing an unknown type. We show that for this case separation into an EM game is always possible as soon as the unknown type receives a favor opportunity (*immediate separation*), we construct bounds on the discount factors that show when immediate separation into more efficient equilibria, in particular HSSGL equilibria, is possible, and we prove that even when these bounds are violated, high types may still reach a HSSGL if they separate and play an EM game for a sufficient number of periods first to discourage the low types from mimicking them. In section 4 we analyze the case of two unknown type agents and how they can form a cooperative favor-trading relationship using a DFFM strategy. Section 5 compares DFFM from the previous section to SS equilibria from Kalla [7]. Section 6 concludes. An appendix follows and references are at the end.

2 The Model

Consider the earlier motivating example: A firm has several parallel divisions that function independently under separate managers. Suppose two new managers have been recruited to head the marketing and finance divisions, respectively. Each manager's job is to maximize productivity within her own division, but occasionally one of the managers receives an opportunity to help the other division at a cost to her own. The ability or opportunity to help is private information, but when possible the cost is known to be less than the benefit. Monetary side payments are not allowed, but reciprocation in similar favors can provide a basis for mutual gains if both managers are sufficiently patient. But the managers do not know how patient the other is, or how likely she is to stay with the firm long-term. To address whether and how they can form a cooperative relationship we analyze the following formal model.

Two agents, a and b, are randomly picked from a population with $\mu_o \in (0, 1)$ of high types with discount factor δ^H and $1 - \mu_o$ of low types with discount factor δ^L. Each agent has utility function $u(x) = x$. They play an infinitely repeated stage game with the following structure. At the beginning of each period nature allocates an opportunity to do a favor (*favor opportunity*) as follows: Either agent a or b receives a favor opportunity with equal probability, $p \in (0, 1/2)$, or neither does with probability $1 - 2p$. Favor opportunities are private information. An agent who receives a favor opportunity may either keep it private and incur no cost, or do a full or partial favor of size $x, y \in (0, 1]$, for agents a and b, respectively, at a cost equal to the favor size. The benefit to the recipient is ky or kx, for agents a and b, respectively, where $k > 1$. For example, if agent a does a favor of size x, flow payoffs to (a, b) are $(1 - x, kx)$. Favors, including their size, are public information. The stage game is repeated in each subsequent period.

To see how favor-trading works consider the following game called *equality matching (EM)*. In EM of level $z \in (0, 1]$, one agent is called *advantaged*, the other *disadvantaged*. The disadvantaged agent is said to owe the advantaged agent a favor of size z. If the disadvantaged agent does a favor of size z, she becomes advantaged and the other disadvantaged. If she does no favor, she remains disadvantaged. Favors of size other than z are not part of equilibrium play and are deterred by the threat of Nash reversion. When $z = 1$, the game is called *full equality matching*.

For the moment, consider a game of full equality matching between two high types in a complete information environment. Suppose agent a is disadvantaged, b advantaged. Let $(\underline{u}_{em}, \bar{u}_{em})$ denote the average discounted payoffs expected by agents (a, b), or more generally by disadvantaged and advantaged agents, respectively. Let $\sigma_{em}(\underline{u}_{em}, \bar{u}_{em}) = (\sigma^a_{em}(\underline{u}_{em}, \bar{u}_{em}), \sigma^b_{em}(\underline{u}_{em}, \bar{u}_{em}))$ denote the EM strategy profile that implements the payoff pair $(\underline{u}_{em}, \bar{u}_{em})$. Under σ_{em} the payoffs are

$$\underline{u}_{em} = p\delta^H \bar{u}_{em} + (1-p)\delta^H \underline{u}_{em}, \tag{1}$$
$$\bar{u}_{em} = p\left(1 - \delta^H + \delta^H \bar{u}_{em}\right) + p\left((1-\delta^H)k + \delta^H \underline{u}_{em}\right) + (1-2p)\delta^H \bar{u}_{em}. \tag{2}$$

The first equation consists of two events: (i) with probability p agent a receives a favor opportunity, does a full favor $(x = 1)$, and becomes the advantaged agent; that is, agent a is promised continuation payoff $\bar{u}_{em}$, (ii) with probability $(1-p)$ agent a receives no favor opportunity so her flow payoff is zero and her continuation promise remains $\underline{u}_{em}$ along with her disadvantaged status. The equation for payoff $\bar{u}_{em}$ consists of three possible events that occur with probabilities p, p and $(1-2p)$, respectively: (i) agent b receives a favor opportunity, does no favor and thereby receives a flow payoff of 1 and her continuation payoff remains $\bar{u}_{em}$ as she is still advantaged, (ii) agent a receives a favor opportunity, does a full favor $(x = 1)$ so agent b receives a flow payoff of k, but her continuation payoff drops to $\underline{u}_{em}$ because she now owes agent a the next favor, and (iii) neither agent receives a favor opportunity, so agent b's flow payoff is zero and her continuation payoff remains $\bar{u}_{em}$. The two previous equations contain two unknowns, $\underline{u}_{em}$ and $\bar{u}_{em}$, solving for these yields

$$\underline{u}_{em} = \frac{\delta^H p^2 (1+k)}{1-\delta^H(1-2p)}, \tag{3}$$
$$\bar{u}_{em} = \frac{p\left(1-\delta^H(1-p)\right)(1+k)}{1-\delta^H(1-2p)}. \tag{4}$$

For the simple EM strategy profile to be a *Nash equilibrium (NE)* in each stage game, neither agent can have a profitable deviation available to them. It is trivial that the advantaged agent has no profitable deviations as she just waits for reciprocation, but does no favors. Public (observable) off-equilibrium path deviations, such as the advantaged agent doing a favor or one of the agents doing the wrong size favor, are deterred by threat of autarky (no more favors). Therefore, we only need to check that it is not profitable for the disadvantaged agent to not do favors. As usual, it is enough to consider a one-shot deviation. Agent a's discount factor has to be high enough that the incentive compatibility constraint below is satisfied.

$$\begin{aligned} ICC^a_{em}: &\quad \delta^H \bar{u}_{em} \geq 1 - \delta^H + \delta^H \underline{u}_{em} \\ \iff &\quad \bar{u}_{em} - \underline{u}_{em} \geq \left(1-\delta^H\right)/\delta^H. \end{aligned}$$

Using equations (3) and (4), ICC^a_{em} may be written as

$$\begin{aligned} &\frac{p\big(1-\delta^H(1-p)\big)(1+k)}{1-\delta^H(1-2p)} - \frac{\delta^H p^2(1+k)}{1-\delta^H(1-2p)} - \frac{1-\delta^H}{\delta^H} \geq 0 \\ \iff &\frac{1-\delta^H}{\delta^H(1-\delta^H(1-2p))}\left(\delta^H p(k-1) - \big(1-\delta^H\big)\right) \geq 0. \end{aligned}$$

It follows that δ^H must satisfy $\delta^H p(k-1) - \big(1-\delta^H\big) \geq 0$ for ICC^a_{em} to hold, or rearranging terms: $\delta^H \geq \frac{1}{1+p(k-1)} \equiv \delta^*$. We use this boundary discount factor to define high and low type agents.

$$\textbf{Condition (5):} \qquad \delta^H \geq \delta^* := \frac{1}{1+p(k-1)} > \delta^L. \tag{5}$$

It is also easy to verify that $\underline{u}_{em} = p$ for $\delta^H = \delta^*$ so the individual rationality constraints of $\bar{u}_{em}, \underline{u}_{em} \geq p$ are satisfied. Therefore, this EM strategy profile is a Nash equilibrium. In fact, we could use the stronger equilibrium concept of *public perfect equilibrium (PPE)* following Fudenberg, Levine and Maskin [4]. A strategy for agent $i \in \{a, b\}$ is public if it depends only on her current period private information, in this case whether or not she received a favor opportunity, and the public history, which consists of favors up to and including the last period. A PPE is a profile of public strategies that form a Nash equilibrium for each period and the corresponding public history.

Since the payoff pair $(\underline{u}_{em}, \bar{u}_{em})$ is enforceable (implementable), it follows by symmetry that $(\bar{u}_{em}, \underline{u}_{em})$ is also enforceable, and therefore any utility pair on the line connecting $(\underline{u}_{em}, \bar{u}_{em})$ and $(\bar{u}_{em}, \underline{u}_{em})$ is enforceable with the use of a public randomization device. AB [1] aptly call these PPE with current and continuation payoffs restricted onto a symmetric line, *symmetric self-generating line (SSGL)* equilibria. The details of SSGL and the corresponding equilibria will be explained in more detail in the next two subsections. For now it suffices to say that AB solve for the highest such line; the *highest symmetric self-generating line (HSSGL)* and they show that condition (5) is exactly the right bound necessary to implement HSSGL equilibria. In fact, the simple EM mechanism is a HSSGL equilibrium for $\delta^H = \delta^*$. When $\delta^H > \delta^*$, we may use the additional wiggle room to obtain a higher total payoff (and thus a higher SSGL) by requiring the advantaged agent to do further small favors while she waits for reciprocation from the disadvantaged agent.[2]

Observe that for the first-best outcome both agents would have to exhibit full trust in terms of x and y. AB [1] (p. 12) call $x + y$ the *level of trust*. Agent a (b) is said to exhibit *more trust* if $x > y$ $(y > x)$. However, if both agents exhibit full trust every period regardless of history, neither agent has any incentive to do costly favors. Thus the first-best outcome cannot be achieved. However, on the HSSGL line, the level of trust is maximized subject to the restriction that continuation payoffs are picked from the same HSSGL.

It is perhaps natural to wonder if it is incentive compatible for low types to trade smaller favors, that is, to cooperate on a lower SSGL. It is not. Decreasing the size of favors and repeating the analysis for the EM mechanism shows that discount factors above or equal to δ^* are still necessary

[2] AB also constructed other types of equilibria that may lead to higher or lower total payoffs than HSSGL equilibria depending on the parameter values. However, we concentrate on HSSGL equilibria because they always exist if condition (5) is satisfied, and loosely speaking outperform other types of equilibria when p is not very high.

to sustain cooperation. Furthermore, cooperation on a lower line would be less efficient. The discount factor required to support EM equilibria is independent of favor size because agents have linear utility functions. They are effectively risk-neutral with respect to the size of favors. In terms of the mathematics, the lower cost of smaller favors is directly proportional by factor one to the resulting lower continuation payoffs. A formal proof will follow after we first introduce additional notation. For future reference, let $\sigma_{em(z)}$ denote the EM strategy profile when the size of exchanged favors is $z \in (0,1]$. Let $\bar{u}_{em(z)}$ and $\underline{u}_{em(z)}$ denote respective continuation payoffs for advantaged and disadvantaged agent. Unless otherwise noted, we use EM to refer to matching of full favors, or simply full EM.

2.1 Summary of notation and structure

The notation that follows is necessary to formally define the equilibrium profiles we will use in the forthcoming sections, but we present it in a format intended to be useful also for reference. Payoffs are in average discounted values.

Model parameters:	
$i, j \in \{a, b\}$:	Agents, where $j \neq i$.
$\omega^i \in \{L, H\}$:	Agent i's type; L = low, H = high.
$p \in (0,1)$:	Probability agent $i \in \{a, b\}$ receives a favor opportunity. Opportunities are either mutually exclusive or independent.
$k > 1$:	Benefit per unit of favor.
$\delta^i \in (0,1)$:	Discount factor of agent $i \in \{a, b\}$.
$\mu_o \in (0,1)$:	Fraction of high type agents in population.
Beliefs:	
$\mu_t^i \in [0,1]$:	Agent i's belief at time t that agent j is a high type (see definition 1 for details).
Actions:	
$x, y \in [0,1]$:	Size of favor by agents a, b, respectively.
Payoffs:	
(u, v) :	Current payoffs to agents (a, b).
(u_o, v_o) :	Continuation payoffs to (a, b) when no one does a favor.
(u_i, v_i) :	Continuation payoffs to (a, b) when $i \in \{a, b\}$ does a favor.

Table 1: Summary of notation

Information structure: Let $t = 1, 2, \ldots$ denote the time index. Let $w_t^i = 1$ if agent i receives a favor opportunity in period t and 0 otherwise. Agent i privately observes $W_t^i = \{w_z^i\}_{z=1}^t$. Let $\tau_t = (x, y)$ denote favors $(x, y) \in (0,1]^2$ agents a and b, respectively, do in period t. If neither agent does a favor, then let $\tau_t = 0$. Both agents observe $T_t = \{\tau_z\}_{z=1}^t$. Private history of agent i and public history up to and including period t are denoted by $h_t^i = W_t^i \in \mathcal{H}_t^i$ and $H_t = T_t \in \mathcal{H}_t$, respectively. A strategy for agent i, denoted by σ^i, consists of a favor making decision, I_t^i, for each period based on i's type, her private history up to period t, and public history up to period $t-1$. More formally, $I_t^i : \{H, L\} \times \mathcal{H}_t^i \times \mathcal{H}_{t-1} \to [0,1]$ s.t. $I_t^i = 0$ when $w_t^i = 0$.

Definition 1 *$\mu \equiv \mu_t \equiv \left(\mu_t^a, \mu_t^b\right)$ where $\mu_0^i \equiv \mu_o$ and $\mu_t^i : \mathcal{H}_t^i \times \mathcal{H}_{t-1} \to [0,1]$ represents agent i's belief. That is, μ_t^i is the probability assigned by i to the event that the other agent is a high type based on i's private history up to period t and public history up to period $t-1$.*

Sometimes we drop the time index for convenience $(\mu \equiv \mu_t)$. The domain of the belief function consists of agent i's private history up to the current period and the public history up to the last period because it refers to agent i's belief at a point in period t when i has observed her private signal (her favor opportunity is 0 or 1) but not the public signal (period t favor, if any, is still pending). That is, μ_t^i captures agent i's updated belief in period t at the point in time when she has either received a favor opportunity and is deciding whether or not to do a favor, or she has received no opportunity and is waiting to see if the other agent does her a favor.

2.2 Strategies and equilibrium concepts

For our solution concept we use *Perfect Bayesian equilibrium (PBE)*. PBE consist of a strategy profile $\left(\sigma = \left(\sigma^a, \sigma^b\right)\right)$ and a belief system $\left(\mu = \left(\mu^a, \mu^b\right)\right)$ such that σ is sequentially rational with respect to μ and μ is consistent with σ. That is, the strategies are optimal at every stage of the game given the beliefs, and the beliefs are updated according to Bayes' rule from equilibrium strategies and observed actions. We should, strictly speaking, also specify beliefs for off-equilibrium path actions, however, these actions are deterred by the threat of autarky play, which is always an equilibrium response, so it is understood that beliefs consistent with autarky exist and would be straightforward if burdensome to specify. Therefore we generally leave out off-equilibrium path beliefs from our belief functions. But this brings us to the following two definitions.

Definition 2 (Autarky strategy) *Let σ_{aut}^i be such that $I_t^i = 0, \forall t$.*

Definition 3 (Public on-equilibrium path histories) *Let $\mathcal{H}_t^*$ be the set of all public on-equilibrium path histories up to and including period t.*

For example, in a full equality matching game, any history that includes a favor by the advantaged agent or a partial favor by the disadvantaged agent would not be in $\mathcal{H}_t^*$. However, histories that include only private deviations, that is, a disadvantaged agent does not do a favor when she has the opportunity, would still be in $\mathcal{H}_t^*$. Next, let us define EM formally.

Definition 4 (Equality matching (EM)) *An EM strategy at level $z \in (0,1]$ for agent i, denoted by $\sigma_{em(z)}^i$ or simply σ_{em}^i when $z = 1$, is such that*

$$I_t^i = \begin{cases} z & \text{if agent } i \text{ is disadvantaged, } w_t^i = 1 \text{ and } h_{t-1} \in \mathcal{H}_{t-1}^*, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5 (Necessary and sufficient condition for EM) *$\delta^H \geq \delta^* \equiv \frac{1}{1+p(k-1)}$ is necessary and sufficient to implement $\sigma_{em(z)}$, $\forall z \in (0,1]$ in a complete information environment.*

Proof. In appendix. ■

While EM is generally not the most efficient way to trade favors, the first-best outcome is not enforceable, and in AB's model [1], the second-best outcome may be intractable. Presumably that is why AB focused on PPE restricted to symmetric lines rather than to the whole space of feasible and individually rational payoffs. While our primary interest is to implement separation efficiently, rather than to optimize subsequent endgames, we do argue that after separation into an EM game, high types can achieve equilibria of higher value. To this end, we explain AB's HSSGL equilibrium concept as it applies to our version of their model. While we do not repeat their proofs, we do provide a basic explanation of how these results were obtained because our model is sufficiently different that a direct transition of results from AB would not be immediate or even possible. Later we show formally that a pair of high types can move to a HSSGL equilibrium after equality matching for a sufficient number of periods to deter the low types from pooling with the high types.

Definition 6 *A self-generating line (SGL) is a line in the payoff space such that any payoffs (u,v) on the line may be implemented using some actions (x,y) and continuation payoffs $(u_a, u_b, u_o, v_a, v_b, v_o)$ subject to $u_o + v_o = T$ and $u_i + v_i = T$ for $i \in \{a,b\}$. A symmetric self-generating line is a SGL such that $(\underline{u}, \bar{u}) \to (\bar{u}, \underline{u})$, and the highest symmetric self-generating line (HSSGL) is a symmetric SGL such that T is maximized [1] (p. 12).*

The HSSGL equilibria are PPE restricted to symmetric lines. As is normal in the literature, AB [1] use the recursive approach introduced by Abreu, Pearce and Stacchetti (APS) [2]. Let operator B produce the largest self-generating set of PPE values, Ψ^*. Then for any set $\Psi \subset \mathbb{R}^2$, let $B(\Psi) = \{(u,v) \in \Psi : \exists\, (u_i, v_i) \in \Psi,\ i \in \{a,b\};\ (u_o, v_o) \in \Psi;\ x, y \in [0,1]$, s.t. (6)-(10) below are satisfied$\}$ [1] (p. 10-11). In autarky each agent's payoff is p, so the individual rationality constraints are the following:

$$IR: u, u_a, u_b, u_o, v, v_a, v_b, v_o \geq p. \tag{6}$$

The relevant incentive compatibility constraints for agents a and b state that when the agent has an opportunity to do a favor, the flow payoff, which reflects the cost of the favor, and the continuation promise for doing the favor exceed the flow payoff without cost and the continuation payoff when neither agent does a favor:

$$ICC_x^a : \left(1-\delta^H\right)(1-x) + \delta^H u_a \geq \left(1-\delta^H\right) + \delta^H u_o, \tag{7}$$
$$ICC_y^b : \left(1-\delta^H\right)(1-y) + \delta^H v_b \geq \left(1-\delta^H\right) + \delta^H v_o. \tag{8}$$

The current period payoffs u and v must be consistent with the flow payoffs and the continuation promises consisting of three possible outcomes: either agent i receives a favor opportunity and does a favor for agent j, agent j receives a favor opportunity and does a favor for agent i, or neither agent receives a favor opportunity:

$$u = p\left(\left(1-\delta^H\right)(1-x) + \delta^H u_a\right) + p\left(\left(1-\delta^H\right)ky + \delta^H u_b\right) + (1-2p)\delta^H u_o, \qquad (9)$$

$$v = p\left(\left(1-\delta^H\right)(1-y) + \delta^H v_b\right) + p\left(\left(1-\delta^H\right)kx + \delta^H v_a\right) + (1-2p)\delta^H v_o. \qquad (10)$$

By restricting their analysis onto symmetric lines, AB [1] essentially reduce the problem to starting with the full EM profile, σ_{em}, and solving for the highest enforceable favor by the advantaged agent, that is a favor that makes her incentive compatibility problem bind, and all continuation payoffs must be chosen from the same symmetric line. That way the value of the game remains the same regardless of who, if anyone, does a favor. In particular, AB [1] (p. 24-25) provide the equivalent of the following characterization of their HSSGL equilibrium, which implies that the corner payoff pair, $(\underline{u}, \bar{u})$, can be implemented with $(x, y, u_a, u_b, u_o, v_a, v_b, v_o)$, such that

$$x = 1,\ y = \tfrac{\delta^H - \delta^*}{\delta^H + \delta^*}, \qquad (11)$$

$$u_a = \underline{u} + (1+y)\tfrac{1-\delta^H}{\delta^H} = \bar{u},\ u_b = \underline{u},\ u_o = \underline{u} + y\tfrac{1-\delta^H}{\delta^H}, \qquad (12)$$

$$v_a = \underline{u},\ v_b = \bar{u},\ v_o = \bar{u} - y\tfrac{1-\delta^H}{\delta^H}, \qquad (13)$$

$$\underline{u} = p + y(p(k-1)+1) = p + \tfrac{\delta^H - \delta^*}{\delta^H + \delta^*}\tfrac{1}{\delta^*}, \qquad (14)$$

$$\bar{u} = pk - y = \underline{u} + \tfrac{2\left(1-\delta^H\right)}{\delta^H + \delta^*}. \qquad (15)$$

and by the threat of autarky if one of the agents publicly deviates off-equilibrium path. The other corner payoff pair, $(\bar{u}, \underline{u})$, can be implemented symmetrically. It then follows that any payoff pair (u, v) between $(\underline{u}, \bar{u})$ and $(\bar{u}, \underline{u})$ may be implemented with a public randomization device at the start of the game. Call this strategy profile $\sigma_{hssgl}(u, v)$.

AB [1] further calculate a deterministic algorithm to enforce $\sigma_{hssgl}(u, v)$. From the corner solution we know the total trust, $x + y = \frac{2\delta^H}{\delta^H + \delta^*}$, and the total payoff, $T = u + v = 2p + \frac{2\delta^H p(k-1)}{\delta^H + \delta^*}$, $\forall x, y, u, v$ on the HSSGL. We also have the corner point values, $(\underline{u}, \bar{u})$ and $(\bar{u}, \underline{u})$, defined by (14) and (15), that are necessary for use as maximal rewards to implement any HSSGL equilibria. In fact, we know that $u_a = \underline{u} + (x+y)\frac{1-\delta^H}{\delta^H} = \bar{u}, u_b = \underline{u}, v_a = \underline{u}$ and $v_b = \bar{u}$. The incentive compatibility constraints must also bind, so we can use these identities, along with (binding) equations (7)-(10) to calculate x, y, u_o and v_o required to implement any payoff pair (u, v) on the HSSGL defined by $(\underline{u}, \bar{u})$ and $(\bar{u}, \underline{u})$. Our equivalents of AB's equations are stated below without proof:[3]

$$x = \tfrac{\delta^H \delta^*}{1-\delta^H}\left[\tfrac{v - \left(1-\delta^H\right)p}{\delta^H} - \underline{u}\right],\ y = \tfrac{\delta^H \delta^*}{1-\delta^H}\left[\tfrac{u - \left(1-\delta^H\right)p}{\delta^H} - \underline{u}\right], \qquad (16)$$

[3] A strategy profile $\sigma(u, v)$ that begins at time $t \in \mathbb{N}$, denotes a strategy profile σ that implements expected payoffs (u, v) at that time. However, σ will not implement (u, v) every period after t. The payoffs to be implemented each period depend on the history and σ. For example, $\sigma_{hssgl}(\underline{u}, \bar{u})$ at $t = 1$, denotes a HSSGL strategy profile that at the beginning of period 1 is expected to generate payoffs $(\underline{u}, \bar{u})$. If neither agent does a favor in the first period, σ_{hssgl} is expected to yield payoffs $(u_o(\underline{u}, \bar{u}), v_o(\underline{u}, \bar{u}))$ at the beginning of period 2, where u_o and v_o are given by equation (17) for $u = \underline{u}$ and $v = \bar{u}$.

$$u_o = \delta^* \left[\frac{u - \left(1-\delta^H\right)p}{\delta^H} + \frac{\underline{u}(1-\delta^*)}{\delta^*} \right],\ v_o = \delta^* \left[\frac{v - \left(1-\delta^H\right)p}{\delta^H} + \frac{\underline{u}(1-\delta^*)}{\delta^*} \right]. \quad (17)$$

2.3 Related literature

Möbius [12] first investigated the type of two-player favor-trading games we study (2001), albeit with complete information and continuous time. He focused on an intuitive "chips mechanism." That is, each player begins with K chips, and each time an agent does a favor, she receives a chip from the other agent. If one agent accumulates all $2K$ chips, she suspends favors until reciprocation. EM is effectively a chips game with only one chip held by the advantaged agent. Hauser and Hopenhayn [6] continue Möbius' favor-trading research by allowing partial favors (first draft in 2004). Consequently, they let the cost of favors vary based on public history of favor exchanges, notably including time passed since the last exchange. They characterize a set of Pareto optimal PPE, and show numerically that partial favors lead to significant efficiency gains over Möbius' chips mechanism. Their findings display similar characteristics to HSSGL equilibria formulated by AB [1] in discrete time (first draft in 2004). Both Hauser and Hopenhayn and AB use PPE as their solution concept and allow partial favors. Both find equilibria that call for larger favors to be followed by unlimited smaller favors until reciprocation. This is in contrast to Möbius who assumed favors were all the same size, and an agent would suspend favors whenever she was owed $2K$ favors. Both Hauser and Hopenhayn and AB discover what the former call "debt forgiveness." That is, the value of favors owed declines; debt is forgiven, unless "interest" in the form of small favors is "paid" by the advantaged agent.

Therefore it seems that favor-trading in a complete information environment is robust to the model's timing structure (continuous versus discrete) and the arrival process of favors (independent versus mutually exclusive). We show that with incomplete information, immediate separation by high types into the more efficient equilibria characterized by multiple consecutive favors may be precluded by the presence of low types, but can always be achieved over time.

A notable difference between Hauser and Hopenhayn [6] and AB [1] is that AB include opportunities for immediate reciprocity with private information. However, AB show that immediate reciprocity is unnecessary for HSSGL equilibria, which is why our streamlined version of their model does not include it. Furthermore, AB describe favor opportunities as income shocks, and favors as investments. We dropped this terminology because favor-trading precludes side payments, and we felt that using monetary language to discuss the topic confused the issue. We also normalized payoffs to average discounted values for convenience.

More recent favor-trading research includes essays by Nayyar [13] (2009) that provide a discrete time version of the Hauser and Hopenhayn [6] model, and further extend some of their analysis within this setup. A very preliminary paper by Lau [9] (2010) looks at favor-trading when benefits and costs are stochastic.

Outside of the favor-trading literature, Watson [18] studied the sustainability of cooperation using a two-player infinitely-repeated prisoner's dilemma model with incomplete information about agents' types. However, in his model deviations from cooperative behavior are publicly observable, whereas in favor-trading games only cooperative actions are observable, and deviations are private. Still, both models have the broad characteristic that agents start cooperation with small stakes, but form more profitable relationships over time if each agent proves her willingness to cooperate.

3 One-sided incomplete information

For the rest of this section, we suppose there is one agent whose type is known; a is a high type, and one agent whose type is unknown; b is a high type with probability μ_o, and a low type with probability $1-\mu_o$. Subsection 3.1 establishes that under this kind of one-sided incomplete information *immediate separation,* that is, separation at first possible opportunity, to an EM endgame is always possible. Subsection 3.2 extends the analysis to endgames of greater value, in particular to HSSGL endgames. Ultimately we are interested in the case of two-sided incomplete information (neither agent knows the other agent's type), but when one of the agents separates in that scenario, we are back to one-sided incomplete information. The main point of this section is to show that once the agents reach this stage they can always further separate into an EM or HSSGL endgame and how that is achieved.

3.1 Separation to an equality matching (EM) endgame

A low type agent b facing a high type agent a would prefer to be seen as a high type in order to receive favors from a even though she would not reciprocate. The high type wants to separate herself from the low type as soon as possible to exchange favors with agent a. The question is whether a separating equilibrium exists, and if so, how quickly and efficiently can a high type separate?

The answer to the first part is clear looking at the EM strategy profile, $\sigma_{em(z)}\left(\bar{u}_{em(z)}, \underline{u}_{em(z)}\right)$, discussed in section 2. Recall that the necessary and sufficient condition for agent b to benefit from EM is $\delta^b \geq \delta^*$. It follows immediately that a high type may separate as soon as she receives a favor opportunity by doing a favor of size z. Lemma 7 formalizes this intuition.

Lemma 7 (Separation with one-sided incomplete information) *A strategy profile (σ, μ) defined by equations (18)-(21) is a PBE for $z \in (0,1]$.*

$$\sigma^a := \sigma^a_{em(z)}\left(\bar{u}_{em(z)}, \underline{u}_{em(z)}\right), \tag{18}$$

$$\sigma^b := \begin{cases} \sigma^b_{em(z)}\left(\bar{u}_{em(z)}, \underline{u}_{em(z)}\right) & \text{if } \omega^b = H, \\ \sigma^b_{aut} & \text{if } \omega^b = L, \end{cases} \tag{19}$$

$$\mu^b_t := 1, \tag{20}$$

$$\mu^a_t := \begin{cases} 0 & \text{if } H_{t-1} \notin \mathcal{H}^*_{t-1} \text{ (off-equilibrium path move)}, \\ 1 & \text{else if } \exists n < t \text{ s.t. } \tau_n = (0, z), \\ \mu^a_{t-1} & \text{else if } w^a_t = 1, \\ \frac{\mu^a_{t-1}(1-2p)}{1-\left(1+\mu^a_{t-1}\right)p} & \text{otherwise.} \end{cases} \tag{21}$$

Proof. The proof follows immediately from lemma 5 as long as σ is consistent with μ. In this case μ was constructed from σ by Bayesian updating, so consistency follows. Recall that part of the EM strategy is to stop doing favors (switch to autarky) if anyone deviates publicly from the equilibrium path. Thereby such moves are deterred. Consequently off-equilibrium path beliefs are moot (first row of μ_t^a). On the equilibrium path, three possibilities exist: (i) agent b does a favor of size z, in which case a believes b is a high type (second row of μ_t^a). This belief is consistent with σ because only a high type would do a favor per σ. (ii) If agent a receives the favor opportunity, b does not because favor opportunities are mutually exclusive, so a's belief about b does not change (third row of μ_t^a). This is trivially consistent with σ. (iii) Agent a receives neither a favor opportunity, nor a favor from b. Either agent b did not receive a favor opportunity (neither did agent), or she received the opportunity but did not do the favor per σ because she is a low type. In this case, agent a's updated belief per Bayes' rule is

$$\mu_t^a = P\left(\omega^b = H : \{\tau_t = 0 \cap w_t^a = 0\}\right) = \frac{P\left(\omega^b = H \cap \tau_t = 0 \cap w_t^a = 0\right)}{P(\tau_t = 0 \cap w_t^a = 0)}$$
$$= \frac{\mu_{t-1}^a(1-2p)}{\mu_{t-1}^a(1-2p) + \left(1-\mu_{t-1}^a\right)(p+1-2p)} = \frac{\mu_{t-1}^a(1-2p)}{1-\left(1+\mu_{t-1}^a\right)p}.$$

Figure 1 depicts paths to events that agent a may observe and her subjective equilibrium beliefs along these paths. For brevity we use w_t to denote the period t favor opportunity recipient. ■

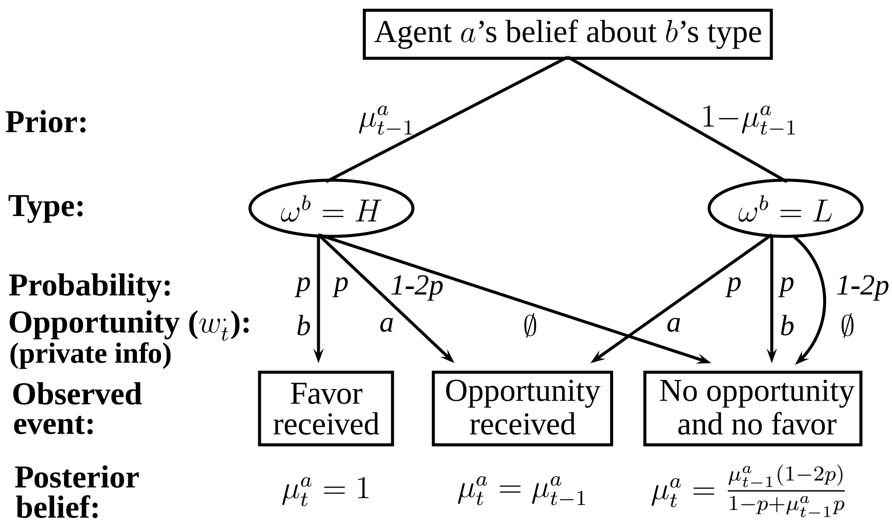

Figure 1: Bayesian updating of beliefs

3.2 Separation to a highest symmetric self-generating line (HSSGL) endgame

Next we investigate whether or not high types can separate immediately into a HSSGL equilibrium profile. If $\delta^H = \delta^*$, full EM and HSSGL equilibria coincide. This was shown in section 2. Therefore, suppose $\delta^H > \delta^*$ for the rest of this subsection. As before we assume agent a is a high type and agent b is an unknown type.

For the moment, suppose agent a's strategy is to wait for a favor of size $z \in (0, 1]$, and if b does such a favor, a will respond with HSSGL play with b as the advantaged player. The point of the exercise is to find out if a HSSGL equilibrium can be directly implemented after separation. To this end, we may set $z = 1$ without compromising the generality of the result. This choice of z minimizes the incentive for low types to mimic high types, yet is incentive compatible for high types per subsection 2.2, in which AB's HSSGL equilibrium results were adapted to our model with complete information. In particular, equations (12)-(15) detail the incentive compatible HSSGL strategy profile to support a corner solution. That is, the continuation payoffs necessary to support a full favor by a disadvantaged high type agent, and a partial favor by the advantaged agent. Recall that each high type agent's incentive compatibility constraint binds in a HSSGL equilibrium, or else we could use the available slack to move to a higher line. Therefore, we cannot lower the continuation promise to agent b to deter low types further unless we deter the high types as well.

In summary, it is enough to calculate if a low type agent b would be willing to do a full favor in return for $\sigma^a_{hssgl}(\underline{u}, \bar{u})$. Equations (16) and (17) specify all actions by agent a, and b follows the autarky strategy after the initial full favor. Therefore we may calculate the expected payoff to agent b and verify that for some parameter values she would have incentive to pool with high types. The lemmas below formalize these claims.

Lemma 8 (HSSGL payoff to advantaged low types) *The expected payoff to a low type agent b under strategy profile $\sigma = \left(\sigma^a_{hssgl}(\underline{u}, \bar{u}), \sigma^b_{aut}\right)$ is*

$$\overline{v}^L_0 = p + pk\frac{\delta^H\left(\delta^H+\delta^*-2\delta^L\delta^*\right)}{(\delta^H+\delta^*)(\delta^H-\delta^L\delta^*(1-p))}. \tag{22}$$

Proof. In appendix. ■

The next proposition defines a strategy for agent a of doing no favors unless b does a full favor first. If the favor is received, a will play according to HSSGL strategy profile that implements the corner payoff pair $(\underline{u}, \bar{u})$ favoring agent b. Given such a strategy, the proposition will prove that condition (5) alone does not guarantee that high types can separate immediately into a HSSGL endgame, and a stronger condition is required.

Proposition 9 (Immediate HSSGL separation 1) *Consider HSSGL payoffs $(\underline{u}, \bar{u})$ defined by (14)-(15) and let t^* denote the first time agent b does a full favor. Then for strategy*

$$\sigma^a := \begin{cases} \sigma^a_{aut} & \text{for } t \le t^* \text{ (no favors until } b \text{ does a full favor)} \\ \sigma^a_{hssgl}(\underline{u}, \bar{u}) & \text{starting at } t = t^*+1 \text{ (HSSGL play if agent } b \text{ does a favor)} \end{cases}$$

$$\text{where } t^* := \inf\{t \in \mathbb{N} : \tau_t = (0,1)\} \text{ and } \inf\{\emptyset\} \equiv \infty,$$

immediate separation is enforceable only if δ^L is low enough or δ^H is high enough. The technical condition they must satisfy is

$$\frac{\delta^H k\ p(1+(k-1)p)\left(1-2\delta^L+\delta^H(1+(k-1)p)\right)}{(1+\delta^H(1+(k-1)p))(\delta^H(1+(k-1)p)-\delta^L(1-p))} \ge \frac{1-\delta^L}{\delta^L}. \tag{23}$$

Proof. Given σ^a, a low type agent b will only do a full favor if the following incentive compatibility constraint is satisfied:

$$ICC_{hssgl}^{L}: \qquad \delta^L \overline{v}_0^L \geq 1 - \delta^L + \delta^L p. \tag{24}$$

Substituting in for $\overline{v}_0^L$ from lemma 8 and simplifying yields condition (23). ■

Lemma 10 (Simple lower bound for $\bar{\delta}^L$) *Let $\bar{\delta}^L$ be defined by (25),*

$$\delta^L \leq \bar{\delta}^L := \frac{2}{\hat{B}+\sqrt{\hat{B}^2-\frac{4\alpha\left(\hat{B}+p(k-1)\right)}{1+\alpha}}} < \delta^* \tag{25}$$
$$\text{where } \hat{B}=1+\alpha(1-p)+pk \text{ and } \alpha\equiv\delta^*/\delta^H.$$

then $\bar{\delta}^L \geq \underline{\delta}^L := 1/\left(1+pk\right).$

Proof. The expression for $\bar{\delta}^L$ is derived in the appendix. The continuation promise for a low type that mimics a high type is in HSSGL is $\overline{v}_0^L = p\left(k+1\right)$ because with probability p she receives a favor of value k from the other agent and with probability p she receives the favor opportunity of value 1. Substituting $\overline{v}_0^L = p\left(k+1\right)$ into ICC_{hssgl}^L (inequality (24)), assuming it binds, and solving for δ^L produces solution $\delta^L = 1/\left(1+pk\right)$. Therefore it suffices to show that $\overline{v}_0^L \leq p\left(k+1\right)$. Rewriting $\overline{v}_0^L$ from lemma 8:

$$\begin{aligned}
\overline{v}_0^L &= p + pk\frac{\delta^H\left(\delta^H+\delta^*\right)-2\delta^H\delta^L\delta^*}{\delta^H\left(\delta^H+\delta^*\right)-\left(1-p\right)\left(\delta^H+\delta^*\right)\delta^L\delta^*} \\
&< p + pk\frac{\delta^H\left(\delta^H+\delta^*\right)-\left(1-p\right)\left(\delta^H+\delta^*\right)\delta^L\delta^*}{\delta^H\left(\delta^H+\delta^*\right)-\left(1-p\right)\left(\delta^H+\delta^*\right)\delta^L\delta^*} \\
&\quad \because 2\delta^H\delta^L\delta^* > (1-p)(\delta^H+\delta^*)\delta^L\delta^* \Longleftarrow 2>(1-p)(1+\delta^*/\delta^H) \\
\Longrightarrow\ \overline{v}_0^L &< p\left(k+1\right).
\end{aligned} \tag{26}$$

$$\Longrightarrow\ \bar{\delta}^L \in \left(\tfrac{1}{1+pk}, \tfrac{1}{1+p(k-1)}\right) \equiv \left(\underline{\delta}^L, \delta^*\right). \ ■$$

The point of presenting lemma 10 was to demonstrate that for k large, $\underline{\delta}^L \rightarrow \delta^*$ and therefore $\bar{\delta}^L \rightarrow \delta^*$. Suppose that $\delta^L \sim \mathcal{U}\left(0,\delta^*\right)$, then the subpopulation of low types with incentive to mimic high types in a game involving separation into HSSGL is bounded from above by $\delta^* - \underline{\delta}^L$ per lemma 10. As a fraction of all low types

$$\begin{aligned}
\frac{\delta^*-\underline{\delta}^L}{\delta^*} &= \frac{\frac{1}{1+p(k-1)}-\frac{1}{1+pk}}{\frac{1}{1+p(k-1)}} = \frac{\frac{p}{(1+pk)(1+p(k-1))}}{\frac{1}{1+p(k-1)}} \\
&= \frac{p}{(1+pk)} \rightarrow 0 \text{ as } k \rightarrow \infty.
\end{aligned}$$

The discount factor bounds, δ^* and $\bar{\delta}^L$, are directly proportional to the expected continuation payoff from $\left(\sigma_{em}^a\left(\underline{u}_{em},\bar{u}_{em}\right),\sigma_{aut}^b\right)$ and $\left(\sigma_{em}^a\left(\underline{u},\bar{u}\right),\sigma_{aut}^b\right)$, respectively. Since the difference between δ^* and $\bar{\delta}^L$ is generally speaking small, it follows that the low type would generally not stand to benefit greatly from HSSGL play over EM play. The reason is that in EM the advantaged low type is guaranteed a favor worth k as soon as agent a receives a favor opportunity, whereas in HSSGL play, low type agent b will only receive a full favor if agent a receives a favor opportunity the very next period after b gains the advantaged status. After that the size of the favor b is owned keeps decreasing unless b keeps doing small favors to remain fully advantaged, and even then her continuation payoff would depreciate each period when neither agent received a favor opportunity.

As for condition (23), we can write $\tilde{\delta}^H = \frac{k}{(1+p(k-1))^2} - \frac{1}{1+p(k-1)}$. It is then easy to see that $\tilde{\delta}^H \to 0$ as $k \to \infty$. The exception to this rule is if $p \to 0$ at a faster rate than $1/\sqrt{k}$. That is, condition (23) is generally violated only if p is significantly smaller than k in relative terms. The reason why δ^H is even a factor in the low types incentive compatibility constraints is because of the way HSSGL equilibria are calculated; namely, the size of partial favors are chosen so that ICC of a high type binds. The higher the high type's discount factor, the greater the favors. We have now established conditions under which low types would not mimic high types. Next we use these results to formally define an equilibrium profile that implements immediate separation if agent b is a high type and $\delta^L \leq \bar{\delta}^L$.

Proposition 11 (Immediate HSSGL separation 2) *Suppose $\delta^L \leq \bar{\delta}^L$ as defined by equations (25). Let $t^* := \inf\{t \in \mathbb{N} : \tau_t = (0,1)\}$ where $\inf\{\emptyset\} \equiv \infty$, and consider strategy profile (σ, μ) defined by equations (27)-(30).*

$$\sigma^a := \begin{cases} \sigma^a_{aut} & \text{for } t \leq t^*, \\ \sigma^a_{hssgl}(\underline{u}, \bar{u}) & \text{for } t = t^* + 1, \end{cases} \tag{27}$$

$$\sigma^b := \begin{cases} \sigma^b_{hssgl}(\bar{u}, \underline{u}) & \text{starting from } t = \inf\left\{s \in \mathbb{N} : w^b_s = 1\right\} \text{ if } \omega^b = H, \\ \sigma^b_{aut} & \text{otherwise,} \end{cases} \tag{28}$$

$$\mu^b_t := 1, \tag{29}$$

$$\mu^a_t := \begin{cases} 0 & \text{if } H_{t-1} \notin \mathcal{H}^*_{t-1} \text{ (off-equilibrium path move),} \\ 1 & \text{if } H_{t-1} \in \mathcal{H}^*_{t-1} \text{ and } \exists n < t \text{ s.t. } \tau_n = (0, z), \\ \mu^a_{t-1} & \text{if } w^a_t = 1, \\ \frac{\mu^a_{t-1}(1-2p)}{1-\left(1+\mu^a_{t-1}\right)p} & \text{otherwise.} \end{cases} \tag{30}$$

Then (σ, μ) is a PBE profile.

Proof. Since $\sigma_{hssgl}(\underline{u}, \bar{u})$ is a PPE profile (restricted to a symmetric line) with complete information, it is also a PBE profile post separation since beliefs become trivial at that point, and $\sigma_{hssgl}(\underline{u}, \bar{u})$ implements a Nash equilibrium at each stage of the game that follows. For $t \geq t^*$, $\mu^a = \mu^b = 1$ on the equilibrium path, so the beliefs are consistent with $\sigma_{hssgl}(\underline{u}, \bar{u})$. And given σ^a, it is incentive compatible for a high type agent b to do a full favor $(y = 1)$ when she first receives a favor opportunity per subsection 2.2 and equations (11)-(15) in particular. The belief system μ is clearly consistent with σ, and given $\delta^L \leq \bar{\delta}^L$, low types would not try to mimic the high types per proposition 9. ■

The next step is to consider what happens if low types have discount factors between $\bar{\delta}^L$ and δ^*. We know immediate separation can always be implemented with an EM endgame per lemma 7, but not with a HSSGL endgame per proposition 9. However, since δ^* is the threshold discount factor required for equality matching to be incentive compatible, and $\delta^L < \delta^*$ by definition, it follows that EM is strictly not incentive compatible for low types. Suppose we promise an agent a payoff T periods from now preceded by full EM. This strategy profile would be incentive compatible for high types for any T in the absence of low types, so we can make T sufficiently large that low types are deterred by the prospect of T periods of EM play prior to the promised higher value payoff.

Proposition 12 (From EM to HSSGL) *Suppose $\delta^L \in (\bar{\delta}^L, \delta^*)$. Let $t^* := \inf\{t \in \mathbb{N} : \tau_t = (0,1)\}$, where $\inf\{\emptyset\} \equiv \infty$, and consider (σ, μ) defined by (31)-(34).*

$$\sigma^a = \begin{cases} \sigma^a_{em}\left(\bar{u}_{em}, \underline{u}_{em}\right) & \text{for } t < t^* + T, \\ \sigma^a_{hssgl}\left(\bar{u}, \underline{u}\right) & \text{starting at } t = t^* + T \text{ if } a \text{ advantaged}, \\ \sigma^a_{hssgl}\left(\underline{u}, \bar{u}\right) & \text{starting at } t = t^* + T \text{ if } b \text{ advantaged}, \end{cases} \tag{31}$$

$$\sigma^b = \begin{cases} \sigma^b_{em}\left(\bar{u}_{em}, \underline{u}_{em}\right) & \text{if } \omega^b = H \text{ starting at } t = \inf\left\{s \in \mathbb{N} : w^b_s = 1\right\} \text{ for } T \text{ periods}, \\ \sigma^b_{hssgl}\left(\bar{u}, \underline{u}\right) & \text{starting at } t = t^* + T \text{ if } a \text{ advantaged and } \omega^b = H, \\ \sigma^b_{hssgl}\left(\underline{u}, \bar{u}\right) & \text{starting at } t = t^* + T \text{ if } b \text{ advantaged and } \omega^b = H, \\ \sigma^b_{aut} & \text{if } \omega^b = L, \end{cases} \tag{32}$$

$$\mu^a_t = \begin{cases} 0 & \text{if } h_{t-1} \notin H^*_{t-1}, \\ 1 & \text{if } t \geq t^* \text{ and } h_{t-1} \in H^*_{t-1}, \\ \mu^a_{t-1} & \text{if } w^a_t = 1, \\ \frac{\mu^a_{t-1}(1-2p)}{1-\left(1+\mu^a_{t-1}\right)p} & \text{otherwise}, \end{cases} \tag{33}$$

$$\mu^b_t = 1. \tag{34}$$

Then (σ, μ) is a PBE for $T \in \mathbb{N}$ large enough.

Proof. In appendix. ■

There are several other methods to deter low types before a move to a HSSGL endgame Some of these methods may be more efficient in some circumstances. For example, if $\delta^H > \delta^*$, there exists some slack in the high type agent b's EM incentive compatibility constraints, that could be used to implement a lower first return favor by agent a. Given a large enough difference between δ^L and δ^H, this tactic might deter the low types after just one round of favor trades, albeit at b's expense. Another method would be to play according to a HSSGL equilibrium profile for some $\hat{\delta} \in \left(\delta^*, \delta^H\right)$. In particular, if the optimal time required to deter a low type were measured in continuous time, then it would land on a $T \in \mathbb{N}$ with probability 0. That is, a small inefficiency is generated by the discrete time structure of our model, which necessitates rounding off to the nearest integer. Such inefficiencies could be removed, at least partly, by playing according to a HSSGL equilibrium profile, in the last period of T, for an appropriately chosen $\hat{\delta} \in \left(\delta^*, \delta^H\right)$. In short, refinements are possible.

4 Two-sided incomplete information: Designated first favor maker (DFFM) equilibria

When incomplete information is two-sided, a high type agent may attempt to separate herself from the low type by doing the first favor. Denote such a favor by z_t at period t. If the initial favor necessary to trigger cooperation between high types is sufficiently large, it will deter low types from mimicking high types, providing a way for high types to separate themselves. After an initial separation, we are back to the one-sided incomplete information case described by the results in the previous section. The question is, can we find a sequence of sufficiently large initial favors to deter the low types from pooling with high types, but still low enough that high types would have incentive to do the favor and risk not receiving reciprocation if the other agent turns out to be a low type, instead of just waiting for the other agent to go first. Recall that in AB's EM and HSSGL equilibria [1], advantaged and disadvantaged designations are determined exogenously, so there is no first mover problem. In this section we take the same approach. We investigate DFFM equilibria in which one agent is designated to do the first favor. Such a strategy takes care of the first mover problem. Subsection 4.1 proves the existence of a DFFM equilibrium. Subsection 4.2 describes DFFM characteristics and solves for the optimal first favor in DFFM equilibria.

4.1 DFFM equilibria: Existence

Consider a strategy of designating an agent at the beginning of the game to do the first favor of pre-agreed size z while the other waits. In separating equilibria z has to be high enough to deter low types from pooling with high types. If the designee does a favor of size z, she earns the advantaged status in an full EM game that follows, provided of course the other agent is a high type. A formal description of this strategy, and a proof are below. For now, suppose without loss of generality that agent a is the designee. At this time we abstract from how the agents determine the designation, but for some later results we will assume agents can randomize, at least with even odds (coin flip), to see who is to do the first favor.

Proposition 13 (Existence of DFFM equilibria) *Let $t^* := \inf\{t \in \mathbb{N} : \tau_t = (z, 0)\}$ and consider strategy profile (σ, μ) be such that*

$$\sigma^a := \begin{cases} \sigma^a_{em(z)}\left(\underline{u}_{em(z)}, \bar{u}_{em(z)}\right) & \textit{if } h_{t-1} \in H^*_{t-1}, \omega^a = H \textit{ and} \\ & \quad t \le \inf\{s \in \mathbb{N} : w^a_s = 1\}, \\ \sigma^a_{em}\left(\bar{u}_{em}, \underline{u}_{em}\right) & \textit{starting at } t = t^* \textit{ if} \\ & \quad h_{t-1} \in H^*_{t-1}, \omega^a = H, \\ \sigma^a_{aut} & \textit{otherwise,} \end{cases} \tag{35}$$

$$\sigma^b := \begin{cases} \sigma^b_{em}\left(\bar{u}_{em}, \underline{u}_{em}\right) & \textit{starting at } t = t^* \textit{ if } h_{t-1} \in H^*_{t-1}, \omega^b = H, \\ \sigma^b_{aut} & \textit{otherwise,} \end{cases} \tag{36}$$

$$\mu_t^a := \begin{cases} 0 & \textit{if } h_{t-1} \notin H_{t-1}^*, \\ 1 & \textit{if } h_{t-1} \in H_{t-1}^* \textit{ and } \exists n < t \textit{ s.t. } \tau_n = (0,1), \\ \mu_o & \textit{if } h_{t-1} \in H_{t-1}^* \textit{ and } \nexists n \leq t \textit{ s.t. } \tau_n = (z,0), \\ \mu_{t-1}^a & \textit{if } h_{t-1} \in H_{t-1}^* \textit{ and } w_t^a = 1, \\ \frac{\mu_{t-1}^a(1-2p)}{1-\left(1+\mu_{t-1}^a\right)p} & \textit{otherwise,} \end{cases} \tag{37}$$

$$\mu_t^b := \begin{cases} 0 & \textit{if } h_{t-1} \notin H_{t-1}^*, \\ 1 & \textit{if } h_{t-1} \in H_{t-1}^* \textit{ and } \exists n < t \textit{ s.t. } \tau_n = (z,0), \\ \mu_{t-1}^b & \textit{if } h_{t-1} \in H_{t-1}^* \textit{ and } w_t^b = 1, \\ \frac{\mu_{t-1}^b(1-2p)}{1-\left(1+\mu_{t-1}^b\right)p} & \textit{otherwise,} \end{cases} \tag{38}$$

$$z \in [\underline{z}, \bar{z}] \equiv \left[\frac{\mu_o p \delta^L k}{1-\delta^L(1-p)},\ \min\left\{1, \frac{\mu_o p \delta^H\left(1-\delta^H\right)k+\delta^H p(k-1)}{(1-\delta^H)(1-\delta^H(1-2p))}\right\}\right]. \tag{39}$$

Then (σ, μ) *is a PBE profile.*

Proof. Off-equilibrium path beliefs do not need to be consistent in a PBE, so they are moot. During publicly observable equilibrium path play, agent b believes a is a high type with probability 1 if she receives the pre-agreed initial favor z. In the meantime she grows more pessimistic according to Bayesian updating when she receives neither a favor from a nor a favor opportunity during the same period. Her beliefs remain unchanged when she receives the favor opportunity because opportunities are mutually exclusive so there is no new information about the other agent. However, even though agent b grows more pessimistic over time if she does not receive a favor, this does not affect the equilibrium outcome because she is not called upon to do any favors until agent a separates first.

Agent a's beliefs on the other hand remain unchanged until she does the first favor unless there is a public off-equilibrium path action by one of the agents. After the first favor, agent a believes b is a high type once the favor is returned according to (full) EM matching. Until such time her beliefs are updated according to Bayesian updating when she receives neither a favor opportunity, nor a favor from b. Agent a's beliefs remain unchanged when she is the one to receive the favor opportunity. These beliefs are clearly sequentially rational on the equilibrium path so it is enough to solve for z such that the incentive compatibility constraints hold for each agent.

The incentive compatibility constraints obviously hold for agent b since she simply waits until the other does a favor and then plays the EM strategy starting as the disadvantaged agent if she is a high type, and the autarky strategy otherwise. A deviation (favor) on her part would clearly no be profitable because it would cost her today and result in autarky play.

For a low type agent a we need to solve for the largest size favor, call it $\underline{z}$, she is willing to do to mimic a high type. Similarly we need to solve for the largest size favor, call it $\bar{z}$, a high type agent a is willing to do to signal her type and begin a full EM game as the advantaged agent provided agent b is also a high type. If not, a receives the autarky payoff instead. Bounds $\underline{z}$ and $\bar{z}$ can be computed from the incentive compatibility constraints for the low and high type of agent a, respectively. To end the proof, we show that $[\underline{z}, \bar{z}] \neq \emptyset$. We show that $\underline{z}$ is increasing in δ^L and $\bar{z}$ in δ^H (as well as all other arguments), so using $\delta^L = \delta^H = \delta^*$ we obtain the maximum value for the lower bound and the minimum value for the upper bound, each of which is μ_o. Hence $[\underline{z}, \bar{z}] \neq \emptyset$, and in particular, $\mu_o \in [\underline{z}, \bar{z}]$ for any values of δ^L, δ^H, p and k. The calculations to prove this are essentially straightforward but tedious and have therefore been relegated to the appendix. ■

4.2 DFFM equilibria: Characteristics

Suppose for the moment that agent a is still, without loss of generality, the designated agent in a DFFM game. If a is a low type, no favors will be exchanged on the equilibrium path, and both agents simply receive their autarky payoff, p. However, if a is a high type facing a low type, her expected payoff would be

$$\begin{aligned} u_D^{HL} &= p\left((1-\delta^H)(1-z)+\delta^H p\right)+(1-p)\delta^H u_D^{HL} \\ &= \tfrac{p\left((1-\delta^H)(1-z)+\delta^H p\right)}{1-\delta^H(1-p)} = p - p\tfrac{1-\delta^H}{1-\delta^H(1-p)}z, \end{aligned} \tag{40}$$

and the corresponding payoff for the low type agent b would be

$$\begin{aligned} v_D^{HL} &= p\left((1-\delta^L)kz+\delta^L p\right)+p\left((1-\delta^L)+\delta^L v_D^{HL}\right)+(1-2p)\delta^L v_D^{HL} \\ &= \tfrac{p\left((1-\delta^L)(1+kz)+\delta^L p\right)}{1-\delta^L(1-p)} = p + pk\tfrac{1-\delta^L}{1-\delta^L(1-p)}z, \end{aligned} \tag{41}$$

and if agent a is facing another high type, then after the initial favor, her continuation payoff will be $\bar{u}_{em}$ instead of p, so her payoff would be

$$\begin{aligned} u_D^{HH} &= p\left((1-\delta^H)(1-z)+\delta^H \bar{u}_{em}\right)+(1-p)\delta^H u_D^{HH} \\ &= \tfrac{p\left((1-\delta^H)(1-z)+\delta^H \bar{u}_{em}\right)}{1-\delta^H(1-p)} = p\tfrac{1-\delta^H+\delta^H\bar{u}_{em}}{1-\delta^H(1-p)} - p\tfrac{1-\delta^H}{1-\delta^H(1-p)}z, \end{aligned}$$

and the corresponding payoff for the high type agent b would be

$$\begin{aligned} v_D^{HH} &= p\left((1-\delta^H)kz+\delta^H \underline{u}_{em}\right)+p\left(1-\delta^H+\delta^H v_D^{HH}\right)+(1-2p)\delta^H v_D^{HH} \\ &= \frac{p\left((1-\delta^H)(1+kz)+\delta^H \underline{u}_{em}\right)}{1-\delta^H(1-p)} = p\frac{1-\delta^H+\delta^H\underline{u}_{em}}{1-\delta^H(1-p)} + p\frac{(1-\delta^H)k}{1-\delta^H(1-p)}z. \end{aligned}$$

The following table summarizes the results with payoffs (u, v) for each type combination.

$\omega^a\omega^b$	$u^{\omega^a\omega^b}$	$v^{\omega^a\omega^b}$
LL	p	p
LH	p	p
HL	$p - p\frac{1-\delta^H}{1-\delta^H(1-p)}z$	$p + p\frac{\left(1-\delta^L\right)k}{1-\delta^L(1-p)}z$
HH	$p\frac{1-\delta^H+\delta^H\bar{u}_{em}}{1-\delta^H(1-p)} - p\frac{1-\delta^H}{1-\delta^H(1-p)}z$	$p\frac{1-\delta^H+\delta^H\underline{u}_{em}}{1-\delta^H(1-p)} + p\frac{(1-\delta^H)k}{1-\delta^H(1-p)}z$

The total value of the game is

$$T_D = (1-\mu_o)\,2p + \mu_o\,(1-\mu_o)\left(u_D^{HL}+v_D^{HL}\right) + \mu_o^2\left(u_D^{HH}+v_D^{H}\right) \tag{42}$$

$$= (1-\mu_o)\,2p + \mu_o(1-\mu_o)\left(2p - p\tfrac{1-\delta^L}{1-\delta^L(1-p)}z + p\tfrac{\left(1-\delta^L\right)k}{1-\delta^L(1-p)}z\right)$$
$$+\,\mu_o^2\left(p\tfrac{1-\delta^H+\delta^H\bar{u}_{em}}{1-\delta^H(1-p)} + p\tfrac{1-\delta^H+\delta^H\underline{u}_{em}}{1-\delta^H(1-p)} + p\tfrac{\left(1-\delta^H\right)(k-1)}{1-\delta^H(1-p)}z\right).$$

Substituting in the values for $\underline{u}_{em}$ and $\bar{u}_{em}$ from (3) and (4) and taking the derivative with respect to z, the initial (separating) favor, yields

$$\frac{\partial T_D}{\partial z} = \mu_o p\left((1\text{-}\mu_o)\left(\tfrac{\left(1-\delta^L\right)k}{1-\delta^L(1-p)} - \tfrac{1-\delta^H}{1-\delta^H(1-p)}\right) + \mu_o\tfrac{\left(1-\delta^H\right)(k-1)}{1-\delta^H(1-p)}\right) > 0$$
$$\Longleftarrow\ \left(\tfrac{\left(1-\delta^L\right)k}{1-\delta^L(1-p)} - \tfrac{1-\delta^H}{1-\delta^H(1-p)}\right) > 0 \text{ and other are terms clearly positive}$$
$$\Longleftarrow\ \tfrac{1-\delta^L}{1-\delta^L(1-p)} > \tfrac{1-\delta^H}{1-\delta^H(1-p)}\ \because\ \delta^L < \delta^H \text{and } k > 1.$$

In words, the total expected payoff in a DFFM equilibrium is increasing in initial favor size (z). This should not be surprising since doing favors is efficient. Of course, increasing z increases the cost faced by the designee, agent a, to elicit a full EM response from agent b, provided that b is a high type, while a's continuation promise remains unchanged at $u = \mu_o\bar{u}_{em} + (1-\mu_o)p$. Naturally, agent b would benefit. But suppose the agents do not know ahead of time whether or not they will be designated to do the first favor, and instead the designation will be determined by a fair coin flip at the beginning of the game. Would the high types want z to be high or low now?

We are interested in this question because each agent who plays this game is either a high type or pretends (at least passively) to be a high type, so any strategies espoused by these agents pre-separation should conform to the high types' interests when those interests are unambiguous. For example, if the agents (claiming to be high types) decide to match pennies to decide who is to do the first favor, and if they choose the size of the initial favor $z \in [\underline{z}, \bar{z}]$, they should both purport to favor the optimal z for high types, or else they would reveal their type.

Proposition 14 (Optimal favor size in DFFM equilibria) *Let (σ, μ) be a DFFM equilibrium profile consistent with proposition 13. Then an initial favor of size*

$$z^i = \begin{cases} \frac{\mu_o\delta^L pk}{1-\delta^L(1-p)} & \text{if } \mu_o k < 1,\\ \text{any point in } [\underline{z}, \bar{z}] & \text{if } \mu_o k = 1,\\ \min\left\{1, \frac{\mu_o\delta^H p\left[\left(1-\delta^H\right)k+\delta^H p(k-1)\right]}{(1-\delta^H)(1-\delta^H(1-2p))}\right\} & \text{if } \mu_o k > 1, \end{cases} \tag{43}$$

maximizes expected payoff to a high type agent i provided that the designee is chosen randomly with even odds.

Proof. The expected payoff a high type agent i, who will be designated to do the first favor (that is, to assume the role of agent a of proposition 13) with probability $1/2$, or to wait for the other agent to do so (role b) otherwise, is

$$u_D^H = \left((1-\mu_o)\,u_D^{HL} + \mu_o u_D^{HH}\right)/2 + \left((1-\mu_o)\,p + \mu_o v_D^{HH}\right)/2$$

$$
\begin{aligned}
&= \frac{1}{2}\left[(1-\mu_o)\left(p - p\tfrac{1-\delta^H}{1-\delta^H(1-p)}z\right) + \mu_o\left(p\tfrac{1-\delta^H+\delta^H\bar{u}_{em}}{1-\delta^H(1-p)} - p\tfrac{1-\delta^H}{1-\delta^H(1-p)}z\right)\right] \\
&+ \frac{1}{2}\left[(1-\mu_o)\,p + \mu_o\left(p\tfrac{1-\delta^H+\delta^H\underline{u}_{em}}{1-\delta^H(1-p)} + p\tfrac{\left(1-\delta^H\right)k}{1-\delta^H(1-p)}z\right)\right] \\
&= \underbrace{\frac{1}{2}p\left[1-\mu_o+\mu_o\tfrac{1-\delta^H+\delta^H\underline{u}_{em}}{1-\delta^H(1-p)} + 1-\mu_o+\mu_o\tfrac{1-\delta^H+\delta^H\underline{u}_{em}}{1-\delta^H(1-p)}\right]}_{\equiv\kappa} \qquad (44) \\
&+ \frac{1}{2}p\left[\mu_o\tfrac{(1-\delta^H)k}{1-\delta^H(1-p)} - (1-\mu_o)\tfrac{1-\delta^H}{1-\delta^H(1-p)} - \mu_o\tfrac{1-\delta^H}{1-\delta^H(1-p)}\right]z \\
&= \kappa + \frac{p\left(1-\delta^H\right)}{2\left(1-\delta^H(1-p)\right)}\left[-(1-\mu_o)-\mu_o+\mu_o k\right]z \\
&= \kappa + \frac{p(1-\delta^H)(\mu_o k-1)}{2\left(1-\delta^H(1-p)\right)}z \\
\therefore \frac{\partial u_D^H}{\partial z} &= \frac{p(1-\delta^H)(\mu_o k-1)}{2\left(1-\delta^H(1-p)\right)} \gtrless 0 \iff \mu_o k \gtrless 1.
\end{aligned}
$$

Since total welfare of high types is either increasing or decreasing, the optimal choice of the initial favor size is one of the endpoints of interval $[\underline{z}, \bar{z}]$, the set of favors consistent with DFFM equilibria of proposition 13. The only exception is $\mu_o k = 1 \Rightarrow \frac{\partial u_D^H}{\partial z} = 0$, for which case a favor of any size in $[\underline{z}, \bar{z}]$ will do. ■

Proposition 14 applies to DFFM equilibria consistent with proposition 13. That proposition implemented a full EM endgame post-separation. However, we showed in section 3 that high types can always move from the full EM line to the Pareto optimal HSSGL line either immediately, or after sufficiently many periods of EM play. For DFFM equilibria beyond proposition 13, in particular for equilibria that implement a HSSGL endgame, proposition 14 does not apply. However, it stands to reason that since low types would have more incentive to pool with high types, the high types would have more incentive to choose a higher initial favor to deter the low types, and in particular, to cut down the number of periods of EM play necessary to move to the HSSGL endgame in instances where immediate separation to HSSGL is not possible.

5 Comparison of DFFM and SS equilibria

In this section we compare these types of symmetric strategies for separation from Kalla [7] to the designated first favor maker strategies from section 4. The appeal of the SS equilibria is that if both agents are high types then separation is twice as fast as in DFFM equilibria conditional on it occurring before the agents become too pessimistic about each other's type to separate. In particular, ignoring a cutoff period for separation that is part of SS strategies when favor opportunities are mutually exclusive[4], the average number of periods to it under DFFM and SS strategies

[4]The details regarding the cutoff period are available in Kalla [7].

respectively, T^{dffm}_{sep} and T^{ss}_{sep}, can be calculated using geometric series:

$$\begin{aligned} T^{dffm}_{sep} &= 1p + 2(1-p)p + 3(1-p)^2 p + ... \\ &= p\sum_{t=1}^{\infty} t(1-p)^{t-1} = \tfrac{1}{p}, \text{ and} \end{aligned} \quad (45)$$

$$\begin{aligned} T^{ss}_{sep} &= 1(2p) + 2(1-2p)(2p) + 3(1-2p)^2(2p) + ... \\ &= 2p\textstyle\sum_{t=1}^{\infty} t(1-2p)^{t-1} = \tfrac{1}{2p}. \end{aligned} \quad (46)$$

But SS equilibria do not guarantee separation and the value of the post-separation endgame depreciates over time during the initial separation phase unlike with the DFFM equilibria. The expected payoffs for the latter have already been examined, but for the former the expected payoff for a high type at the start of the game would be

$$u^H_{sym} = (1-\mu_o)\, u^{HL}_{sym} + \mu_o u^{HH}_{sym},$$

where u^{HL}_{sym} and u^{H}_{sym} are the expected payoffs under σ when facing a low type and a high type, respectively. From Kalla [7]:[5]

$$\begin{aligned} u^{HL}_{sym} &= p - S^*_1 \text{ where } S^*_1 = p\left(1-\delta^H\right)\textstyle\sum_{i=0}^{\infty}\left(\delta^H(1-p)\right)^i z_{1+i}, \text{ and} \\ u^{HH}_{sym} &= p + S^*_2 \text{ where } S^*_2 = p\,(k-1)\textstyle\sum_{i=0}^{\infty}\left(\delta^H(1-2p)\right)^i \left(\left(1-\delta^H\right) z_{1+i} + p\delta^H m\left(z_{1+i}\right)\right). \end{aligned}$$

In the last step we used that $A + B = p(k-1)$, where A and B are defined by (56) and (58), respectively. Then

$$\begin{aligned} u^H_{sym} &= (1-\mu_o)(p - S^*_1) + \mu_o(p + S^*_2) \\ &= p - S^*_1 + \mu_o\left(S^*_1 + S^*_2\right). \end{aligned} \quad (47)$$

Next we will go over two examples, the main point of which is that depending on the parameter values either SS or DFFM equilibrium may dominate the other.

Example 15 (SS dominates DFFM) *Consider a game with two unknown type agents. Suppose* $p = 0.45,\ \delta^L = 0.6,\ \delta^H = 0.8,\ k = 2,\ \mu_o = 0.7$. *Then* $\delta^* = \frac{1}{1+p(k-1)} = 0.690 \implies \delta^L < \delta^* \leq \delta^H$.

Consider a DFFM equilibrium first and suppose that the agents randomize evenly over who is to be the first favor maker. Since $\mu_o k > 1$, this means that z should be maximized per equation (43), so $\bar{z} = \min\left\{1, \frac{\mu_o\delta^H p\left(\left(1-\delta^H\right)k + \delta^H p(k-1)\right)}{(1-\delta^H)(1-\delta^H(1-2p))}\right\} = 1$. And plugging in the values to u^H_D from equation (44) yields $u^H_D = 0.583$. Assuming the designated first favor maker is a high type, separation will

[5] u^{HL}_{sym} and u^{HH}_{sym} are symmetric to $u^{HL}_{-z_t}$ and $u^{HH}_{-z_t}$ from the proof of proposition 10 (**??**) in Kalla [7], except that z_t starts from z_1 instead of z_{t+1}.

take on average 2.22 rounds but it happens for sure and the endgame will be a full EM game if both agents are high types.

Now compare these results to the SS equilibrium. From a proposition in Kalla [7] we have an upper bound for the number of periods that separation may take

$$T = \sup \left\{ t \in \mathbb{N} : \frac{\mu_o \left(1-2p\right)^{t-1}}{\mu_o \left(1-2p\right)^{t-1} + \left(1-\mu_o\right)\left(1-p\right)^{t-1}} \geq \frac{1-\delta^H}{\delta^H BM} \right\}. \tag{48}$$

Given the parameters, the right-hand side of supremum function is 0.3794 while $\mu_0^a = 0.7$ and $\mu_1^a = 0.2979$ so $T = 1$. It is then straightforward to verify that $z = (1, 0, 0, 0, ...)$ maximizes the agents' ex-ante payoffs. In a pure EM game with no designated first favor maker neither agent would make the first full favor by choice, however, the difference in this example is that if the agent who receives the favor opportunity does not do the favor, the chance to separate is lost for ever, so in essence she became the disadvantaged agent as soon as she received the favor opportunity, and that is still better than an autarky continuation payoff for the given parameters.

So in this case separation occurs with probability of 0.9 conditional on both agents being high types and will lead to an endgame of full equality matching. The incentive compatibility constraint is satisfied and the expected payoff from (47) is

$$u_{sym}^H = p - p\left(1-\delta^H\right) + \mu_o p\left(\left(1-\delta^H\right) + (k-1)\left(1-\delta^H(1-p)\right)\right) = 0.599.$$

So in this example the SS equilibrium is clearly better than the best available DFFM equilibrium, which makes sense since separation is 0.9 likely during the first period instead of just 0.45 likely, and the endgames are the same conditional on separation.

Example 16 (DFFM dominates SS) *Consider two agents of unknown type who wish to play a favor-trading game. Suppose* $p = 0.1$, $\delta^L = 0.9$, $\delta^H = 0.91$, $k = 2$ *and* $\mu_o = 0.45$. *Then* $\delta^* = \frac{1}{1+p(k-1)} = 0.9091 \implies \delta^L < \delta^* \leq \delta^H$.

Consider a DFFM equilibrium first and suppose that the agents randomize over who is to be the first favor maker. Since $\mu_o k < 1$, this means that z should be minimized per (43), so $\underline{z} = \frac{\mu_o \delta^H pk}{1-\delta^L(1-p)} = 0.4263$. And plugging in the values to u_D^H from (44) yields $u_D^H = 0.1103$, which is just above the autarky payoff of $p = 0.1$. Assuming the designated first favor maker is a high type, separation will take on average 10 rounds but it happens with probability 1 at some point, and the endgame will be a full EM game if both agents are high types.

Now compare these results to the SS equilibrium. From (48) we have an upper bound for the number of periods that separation may take. Given the parameters the right-hand side of supremum function $\frac{1-\delta^H}{\delta^H BM} = 0.4232$ while $\mu_0^a = 0.45$ and $\mu_1^a = 0.4211$ assuming no favor opportunity, and no favor was received, so $T = 1$. That is, separation has to occur during the first period or not at all. It is then straightforward to verify that $z = (1/M, 0, 0, ...) = (0.4263, 0, 0, ...)$ per condition

(49) maximizes the agents ex-ante payoff.[6]

So in this case separation occurs with probability of 0.2 conditional on both agents being high types and will lead to an endgame of full equality matching. The incentive compatibility constraint is satisfied and the expected payoff from (47) is

$$u_{sym}^{H} = p - p\left(1-\delta^{H}\right) + \mu_{o}p\left(\left(1-\delta^{H}\right) + (k-1)\left(1-\delta^{H}\left(1-p\right)\right)\right) = 0.1037.$$

In this example the DFFM equilibrium is clearly better than any SS equilibrium, which makes sense since separation is possible with only 0.2 probability in the latter case whereas in the DFFM equilibrium it happens with probability 1 sooner or later and the agents are fairly patient at $\delta^{H} = 0.91$.

6 Conclusion

We have shown that using a simple EM mechanism, immediate separation can be implemented in the favor-trading games with one-sided incomplete information without the need to impose any conditions on the discount factors or parameters of the model. We also describe the necessary conditions for immediate separation into more profitable HSSGL equilibria, and prove that even when these conditions are not met, HSSGL equilibria can be achieved later on in the game by playing an EM game sufficiently many periods. We also constructed and characterized a DFFM equilibrium that provides a robust way for high types to separate in a favor-trading game with two-sided incomplete information. We compared the payoffs from DFFM equilibria to SS equilibria analyzed in a companion paper, and showed that either one may dominate depending on the parameter values.

[6]From Kalla [7]: In equilibrium, $\bar{z}_t$ and $m\left(z_t\right)$ satisfy the following relationship,

$$\frac{m\left(\bar{z}_t\right)}{\bar{z}_t} - 1 \leq \frac{1-\delta^{L}\left(1-p\right)}{\mu_{o}\delta^{L}pk} \equiv M \in \left(\frac{1}{\mu_{o}}, \infty\right), \forall t \leq \bar{n}, \quad (49)$$

7 Appendix

Proof. (Lemma 5: Necessary and sufficient condition for EM) Suppose we start the game with agent b as the advantaged agent and the level of trust is $z \in (0,1]$, that is agent a does a favor of size $x = z$ if she receives a favor opportunity, and then does no further favors until the other agent, in this case agent b, reciprocates by doing a favor of size $y = z$. It follows that in terms of average discounted payoffs

$$\begin{aligned}
\underline{u}_{em(z)} &= p\left(\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em(z)}\right)+(1-p)\delta^H\underline{u}_{em(z)} \\
&= p\tfrac{\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em(z)}}{1-\delta^H(1-p)} \\
\bar{u}_{em(z)} &= p\left(\left(1-\delta^H\right)+\delta^H\bar{u}_{em(z)}\right) \\
&\quad + p\left(\left(1-\delta^H\right)kz+\delta^H\underline{u}_{em(z)}\right)+(1-2p)\delta^H\bar{u}_{em(z)} \\
&= p\tfrac{\left(1-\delta^H\right)(1+kz)+\delta^H\underline{u}_{em(z)}}{1-\delta^H(1-p)}.
\end{aligned}$$

The two equations above are in two unknowns, $\underline{u}_{em(z)}$ and $\bar{u}_{em(z)}$. Solving for these yields,

$$\underline{u}_{em(z)} = p + p\tfrac{-\left(1-\delta^H\right)+\delta^H p(k-1)}{1-\delta^H(1-2p)}z \tag{50}$$

$$= p + Az \text{ where } A \equiv p\tfrac{-\left(1-\delta^H\right)+\delta^H p(k-1)}{1-\delta^H(1-2p)} \tag{51}$$

$$\bar{u}_{em(z)} = p + p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}z \tag{52}$$

$$= p + Bz \text{ where } B \equiv p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}. \tag{53}$$

For z fixed, as δ^H ranges from δ^* to 1, $\underline{u}_{em(z)}$ ranges from p to $p+\frac{1}{2}p(k-1)z$ and $\bar{u}_{em(z)}$ from $p+p(k-1)z$ to $p+\frac{1}{2}p(k-1)z$. In particular, for any $\delta^H \in [\delta^*, 1)$,

$$\begin{aligned}
\underline{u}_{em(z)} + \bar{u}_{em(z)} &= 2p + p(k-1)z, \text{ or} \\
&= p\,(k+1) \text{ for } z = 1.
\end{aligned} \tag{54}$$

Agent a's incentive compatibility constraint is

$$\begin{aligned}
ICC^a_{em(z)} : \left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em(z)} &\geq \left(1-\delta^H\right)+\delta^H\underline{u}_{em(z)} \\
\iff \bar{u}_{em(z)} - \underline{u}_{em(z)} - \tfrac{\left(1-\delta^H\right)}{\delta^H}z &\geq 0,
\end{aligned}$$

which using equations (55) and (57) is equivalent to

$$\begin{aligned}
&p + p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}z - p - p\tfrac{-\left(1-\delta^H\right)+\delta^H p(k-1)}{1-\delta^H(1-2p)}z \geq \tfrac{\left(1-\delta^H\right)}{\delta^H}z \\
&\text{solving for } \delta^H \implies \tfrac{1}{1+p(k-1)} \leq \delta^H.
\end{aligned}$$

Recall that $\delta^* = \dfrac{1}{1+p(k-1)}$ so $\delta^* \leq \delta^H$ is necessary and sufficient to implement any simple EM strategy profile. $x = y = 1$ ensures the greatest gains from cooperation. ■

Proof. (Lemma 7: Separation with one-sided incomplete information) Suppose we start the game with agent b as the advantaged agent, and the level of trust is $z \in (0,1]$; that is agent a does a favor of size $x = z$ if she receives a favor opportunity, and does no further favors until the other agent, in this case agent b, reciprocates by doing a favor of size $y = z$. It follows that in terms of average discounted payoffs

$$\begin{aligned}
\underline{u}_{em(z)} &= p\left(\left(1-\delta^H\right)(1-z) + \delta^H \bar{u}_{em(z)}\right) + (1-p)\delta^H \underline{u}_{em(z)} \\
&= p\tfrac{\left(1-\delta^H\right)(1-z)+\delta^H \bar{u}_{em(z)}}{1-\delta^H(1-p)}. \\
\bar{u}_{em(z)} &= p\left(\left(1-\delta^H\right) + \delta^H \bar{u}_{em(z)}\right) \\
&\quad + p\left(\left(1-\delta^H\right)kz + \delta^H \underline{u}_{em(z)}\right) + (1-2p)\delta^H \bar{u}_{em(z)} \\
&= p\tfrac{\left(1-\delta^H\right)(1+kz)+\delta^H \underline{u}_{em(z)}}{1-\delta^H(1-p)}.
\end{aligned}$$

The two equations above are in two unknowns, $\underline{u}_{em(z)}$ and $\bar{u}_{em(z)}$. Solving for these yields,

$$\begin{aligned}
\underline{u}_{em(z)} &= p + p\tfrac{-\left(1-\delta^H\right)+\delta^H p(k-1)}{1-\delta^H(1-2p)} z && (55) \\
&= p + Az \text{ where } A \equiv p\tfrac{-\left(1-\delta^H\right)+\delta^H p(k-1)}{1-\delta^H(1-2p)} && (56) \\
\bar{u}_{em(z)} &= p + p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)} z && (57) \\
&= p + Bz \text{ where } B \equiv p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}. && (58)
\end{aligned}$$

For z fixed, as δ^H ranges from δ^* to 1, $\underline{u}_{em(z)}$ ranges from p to $p + \frac{1}{2}p(k-1)z$ and $\bar{u}_{em(z)}$ from $p + p(k-1)z$ to $p + \frac{1}{2}p(k-1)z$. In particular, for any $\delta^H \in [\delta^*, 1)$,

$$\begin{aligned}
\underline{u}_{em(z)} + \bar{u}_{em(z)} &= 2p + p(k-1)z, \text{ or} \\
&= p\,(k+1) \text{ for } z = 1. && (59)
\end{aligned}$$

Agent a's incentive compatibility constraint is

$$\begin{aligned}
ICC^a_{em(z)} : \left(1-\delta^H\right)(1-z) + \delta^H \bar{u}_{em(z)} &\geq \left(1-\delta^H\right) + \delta^H \underline{u}_{em(z)} \\
\iff \bar{u}_{em(z)} - \underline{u}_{em(z)} - \tfrac{\left(1-\delta^H\right)}{\delta^H} z &\geq 0,
\end{aligned}$$

which using equations (55) and (57) is equivalent to

$$\begin{aligned}
&p + p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)} z - p - p\tfrac{-\left(1-\delta^H\right)+\delta^H p(k-1)}{1-\delta^H(1-2p)} z \geq \tfrac{\left(1-\delta^H\right)}{\delta^H} z \\
&\text{solving for } \delta^H \implies \tfrac{1}{1+p(k-1)} \leq \delta^H.
\end{aligned}$$

Recall that $\delta^* = \dfrac{1}{1+p(k-1)}$ so $\delta^* \leq \delta^H$ is necessary and sufficient to implement any simple EM strategy profile. $x = y = 1$ ensures the greatest gains from cooperation. ■

Proof. (Lemma 8: HSSGL payoff to advantaged low types) Let $\overline{v}_s^L$ denote agent b's continuation payoff s periods after the game starts conditional on b not having received a favor yet. Let $\underline{v}_s^L$ denote agent b's continuation payoff s periods after she has received a favor. And let $\bar{x}_s$ and $\underline{x}_s$ denote the favors as specified by (16) and (17) that agent a will do following states associated with $\overline{v}_s^L$ and $\underline{v}_s^L$, respectively. From equation (11) we already know that $\overline{x}_0 = 1$ and $\underline{x}_0 = \frac{\delta^H - \delta^*}{\delta^H + \delta^*}$. Given the notation, we may write agent b's expected payoff as follows,

$$
\begin{aligned}
\overline{v}_0^L &= p\left(\left(1-\delta^L\right)k\overline{x}_0 + \delta^L \underline{v}_0^L\right) + (1-p)\left(\left(1-\delta^L\right)\tfrac{p}{1-p} + \delta^L \overline{v}_1^L\right) \\
&= p\left(1-\delta^L\right) + p\delta^L \underline{v}_0^L + p\left(1-\delta^L\right)k\overline{x}_0 + (1-p)\,\delta^L \overline{v}_1^L \\
&= p\left(1-\delta^L\right) + p\delta^L \underline{v}_0^L + p\left(1-\delta^L\right)k\overline{x}_0 \\
&+ (1-p)\delta^L\left(p\left(1-\delta^L\right) + p\delta^L\underline{v}_0^L + p\left(1-\delta^L\right)k\overline{x}_1 + (1-p)\delta^L\overline{v}_2^L\right) \\
&= p\left(1-\delta^L\right)\left(1+(1-p)\delta^L\right) + p\delta^L\underline{v}_0^L\left(1+(1-p)\delta^L\right) \\
&+ p\left(1-\delta^L\right)k\left(\overline{x}_0 + (1-p)\delta^L\overline{x}_1\right) + \left((1-p)\delta^L\right)^2\overline{v}_2^L \\
&= p\left(1-\delta^L\right)\left(1+d+d^2+...\right) + p\delta^L\underline{v}_0^L\left(1+d+d^2+...\right) \\
&+ p\left(1-\delta^L\right)k\left(\overline{x}_0 + d\overline{x}_1 + d^2\overline{x}_2 + ...\right) \text{ where } d = (1-p)\delta^L \\
&= p\tfrac{1-\delta^L}{1-d} + \tfrac{p\delta^L}{1-d}\underline{v}_0^L + p\left(1-\delta^L\right)k\sum_{t=0}^{\infty} d^t\overline{x}_t. \qquad (60)
\end{aligned}
$$

To proceed further, we need to calculate $\underline{v}_0^L$ and $\overline{x}_t$.

Claim 8a: Let $\bar{u}\,(t)$ denote the HSSGL continuation promise to an advantaged agent t periods since the last favor in a game of complete information. Then,

$$\bar{u}(t) = p + \tfrac{\left(\delta^H\right)^{t+1}(1-\delta^*)+\left(1-\delta^H\right)(\delta^*)^{t+1}}{\left(\delta^H\right)^t\delta^*(\delta^H+\delta^*)}, \qquad (61)$$

$$\overline{x}_t = \tfrac{\delta^H + \left(\delta^*/\delta^H\right)^t\delta^*}{\delta^H+\delta^*}. \qquad (62)$$

Proof of claim 8a: By (14) and (15),

$$
\begin{aligned}
\bar{u} &= \underline{u} + \tfrac{2\left(1-\delta^H\right)}{\delta^H+\delta^*} = p + \tfrac{\delta^H-\delta^*}{\delta^H+\delta^*}\tfrac{1}{\delta^*} + \tfrac{2\left(1-\delta^H\right)}{\delta^H+\delta^*} \\
&= p + \tfrac{\delta^H(1-\delta^*)+\left(1-\delta^H\right)\delta^*}{\delta^*(\delta^H+\delta^*)} = \bar{u}\,(0)\,.
\end{aligned}
$$

Suppose (61) holds for some $s \in \mathbb{N}$, then by (17),

$$\bar{u}\,(s+1) = \delta^*\left(\tfrac{\bar{u}(s)-\left(1-\delta^H\right)p}{\delta^H} + \tfrac{\underline{u}(1-\delta^*)}{\delta^*}\right)$$

$$= \delta^* \left(\frac{1}{\delta^H} \left(p + \frac{\left(\delta^H\right)^{s+1}(1-\delta^*) + \left(1-\delta^H\right)(\delta^*)^{s+1}}{\left(\delta^H\right)^s \delta^* (\delta^H+\delta^*)} \right) - p \frac{1-\delta^H}{\delta^H} + \left(p + \frac{\delta^H - \delta^*}{\delta^H+\delta^*} \frac{1}{\delta^*} \right) \frac{1-\delta^*}{\delta^*} \right)$$
$$= \delta^* \left(\frac{1}{\delta^H} - \frac{1-\delta^H}{\delta^H} + \frac{1-\delta^*}{\delta^*} \right) p + \frac{\delta^*}{\delta^H} \frac{\left(\delta^H\right)^{s+1}(1-\delta^*) + \left(1-\delta^H\right)(\delta^*)^{s+1}}{\left(\delta^H\right)^s \delta^* (\delta^H+\delta^*)} + (1-\delta^*) \frac{\delta^H - \delta^*}{\delta^H+\delta^*} \frac{1}{\delta^*}$$
$$= p + \frac{\left(\delta^H\right)^{s+2}(1-\delta^*) + \left(1-\delta^H\right)(\delta^*)^{s+2}}{\left(\delta^H\right)^{s+1} \delta^* (\delta^H+\delta^*)} = \bar{u}\,(s+1)\,.$$

Given that equation (61) holds for $s+1$ if it holds for s, and we know it holds for $s=0$, then by induction (61) must hold for all $s \in \mathbb{N}$. Last, it is a straightforward computation to verify that if we apply (16), the equation to compute HSSGL favors from payoffs, to (61), the hypothetical payoff to a high type, and simplify, formula (62) results. [■]

Claim 8b: Let $\underline{u}\,(t)$ denote the HSSGL continuation promise to a disadvantaged agent t periods since the last favor in the complete information game. Then,

$$\underline{u}\,(t) = p + \frac{\left(\delta^H\right)^{t+1}(1-\delta^*) - \left(1-\delta^H\right)(\delta^*)^{t+1}}{\left(\delta^H\right)^t \delta^* (\delta^H+\delta^*)} \tag{63}$$

$$\underline{x}_t = \frac{\delta^H - \left(\delta^*/\delta^H\right)^t \delta^*}{\delta^H+\delta^*} \tag{64}$$

Proof of claim 8b: By (14), $\underline{u} = p + \frac{\delta^H-\delta^*}{\delta^H+\delta^*}\frac{1}{\delta^*} = \underline{u}\,(0)$. Suppose (63) holds for some $s \in \mathbb{N}$, then by (17)

$$\underline{u}\,(s+1) = \delta^* \left(\frac{\underline{u}(s) - \left(1-\delta^H\right)p}{\delta^H} + \frac{\underline{u}(1-\delta^*)}{\delta^*} \right)$$
$$= \frac{\delta^*}{\delta^H} \left(p + \frac{\left(\delta^H\right)^{s+1}(1-\delta^*) - \left(1-\delta^H\right)(\delta^*)^{s+1}}{\left(\delta^H\right)^s \delta^* (\delta^H+\delta^*)} \right) - \frac{1-\delta^H}{\delta^H} p + \left(p + \frac{\delta^H - \delta^*}{\delta^H+\delta^*} \frac{1}{\delta^*} \right) \frac{1-\delta^*}{\delta^*}$$
$$= \delta^* \left(\frac{1}{\delta^H} - \frac{1-\delta^H}{\delta^H} + \frac{1-\delta^*}{\delta^*} \right) p + \frac{\delta^*}{\delta^H} \frac{\left(\delta^H\right)^{s+1}(1-\delta^*) - \left(1-\delta^H\right)(\delta^*)^{s+1}}{\left(\delta^H\right)^s \delta^* (\delta^H+\delta^*)} + \frac{\delta^H - \delta^*}{\delta^H+\delta^*} \frac{1-\delta^*}{\delta^*}$$
$$= p + \frac{\left(\delta^H\right)^{s+2}(1-\delta^*) + \left(1-\delta^H\right)(\delta^*)^{s+2}}{\left(\delta^H\right)^{s+1} \delta^* (\delta^H+\delta^*)}.$$

Given that (63) holds for $s+1$ if it holds for s, and we know it holds for $s=0$, then by induction (63) must hold for all $s \in \mathbb{N}$. Last, it is a straightforward computation to verify that if we apply (16), the equation to calculate favors from payoffs, to (63), the hypothetical payoff to a high type, and simplify, and simplify, formula (64) results. [■]

Claim 8c: Let $d = (1-p)\delta^L$, then

$$\underline{v}_0^L = p + pk\,(1-d) \sum_{t=0}^{\infty} d^t \underline{x}_t \tag{65}$$

Proof of claim 8c: Proceeding as before.

$$\underline{v}_0^L = p\left(\left(1-\delta^L\right) k \underline{x}_0 + \delta^L \underline{v}_0^L\right) + (1-p)\left(\left(1-\delta^L\right) \frac{p}{1-p} + \delta^L \underline{v}_1^L\right)$$

$$
\begin{aligned}
&= p\left(1-\delta^L\right) + p\delta^L \underline{v}_0^L + p\left(1-\delta^L\right)k\underline{x}_0 + (1-p)\delta^L \underline{v}_1^L \\
&= p\left(1-\delta^L\right) + p\delta^L \underline{v}_0^L + p\left(1-\delta^L\right)k\underline{x}_0 \\
&+ (1-p)\delta^L\left(\left(1-\delta^L\right)p + p\delta^L \underline{v}_0^L + p\left(1-\delta^L\right)k\underline{x}_1 + (1-p)\delta^L \underline{v}_2^L\right) \\
&= p\left(1-\delta^L\right)\left(1+(1-p)\delta^L\right) + p\delta^L \underline{v}_0^L\left(1+(1-p)\delta^L\right) \\
&+ p\left(1-\delta^L\right)k\left(\underline{x}_0 + (1-p)\,\delta^L \underline{x}_1\right) + \left((1-p)\delta^L\right)^2 \underline{v}_2^L \\
&= p\left(1-\delta^L\right)\left(1+d+d^2+...\right) + p\delta^L \underline{v}_0^L\left(1+d+d^2+...\right) \\
&+ p\left(1-\delta^L\right)k\left(\underline{x}_0 + d\underline{x}_1 + d^2\underline{x}_2 + ...\right) \text{ where } d = (1-p)\delta^L \\
&= \tfrac{p\left(1-\delta^L\right)}{1-d} + \tfrac{p\delta^L}{1-d}\underline{v}_0^L + p\left(1-\delta^L\right)k\textstyle\sum_{t=0}^{\infty} d^t \underline{x}_t \\
&= p + pk(1-d)\textstyle\sum_{t=0}^{\infty} d^t \underline{x}_t
\end{aligned}
$$

This proves (65). [■]
We can now return to equation (60) for $\overline{v}_0^L$ and substituting in for $\underline{v}_0^L$ from (65) yields,

$$
\begin{aligned}
\overline{v}_0^L &= \tfrac{p\left(1-\delta^L\right)}{1-d} + \tfrac{p\delta^L}{1-d}\left(p + pk\,(1-d)\sum_{t=0}^{\infty} d^t \underline{x}_t\right) + p\left(1-\delta^L\right)k\sum_{t=0}^{\infty} d^t \overline{x}_t \\
&= \tfrac{p\left(1-\delta^L\right)}{1-d} + \tfrac{p^2\delta^L}{1-d} + p^2 k\delta^L \sum_{t=0}^{\infty} d^t \underline{x}_t + p\left(1-\delta^L\right)k\sum_{t=0}^{\infty} d^t \overline{x}_t \\
&= p + pk\sum_{t=0}^{\infty} d^t\left(p\delta^L \underline{x}_t + \left(1-\delta^L\right)\overline{x}_t\right)
\end{aligned}
$$

Using (62) and (64) for $\overline{x}_t$ and $\underline{x}_t$, respectively, we can write the last equation as

$$
\begin{aligned}
\overline{v}_0^L &= p + pk\sum_{t=0}^{\infty} d^t\left(p\delta^L \tfrac{\delta^H - \left(\delta^*/\delta^H\right)^t \delta^*}{\delta^H+\delta^*} + \left(1-\delta^L\right)\tfrac{\delta^H + \left(\delta^*/\delta^H\right)^t \delta^*}{\delta^H+\delta^*}\right) \\
&= p + pk\sum_{t=0}^{\infty} d^t \tfrac{\left(\delta^H\right)^{t+1}\left(1-\delta^L(1-p)\right) + \left(\delta^*\right)^{t+1}\left(1-\delta^L-\delta^L p\right)}{\left(\delta^H\right)^t\left(\delta^H+\delta^*\right)} \\
&= p + pk\sum_{t=0}^{\infty} d^t \tfrac{1-\delta^L(1-p) + \alpha^{t+1}\left(1-\delta^L-\delta^L p\right)}{1+\alpha} \text{ where } \alpha = \tfrac{\delta^*}{\delta^H} \\
&= p + pk\tfrac{1-\delta^L(1-p)}{(1+\alpha)(1-d)} + pk\tfrac{\alpha\left(1-\delta^L-\delta^L p\right)}{(1+\alpha)(1-\alpha d)}.
\end{aligned}
$$

Next we substitute $d = (1-p)\,\delta^L$ and $\alpha = \delta^*/\delta^H$ back in,

$$
\begin{aligned}
\overline{v}_0^L &= p + pk\tfrac{1-\delta^L(1-p)}{(1+\delta^*/\delta^H)(1-(1-p)\delta^L)} + pk\tfrac{\left(\delta^*/\delta^H\right)\left(1-\delta^L-\delta^L p\right)}{(1+\delta^*/\delta^H)(1-(\delta^*/\delta^H)(1-p)\delta^L)} \\
&= p + pk\tfrac{\delta^H\left(\delta^H+\delta^*-2\delta^L\delta^*\right)}{(\delta^H+\delta^*)(\delta^H-\delta^L\delta^*(1-p))}.
\end{aligned}
$$

This concludes the proof. ■

Proof. (Lemma 10: Simple lower bound for $\bar{\delta}^L$) Suppose agent b is a low type, then b's incentive compatibility constraint to do the first favor is

$$ICC^L_{hssgl} : \delta^L \overline{v}^L_0 \geq \left(1 - \delta^L\right) + \delta^L p$$

Substituting in for $\overline{v}^L_0$ from equation (22) yields

$$\delta^L \left(p + pk \frac{\delta^H \left(\delta^H + \delta^* - 2\delta^L \delta^*\right)}{(\delta^H + \delta^*)(\delta^H - \delta^L \delta^* (1-p))} \right) \geq 1 - \delta^L + \delta^L p \tag{66}$$

The above inequality implicitly defines the exact upper bound necessary to deter low types from pooling with high types. Call this bound $\bar{\delta}^L$. Cancel $\delta^L p$ on both sides of inequality (66), multiply what is left on the right side by the denominator of the left side, take everything to the left side and write the inequality as a polynomial of δ^L, then

$$\begin{aligned}
&\delta^* \left(-\delta^H - \delta^* + \delta^H p + \delta^* p - 2\delta^H pk\right) \left(\delta^L\right)^2 \\
&+ \left(\delta^H + \delta^*\right) \left(\delta^H + \delta^* - \delta^* p + \delta^H pk\right) \delta^L - \delta^H \left(\delta^H + \delta^*\right) \geq 0 \\
&\iff -\frac{\delta^* \left(\delta^H + \delta^* - \delta^H p - \delta^* p + 2\delta^H pk\right)}{\delta^H (\delta^H + \delta^*)} \left(\delta^L\right)^2 + \frac{\left(\delta^H + \delta^*\right)\left(\delta^H + \delta^* - \delta^* p + \delta^H pk\right)}{\delta^H (\delta^H + \delta^*)} \delta^L - 1 \geq 0 \\
&\iff \overbrace{-\frac{\alpha(1+\alpha(1-p)+p(2k-1))}{(1+\alpha)}}^{\equiv \hat{A}} \left(\delta^L\right)^2 + \overbrace{(1+\alpha\,(1-p)+pk)}^{\equiv \hat{B}} \delta^L - 1 \geq 0 \\
&\qquad \text{where } \alpha = \delta^* / \delta^H
\end{aligned} \tag{67}$$

$$\text{Let } Q\left(\delta^L\right) := \left(\delta^L\right)^2 \hat{A} + \delta^L \hat{B} - 1 \text{ for } \hat{A} \text{ and } \hat{B} \text{ defined above.} \tag{68}$$

If the inequality (67) binds, expressions (67)-(68) define a quadratic equation for the upper bound of δ^L with the following two solutions,

$$\bar{\delta}^L = \frac{2}{\hat{B} \pm \sqrt{\hat{B}^2 + 4\hat{A}}} \tag{69}$$

where $\hat{A}$ and $\hat{B}$ refer to expressions from equation (67). First we need to verify that $\hat{B}^2 + 4\hat{A} > 0$ so that our solutions are real numbers

$$\begin{aligned}
\hat{B}^2 + 4\hat{A} &= (1 + \alpha(1-p) + pk) - 4\frac{\alpha(1+\alpha(1-p)+p(2k-1))}{1+\alpha} > 0 \\
&= \tfrac{1}{1+\alpha}(1 - \alpha - \alpha^2 + \alpha^3 + 2\alpha p - 2\alpha^3 p + 2kp - 4\alpha kp) \\
&+ \alpha^2 kp + \alpha^2 p^2 + \alpha^3 p^2 - 2\alpha kp^2 - 2\alpha^2 kp^2 + k^2 p^2 + \alpha k^2 p^2) > 0 \\
&= \tfrac{1}{1+\alpha} \left((1-\alpha)^2 (1+\alpha) + 2(1-\alpha)\left(\alpha(1+\alpha)\right.\right. \\
&\left.\left. + k(1-\alpha)\right) p + (1+\alpha)(k-\alpha)^2 p^2\right) > 0,
\end{aligned}$$

the above expression is positive since every term in it is clearly positive. In addition, $\hat{A} < 0$,

$\hat{B} > 0$, so the denominator, and hence both roots defined by (69), are also real and positive. Since $\hat{A} < 0$, we know the quadratic equation $Q\left(\delta^L\right)$ defined by (68) is strictly concave with two positive, real roots. Since we are interested in the least upper bound for δ^L, the appropriate solution to $Q\left(\delta^L\right) = 0$, or alternatively, the left side of ICC^L_{hssgl} (the deviation), exceeds the right side after if δ^L is greater than

$$\bar{\delta}^L = \frac{2}{\hat{B} + \sqrt{\hat{B}^2 + 4\hat{A}}}. \tag{70}$$

where $\hat{A}$ and $\hat{B}$ are defined in expression (67). ■

Proof. (Proposition 12: From EM to HSSGL) If a low type does a full favor $y = 1$, then for the next T periods it will be optimal for her to play the autarky strategy instead of exchanging favors according to the simple EM mechanism as was shown in section 2. Therefore we may ignore favor opportunities she receives during those T periods as far as the incentive compatibility constraint is concerned because she would have received these opportunities had she chosen not to deviate by doing a favor of size $y = 1$. However, we do have to calculate the expected amount she will receive in reciprocation from agent a. Namely, in the first period after separation agent a will receive a favor opportunity with probability p and do a full favor worth k, but she will not do any further favors during the rest of the T periods since agent b does not reciprocate. With probability $(1-p)p$ agent a will not receive a favor opportunity during period 1, but will do so in period 2, and thus does a full favor, but no more until reciprocation for the rest of the T periods. And so forth for the other T periods. Let v^L be her expected payoff from deviating minus the (autarky) favor costs saved during the first T periods,

$$\begin{aligned} v^L = &\left(1-\delta^L\right)\Big(pk + \delta^L(1-p)p + \left(\delta^L\right)^2(1-p)^2pk + \ldots \\ &+ \left(\delta^L\right)^{T-1}(1-p)^{T-1}pk\Big) + \left(\delta^L\right)^T(1-p)^T\overline{v}^L + \left(\delta^L\right)^T\left(1-(1-p)^T\right)\underline{v}^L, \end{aligned}$$

where the $\left(\delta^L\right)^T$-terms are the continuation payments after T periods pass. With probability $(1-p)^T$ agent a did not receive income during any of the T periods and was thus not able to reciprocate which is why agent b remains advantaged and receives the continuation promise $\overline{v}^L$. Otherwise a has reciprocated and is currently the advantaged agent, so b's continuation promise is $\underline{v}^L$.

Note that the above equation contains a geometric series that can be written more compactly, and using the fact that $\underline{v}^L \leq \overline{v}^L$ we know

$$\begin{aligned} v^L &\leq \left(1-\delta^L\right)pk\textstyle\sum_{t=0}^{T-1}\left(\delta^L-\delta^Lp\right)^t + \left(\delta^L\right)^T(1-p)^T\overline{v}^L + \left(\delta^L\right)^T\overline{v}^L - \left(\delta^L\right)^T(1-p)^T\overline{v}^L \\ &= pk\tfrac{\left(1-\delta^L\right)\left(1-\left(\delta^L-\delta^Lp\right)^T\right)}{1-\delta^L(1-p)} + \left(\delta^L\right)^T\overline{v}^L. \end{aligned} \tag{71}$$

The incentive compatibility constraint for the low type not to pool is $\delta^Lv^L \leq 1-\delta^L + \left(\delta^L\right)^{T+1}p$. Note that both sides of the inequality exclude T-terms after the initial favor by agent b since they cancel each other out. Then according to condition (71) it is sufficient to show that

$$\delta^L\tfrac{\left(1-\delta^L\right)pk\left(1-\left(\delta^L-\delta^Lp\right)^T\right)}{1-\delta^L(1-p)} + \left(\delta^L\right)^{T+1}\overline{v}^L \leq 1-\delta^L + \left(\delta^L\right)^{T+1}p.$$

Since $\left(1-\left(\delta^L-\delta^L p\right)^T\right)<1$ and $\overline{v}^L<p\,(k+1)$ by (26) from the proof of lemma 10 it is enough to show that

$$\begin{aligned}&\delta^L\tfrac{\left(1-\delta^L\right)pk}{1-\delta^L(1-p)}+\left(\delta^L\right)^{T+1}p\,(1+k)\leq\left(1-\delta^L\right)+\left(\delta^L\right)^{T+1}p\\ \iff\;&\delta^L\tfrac{\left(1-\delta^L\right)pk}{1-\delta^L(1-p)}\leq 1-\delta^L-\left(\delta^L\right)^{T+1}pk\\ \iff\;&\delta^L\tfrac{pk}{1-\delta^L(1-p)}\leq 1-\tfrac{\left(\delta^L\right)^{T+1}}{1-\delta^L}pk. \end{aligned}\tag{72}$$

To enforce the inequality we need to construct a T sufficiently high that it holds. To this end observe that since $\delta^L<\delta^*$ there exists $\varepsilon>0$ such that

$$\delta^L\tfrac{pk}{1-\delta^L(1-p)}+\varepsilon=\delta^*\tfrac{pk}{1-\delta^L(1-p)}=\tfrac{pk}{(1+p(k-1))(1-\delta^L(1-p))}.$$

Choose T to be the least integer such that $pk\left(\delta^L\right)^{T+1}/\left(1-\delta^L\right)\leq\varepsilon/2$. Simplifying the last expression, it is straightforward to show that

$$\tfrac{pk}{(1+p(k-1))(1-\delta^L(1-p))}<\tfrac{pk}{(1+p(k-1))(1-\delta^*(1-p))}=1,$$

so we have that the left-hand side of (72) is strictly than $1-\varepsilon$ and the right-hand side is greater than $1-\varepsilon/2$, therefore (72) holds, which implies that the incentive compatibility constraint is satisfied and therefore the low type will not pool for this choice of T. ■

Proof. (Proposition 13: Existence of DFFM equilibria) Consider the agent designated to do the first favor of size z. Let u_z^H and u_{-z}^H denote the expected payoffs, respectively, for a high type who did and did not do the initial z favor. Similarly, for a low type let u_z^L and $u_{-z}^L=p$ denote the analogous payoffs. The latter payoff is equal to p, the autarky payoff, since we are considering just one shot deviations and if the low type does not deviate in the given period by mimicking the high type then she just falls back to the specified equilibrium autarky strategy. Then for the specified equilibrium to work the incentive compatibility constraints for the designated high and low types, respectively, require that

$$ICC_D^H:\;\left(1-\delta^H\right)(1-z)+\delta^H u_z^H\geq 1-\delta^H+\delta^H u_{-z}^H \tag{73}$$
$$ICC_D^H:\;\left(1-\delta^L\right)(1-z)+\delta^L u_z^L\leq 1-\delta^L+\delta^L p \tag{74}$$

and the continuation payoffs u_z^H, u_{-z}^H and u_z^L can be expanded into the component when facing a low type and the component when facing a high type. Namely,

$$u_z^H=(1-\mu_o)\,p+\mu_o\bar{u}_{em}, \tag{75}$$
$$u_{-z}^H=(1-\mu_o)\,u_{-z}^{HL}+\mu_o u_{-z}^{HH}, \tag{76}$$
$$u_z^L=(1-\mu_o)\,p+\mu_o u_z^{LH}. \tag{77}$$

As before, $\bar{u}_{em}$ denotes the equality matching payoff expected by an advantaged high type facing another high type. So for example, if a high type does a favor of size z this period, then her continuation payoff is just the autarky payoff p with probability $(1-\mu_o)$ since that is the likelihood he is facing a low type that will not return any favors, and with probability μ_o he is facing another high type agent so her continuation value is $\bar{u}_{em}$. Payoffs u^{HL}_{-z} and u^{H}_{-z} denote the continuation values for a high type agent who didn't do the favor this turn (i.e. deviated from the proposed equilibrium strategy), but will at the next available opportunity when facing a low type agent and a high type agent, respectively, since we are just considering one-shot deviations. Finally, u^{LH}_{z} denotes the expected payoff for a low type who mimicked a high type by doing a favor of size z; namely the expected value of a one-time full favor from the high type agent at the next available opportunity combined with the agent's own favor opportunities for the rest of the game. In other words

$$\begin{aligned} u^{HL}_{-z} &= p\left(\left(1-\delta^H\right)(1-z)+\delta^H p\right)+(1-p)\delta^H u^{HL}_{-z} \\ &= p\tfrac{\left(1-\delta^H\right)(1-z)+\delta^H p}{1-\delta^H(1-p)}, && (78)\\ u^{HH}_{-z} &= p\left(\left(1-\delta^H\right)(1-z)+\delta^H \bar{u}_{em}\right)+(1-p)\delta^H u^{HH}_{-z} \\ &= p\tfrac{\left(1-\delta^H\right)(1-z)+\delta^H \bar{u}_{em}}{1-\delta^H(1-p)}, && (79)\\ u^{LH}_{z} &= p\left(\left(1-\delta^L\right)k+\delta^L p\right)+p\left(1-\delta^L+\delta^L u^{LH}_{z}\right)+(1-2p)\delta^L u^{LH}_{z} \\ &= p\tfrac{\left(1-\delta^L\right)(k+1)+\delta^L p}{1-\delta^L(1-p)}. && (80)\end{aligned}$$

To find the lower bound for z, substitute the expression for u^L_z from (77) and substitute it into the incentive compatibility constraint for the low type given by (74), which yields

$$\left(1-\delta^L\right)(1-z)+\delta^L\left((1-\mu_o)\,p+\mu_o u^{LH}_{z}\right)\leq 1-\delta^L+\delta^L p.$$

We use (80) to replace u^{LH}_{z},

$$\left(1-\delta^L\right)(1-z)+\delta^L\left((1-\mu_o)\,p+\mu_o\left(p\tfrac{\left(1-\delta^L\right)(k+1)+\delta^L p}{1-\delta^L(1-p)}\right)\right)\leq 1-\delta^L+\delta^L p.$$

And solve for z and simplify to obtain the following lower bound,

$$z \geq \tfrac{\mu_o \delta^L p k}{1-\delta^L(1-p)} \equiv \underline{z}. \qquad (81)$$

At first it might look like the above lower bound could violate feasibility constraints for k large enough, but condition (5), $\delta^* > \delta^L = \frac{1}{1+p(k-1)}$, rules this out, which will be shown later, since as k grows large, the upper bound for δ^L grows proportionally small. So we need $z \geq \underline{z}$ to deter the low types from mimicking the high types. Next we need to solve for the highest z a designated high type agent would be willing to do a favor in order to signal her type and become the advantaged agent in an equality matching game if the other agent also turns out to be a high type. To this end we need to substitute the expressions for u^H_z and u^H_{-z} from (75) and (76), respectively, into the incentive compatibility constraint for the high type (73) which yields

$$\left(1-\delta^H\right)(1-z)+\delta^H\left((1-\mu_o)\,p+\mu_o\bar{u}_{em}\right)\geq 1-\delta^H+\delta^H\left((1-\mu_o)\,u^{HL}_{-z}+\mu_o u^{HH}_{-z}\right)$$

Next substitute in expressions for u^{HL}_{-z} and u^{HH}_{-z} from (78) and (79), respectively,

$$\begin{aligned}&\left(1-\delta^H\right)(1-z)+\delta^H\left((1-\mu_o)\,p+\mu_o\bar{u}_{em}\right)\geq 1-\delta^H\\&+\delta^H\left((1-\mu_o)\,p\frac{\left(1-\delta^H\right)(1-z)+\delta^H p}{1-\delta^H(1-p)}+\mu_o p\frac{\left(1-\delta^H\right)(1-z)+\delta^H\bar{u}_{em}}{1-\delta^H(1-p)}\right)\end{aligned}$$

And also substitute in for $\bar{u}_{em}$ from (4) to get

$$\begin{aligned}&\left(1-\delta^H\right)(1-z)+\delta^H\left((1-\mu_o)\,p+\mu_o\frac{p\left(1-\delta^H(1-p)\right)(1+k)}{1-\delta^H(1-2p)}\right)\geq 1-\delta^H\\&+\delta^H\left((1-\mu_o)\,p\frac{\left(1-\delta^H\right)(1-z)+\delta^H p}{1-\delta^H(1-p)}+\mu_o p\frac{\left(1-\delta^H\right)(1-z)+\delta^H\frac{p\left(1-\delta^H(1-p)\right)(1+k)}{1-\delta^H(1-2p)}}{1-\delta^H(1-p)}\right)\end{aligned}$$

Finally solve for the z and simplify, which results in

$$z\leq\min\left\{1,\frac{\mu_o\delta^H p\left(\left(1-\delta^H\right)k+\delta^H p(k-1)\right)}{(1-\delta^H)(1-\delta^H(1-2p))}\right\}\equiv\bar{z}. \tag{82}$$

The last step is to verify that $\bar{z}\geq\underline{z}$. To this end, we show that $\underline{z}$ from (81) is increasing in δ^L :

$$\begin{aligned}&\frac{\partial\underline{z}}{\partial\delta^L}=\frac{\mu_o pk}{\left(1-\delta^L(1-p)\right)^2}>0\\&\implies\ \underline{z}\leq\underline{z}|_{\delta^L=\delta^*}=\left.\frac{\mu_o\delta^L pk}{1-\delta^L\,(1-p)}\right|_{\delta^L=\frac{1}{1+p(k-1)}}=\mu_o.\end{aligned}$$

So it is enough to verify that $\bar{z}\geq\mu_o$ and incidentally this also proved that $\underline{z}\leq 1$.To show that $\bar{z}\geq\mu_o$ proceed as before:

$$\begin{aligned}&\frac{\partial\bar{z}}{\partial\delta^H}=\mu_o p\frac{\left(1-\delta^H\right)^2k+2\delta^H p\left(1-\delta^H(1-p)\right)(k-1)}{(1-\delta^H)^2\left(1-\delta^H(1-2p)\right)^2}>0\\&\implies\ \bar{z}\geq\ \bar{z}|_{\delta^L=\delta^*}=\mu_o\delta^H p\left.\frac{\left(1-\delta^H\right)k+\delta^H p(k-1)}{(1-\delta^H)(1-\delta^H(1-2p))}\right|_{\delta^L=\frac{1}{1+p(k-1)}}=\mu_o.\end{aligned}$$

In other words, $\bar{z}\geq\underline{z}$, so this equilibrium is incentive compatible for both types for any pre-agreed $z\in[\underline{z},\bar{z}]$ as specified by equations (81) and (82) ■

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